Ambiguities in one-dimensional phase retrieval from Fourier magnitudes

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Phase retrieval

Formulation of the problem

Problem (Phase retrieval)

Recover the unknown complex-valued signal

$$x := (x[n])_{n\in\mathbb{Z}}$$

with finite support from the Fourier intensity

$$|\widehat{x}(\omega)| \qquad (\omega \in \mathbb{R}).$$

Definition (Discrete-time Fourier transform)

$$\widehat{x}(\omega) := \sum_{n\in\mathbb{Z}} x[n]\, \mathrm{e}^{-\mathrm{i}\omega n} \qquad (\omega \in \mathbb{R})$$

Trivial ambiguities

Example

Let x be a complex-valued signal. Then

- the rotated signal
$$(y[n]) := (\mathrm{e}^{\mathrm{i}\alpha}\, x[n]),$$
- the shifted signal
$$(y[n]) := (x[n - n_0]),$$
- the reflected, conjugated signal
$$(y[n]) := \left(\overline{x[-n]}\right)$$

have the same Fourier intensity $|\widehat{x}|$.

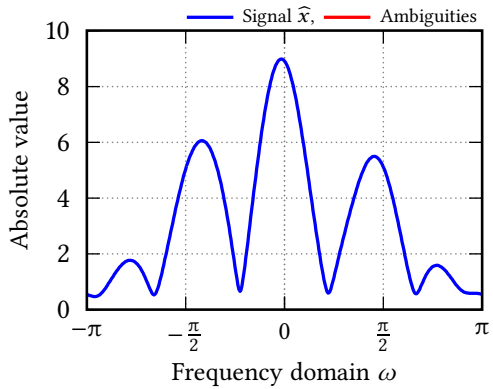
Definition (Trivial and non-trivial ambiguities)

A *trivial ambiguity* is caused by rotation, shift, or reflection and conjugation.

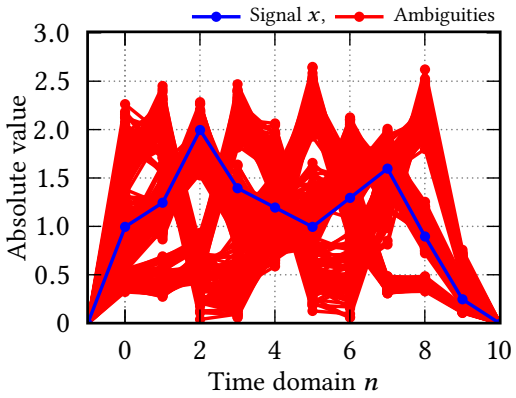
All other occurring ambiguities are called *non-trivial*.

Non-trivial ambiguities

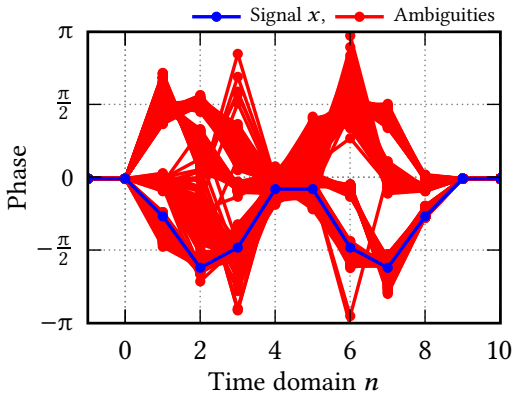
Example



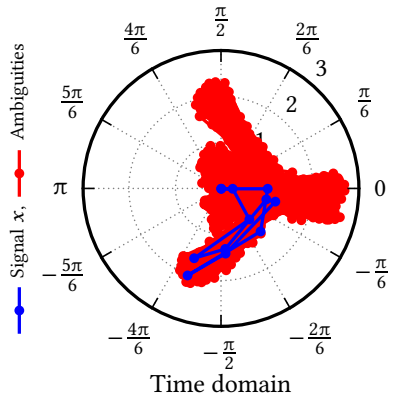
Fourier intensity: $|\widehat{x}(\omega)|$



