PHASE RETRIEVAL VIA POLARIZATION IN DYNAMICAL SAMPLING

Robert Beinert, Marzieh Hasannasab

Technische Universität Berlin

Problem Formulation

Phaseless Dynamical Samples

Recover an unknown vector $\boldsymbol{x} \in \mathbb{C}^d$ that evolves under the action of a matrix $\boldsymbol{A} \in \mathbb{C}^{d \times d}$ from its *phaseless dynamical measurements*

$$|\langle \boldsymbol{x}, \boldsymbol{A}^\ell \boldsymbol{\phi} \rangle|, \quad \ell = 0, \dots, L-1, \tag{1}$$

with $L \geq d$.

- More precisely, at time $\ell \in \mathbb{N}$ the signal becomes $\boldsymbol{x}_\ell = (\boldsymbol{A}^*)^\ell \boldsymbol{x}$. Therefore dynamical samples in (**??**) corresponds to sampling $\boldsymbol{x}$ at different times:
$$|\langle \boldsymbol{x}_\ell, \boldsymbol{\phi} \rangle| = |\langle (\boldsymbol{A}^*)^\ell \boldsymbol{x}, \boldsymbol{\phi} \rangle| = |\langle \boldsymbol{x}, \boldsymbol{A}^\ell \boldsymbol{\phi} \rangle|.$$
- To ensure that we can do phase retrieval, we assume that the sequence $\{\boldsymbol{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ is a frame.

Dynamical Frames

An arbitrary vector $\boldsymbol{x} \in \mathbb{C}^d$ can be recovered from the set $\{\langle \boldsymbol{A}^\ell \boldsymbol{x}, \boldsymbol{\phi} \rangle\}_{\ell=0}^{L-1}$ in an stable way if there exists $\alpha, \beta > 0$ such that

$$\alpha \|\boldsymbol{y}\|^2 \leq \|\{\langle \boldsymbol{A}^\ell \boldsymbol{y}, \boldsymbol{\phi} \rangle\}_{\ell=0}^{L-1}\|^2 \leq \beta \|\boldsymbol{y}\|^2, \quad \text{for all } \boldsymbol{y} \in \mathbb{C}^d,$$

i.e., when the set $\{(\boldsymbol{A}^*)^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ is a frame for $\mathbb{C}^d$.

Consider the Jordan decomposition of the matrix $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{J}\boldsymbol{S}^{-1}$ into an invertible matrix $\boldsymbol{S} \in \mathbb{C}^{d \times d}$ and the blocked diagonal Jordan matrix $\boldsymbol{J} \in \mathbb{C}^{d \times d}$ of the form

$$\boldsymbol{J} = \operatorname{diag}(\boldsymbol{J}_0, \dots, \boldsymbol{J}_{M-1}) \qquad \text{with} \qquad \boldsymbol{J}_j = \begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & 1 \\ & & \lambda_j \end{pmatrix} \in \mathbb{C}^{m_j \times m_j},$$

where λ_j is the jth eigenvalue of $\boldsymbol{A}$ and m_j the corresponding algebraic multiplicity. We say that $\boldsymbol{\phi}$ depends on the jth Jordan generator if $(\boldsymbol{S}^{-1}\boldsymbol{\phi})_{k-1} \neq 0$ where $k = \sum_{i=0}^{j-1} m_i$.

Theorem (Characterization Based on Jordan Matrix)

The sequence $\{\boldsymbol{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ is a frame if and only if the eigenvalues of the Jordan blocks of $\boldsymbol{A}$ are pairwise distinct and $\boldsymbol{\phi}$ depends on all Jordan generators.

- For a diagonalizable matrix $\boldsymbol{A} \in \mathbb{C}^{d \times d}$, the sequence $\{\boldsymbol{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ is a frame if and only if the eigenvalues of $\boldsymbol{A}$ are pairwise distinct and $\boldsymbol{\phi}$ depends on all eigenvectors.

Repeated Convolution

Let $\boldsymbol{\phi}, \boldsymbol{a} \in \mathbb{C}^d$ be arbitrary. Then the family

$$\{\underbrace{\boldsymbol{a} * \cdots * \boldsymbol{a}}_{\ell \text{ times}} * \boldsymbol{\phi}\}_{\ell=0}^{L-1}$$

is a frame for $\mathbb{C}^d$ if and only if the coordinates of $\hat{\boldsymbol{\phi}}$ do not vanish and the coordinates of $\hat{\boldsymbol{a}}$ are pairwise distinct.

Full-Spark Dynamical Frames

A frame $\{\boldsymbol{f}_k\}_{k=0}^{L-1}$ has full spark if every subset embracing d elements is again a frame for $\mathbb{C}^d$.

Theorem (Characterization Based on Vandermonde Matrix)

Let $\boldsymbol{A} \in \mathbb{C}^{d \times d}$ be diagonalizable with eigenvalues $\boldsymbol{\lambda}$. For every $L \geq d$, the set $\{\boldsymbol{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ is a full spark frame if and only if $\boldsymbol{\phi}$ depends on all eigenvectors and the Vandermonde matrix $\boldsymbol{V}_{\boldsymbol{\lambda}} \in \mathbb{C}^{d \times L}$ generated by $\boldsymbol{\lambda}$ has full spark.

Special cases:

- $\boldsymbol{A}$ has distinct eigenvalues of the form λ^k for $\lambda \in \mathbb{C}$, and $\boldsymbol{\phi}$ depends on all eigenvectors.
- $\boldsymbol{A}$ has distinct non-negative eigenvalues, and $\boldsymbol{\phi}$ depends on all eigenvectors.