Absolute value: $|x[n]|$



Phase: $\arg(x[n])$



Polar representation: $x[n]$

Characterizing the solutions

Autocorrelation signal and function

Definition (Autocorrelation signal)

$$a[n] := \sum_{k \in \mathbb{Z}} \overline{x[k]}\, x[k+n] \qquad (n \in \mathbb{Z}).$$

- The autocorrelation signal is conjugate symmetric, i.e.

$$\overline{a[-n]} = \sum_{k \in \mathbb{Z}} x[k]\, \overline{x[k-n]} = \sum_{k \in \mathbb{Z}} x[k+n]\, \overline{x[k]} = a[n] \quad (n \in \mathbb{Z}).$$

Definition (Autocorrelation function)

$$A(\omega) := \sum_{n \in \mathbb{Z}} a[n]\, \mathrm{e}^{-\mathrm{i}\omega n} = \sum_{n=-N+1}^{N-1} a[n]\, \mathrm{e}^{-\mathrm{i}\omega n}.$$

- The autocorrelation function is a non-negative trigonometric polynomial of degree $N-1$.

Phase retrieval in the frequency domain

- Relationship to the Fourier transform:

$$|\widehat{x}(\omega)|^2 = \left(\sum_{n\in\mathbb{Z}} x[n]\,\mathrm{e}^{-\mathrm{i}\omega n}\right)\left(\sum_{k\in\mathbb{Z}} \overline{x[k]}\,\mathrm{e}^{\mathrm{i}\omega k}\right)$$

$$= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x[k+n]\,\overline{x[k]}\,\mathrm{e}^{-\mathrm{i}\omega n} = A(\omega).$$

Equivalent problem

Find a trigonometric polynomial B such that

$$|B(\omega)|^2 = A(\omega).$$

Definition (Associated polynomial)

$$P_A(\mathrm{e}^{-\mathrm{i}\omega}) = \mathrm{e}^{-\mathrm{i}\omega(N-1)}\, A(\omega)$$

- The algebraic polynomial P_A is thus defined by

$$P_A(z) := \sum_{n=0}^{2N-2} a[n-N+1]\, z^n \qquad \text{with} \qquad a[-n] = \overline{a[n]}.$$

- Obviously, we have

$$A(\omega) = \left| P_A(\mathrm{e}^{-\mathrm{i}\omega}) \right|.$$

- Factorize A with respect to the roots of the polynomial P_A.

- If γ is a root of P_A, then we have

$$\begin{aligned} P_A(\overline{\gamma}^{-1}) &= \sum_{n=0}^{2N-2} a[n-N+1]\,\overline{\gamma}^{-n} \\ &= \overline{\gamma}^{-2N+2} \sum_{n=0}^{2N-2} \overline{a[N-1-n]}\,\overline{\gamma}^{2N-2-n} \\ &= \overline{\gamma}^{-2N+2} \sum_{n=0}^{2N-2} \overline{a[n-N+1]}\,\overline{\gamma}^{n} \\ &= \overline{\gamma}^{-2N+2}\,\overline{P_A(\gamma)} \end{aligned}$$

- P_A has the factorization

$$P_A(z) = a[N-1] \prod_{j=1}^{N-1} (z-\gamma_j)(z-\overline{\gamma}_j^{-1}).$$

- For $z := \mathrm{e}^{-\mathrm{i}\omega}$, the absolute value of the linear factors is

$$\left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{-1}\right)\right| = \left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|\left|\overline{\gamma}_j^{-1}\right|\left|\overline{\gamma}_j - \mathrm{e}^{\mathrm{i}\omega}\right|$$

$$= \left|\gamma_j\right|^{-1}\left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|^2 .$$

- A has the factorization

$$A(\omega) = \left|P_A\left(\mathrm{e}^{-\mathrm{i}\omega}\right)\right| = |a[N-1]| \prod_{j=1}^{N-1}\left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{-1}\right)\right|$$

$$= |a[N-1]| \prod_{j=1}^{N-1}\left|\beta_j\right|^{-1}\left|\prod_{j=1}^{N-1}\left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right)\right|^2 = |B(\omega)|^2$$

with $\beta_j \in (\gamma_j, \overline{\gamma}_j^{-1})$.

Theorem (Beinert, Plonka [2015])

Let A be a non-negative trigonometric polynomial. Then the problem

$$|B(\omega)|^2 = A(\omega)$$

has at least one solution. Every solution has a representation of the form

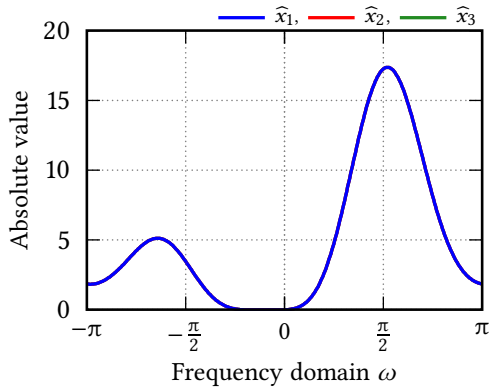
$$B(\omega) = \mathrm{e}^{\mathrm{i}\alpha+\mathrm{i}\omega n_0} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}} \cdot \prod_{j=1}^{N-1} (\mathrm{e}^{-\mathrm{i}\omega} - \beta_j),$$

where β_j can be chosen from the zero pair $(\gamma_j, \overline{\gamma}_j^{-1})$ of the associated polynomial P_A.

- Each non-trivial solution is completely determined by its *corresponding zero set* $\{\beta_1, \ldots, \beta_{N-1}\}$.

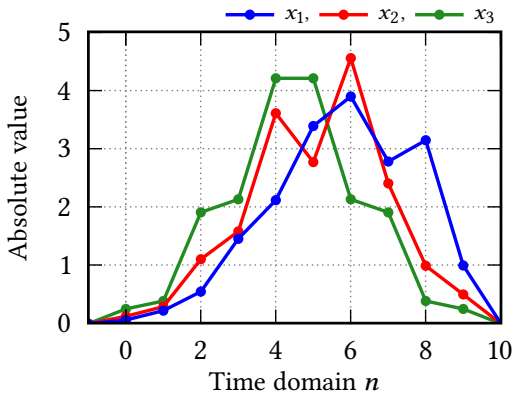
Number of non-trivial ambiguities

Example

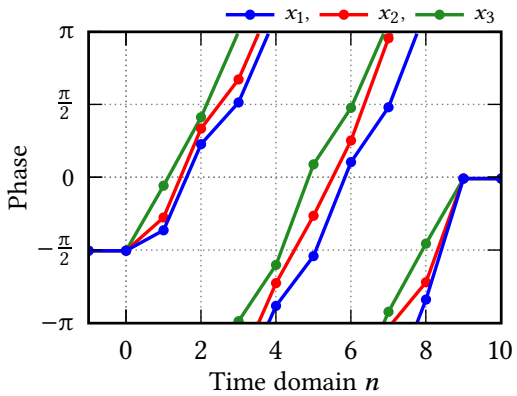


$$|\widehat{x}(\omega)| = \left|\left(e^{-i\omega} + \frac{1}{2}\right)\left(e^{-i\omega} + 2\right)\right|^4 \cdot \left|e^{-i\omega} + e^{i\,\frac{\pi}{10}}\right|^{10}$$

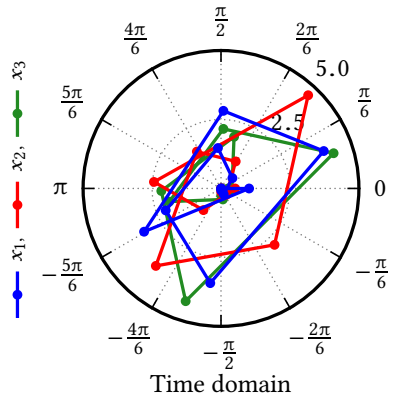
Fourier intensity: $|\widehat{x}(\omega)|$



Absolute value: $|x[n]|$



Phase: $\arg(x[n])$



Polar representation: $x[n]$

Number of non-trivial ambiguities

Corollary

The number of non-trivial ambiguities may vary from 1 *up to* 2^{N-2}.