Phase Retrieval in Dynamical Sampling

Theorem (Polarization Identity)

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ satisfy $\alpha_1 - \alpha_2 \notin \pi\mathbb{Z}$. Then, for every $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, the product $\bar{z}_1 z_2$ is uniquely determined by

$$|z_1|, \quad |z_2|, \quad |z_1 + \mathrm{e}^{\mathrm{i}\alpha_1} z_2|, \quad |z_1 + \mathrm{e}^{\mathrm{i}\alpha_2} z_2|.$$

Extending the measurement set and using the above polarization allow the extraction of relative phases for almost all vectors.

Theorem (Recovery Guarantee for Dynamical Frames)

Let $\{\boldsymbol{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ be a frame for $\mathbb{C}^d$, and let $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \notin \pi\mathbb{Z}$. Then almost all $\boldsymbol{x} \in \mathbb{C}^d$ can be recovered from

$$\left\{|\langle \boldsymbol{x}, \boldsymbol{A}^\ell \boldsymbol{\phi} \rangle|\right\}_{\ell=0}^{L-1} \cup \left\{|\langle \boldsymbol{x}, \boldsymbol{A}^\ell (\boldsymbol{\phi} + \mathrm{e}^{\mathrm{i}\alpha_k} \boldsymbol{A}\boldsymbol{\phi}) \rangle|\right\}_{\ell=0,k=1}^{L-2,2}$$

up to global phase.

- If $\{\boldsymbol{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ is a frame for $\mathbb{R}^d$, and $\alpha \in \{-1, 1\}$, then almost every $\boldsymbol{x} \in \mathbb{R}^d$ can be recovered from
$$\left\{|\langle \boldsymbol{x}, \boldsymbol{A}^\ell \boldsymbol{\phi} \rangle|\right\}_{\ell=0}^{L-1} \cup \left\{|\langle \boldsymbol{x}, \boldsymbol{A}^\ell (\boldsymbol{\phi} + \alpha \boldsymbol{A}\boldsymbol{\phi}) \rangle|\right\}_{\ell=0}^{L-2}$$
up to sign.
- The idea of the proof is to use polarization identity to propagate the phase from $\langle \boldsymbol{x}, \boldsymbol{A}^\ell \boldsymbol{\phi} \rangle$ to $\langle \boldsymbol{x}, \boldsymbol{A}^{\ell+1} \boldsymbol{\phi} \rangle$.

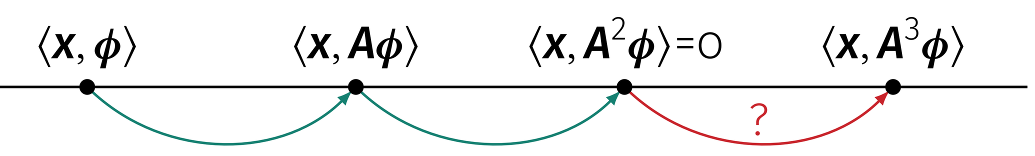

Figure: Main idea: propagating phases between coefficients

- The procedure fails if some of the coefficient $|\langle x, \boldsymbol{A}^\ell \boldsymbol{\phi} \rangle|$ becomes zero.
- We can resolve this problem by considering full-spark frames.

Theorem (Recovery Guarantee for Full-Spark Dynamical Frames)

Let $\{\boldsymbol{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ be a full-spark frame, and let $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \notin \pi\mathbb{Z}$. If $L \geq d^2/4 + d/2$, then every $\boldsymbol{x} \in \mathbb{C}^d$ can be recovered from the samples

$$\left\{|\langle \boldsymbol{x}, \boldsymbol{A}^\ell \boldsymbol{\phi} \rangle|\right\}_{\ell=0}^{L-1} \cup \left\{|\langle \boldsymbol{x}, \boldsymbol{A}^\ell (\boldsymbol{\phi} + \mathrm{e}^{\mathrm{i}\alpha_k} \boldsymbol{A}\boldsymbol{\phi}) \rangle|\right\}_{\ell=0,k=1}^{L-2,2}$$

up to global phase.

Following result provides trade-off between the number of spatial and temporal samples.

Theorem (Jumping Over Zeros)

Let $\{\boldsymbol{A}^\ell \boldsymbol{\phi}\}_{\ell=0}^{L-1}$ be a full-spark frame, let $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \notin \pi\mathbb{Z}$, and let $J \in \{0, \dots, d-2\}$. If $L \geq d^2/4(J+1) + d$, then every $\boldsymbol{x} \in \mathbb{C}^d$ can be recovered from the samples

$$\left\{|\langle \boldsymbol{x}, \boldsymbol{A}^\ell \boldsymbol{\phi} \rangle|\right\}_{\ell=0}^{L-1} \cup \left\{|\langle \boldsymbol{x}, \boldsymbol{A}^\ell (\boldsymbol{\phi} + \mathrm{e}^{\mathrm{i}\alpha_k} \boldsymbol{A}^j \boldsymbol{\phi}) \rangle|\right\}_{\ell=0,k=1,j=1}^{L-2,2,J+1}$$

up to global phase.

Conclusion

- We start with one sampling vector $\boldsymbol{\phi}$ and provide conditions to ensure uniqueness.
- By adding a few more specially designed sampling vectors, almost all signals can be recovered from the phaseless dynamical samples uniquely and stably.
- For full-spark dynamical frames, the reconstruction of all signals is possible.
- A relation between the required number of measurements in time and space is presented.

Referenzen

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{beinert,hasannas}@math.tu-berlin.de