Proposition (Beinert, Plonka [2015])

Let L *be the number of distinct zero pairs* $(\gamma_\ell, \overline{\gamma}_\ell^{-1})$ *of* P_A *not lying on the unit circle, and let* m_ℓ *be the multiplicity of these zero pairs. The corresponding phase retrieval problem has*

$$\left\lceil \frac{1}{2} \prod_{\ell=1}^{L} (m_\ell + 1) \right\rceil$$

non-trivial ambiguities.

Representation of the ambiguities in time domain

Definition (Convolution of signals)

$$(x_1 * x_2)[n] := \sum_{k \in \mathbb{Z}} x_1[k]\, x_2[n-k].$$

Theorem (Beinert, Plonka [2015])

Let x be a signal with finite support and factorization

$$x = x_1 * x_2.$$

Then the signal

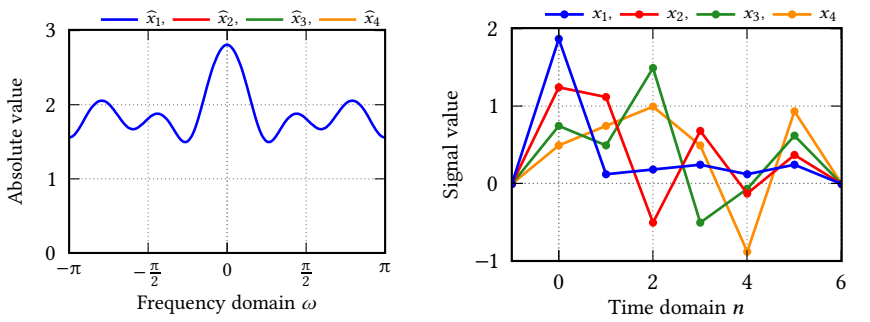
$$y := \mathrm{e}^{\mathrm{i}\alpha} \left(\overline{x_1[-\cdot]}\right) * (x_2[\cdot - n_0])$$

has the same Fourier intensity $|\widehat{x}|$ and all signals with the Fourier intensity $|\widehat{x}|$ can be represented in this manner.

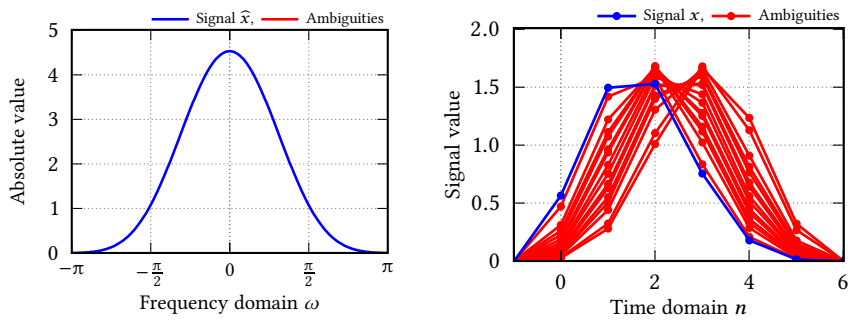
Ensuring uniqueness

Phase retrieval of non-negative signals

Example



Unique non-negative solution



Full non-negative solution set

- The solutions of the phase retrieval problem have the form

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\alpha+\mathrm{i}\omega n_0}\sqrt{|a[N-1]|\prod_{j=1}^{N-1}|\beta_j|^{-1}\cdot\prod_{j=1}^{N-1}(\mathrm{e}^{-\mathrm{i}\omega}-\beta_j)}.$$

- A solution is non-negative if and only if all coefficients of

$$Q(z) := \prod_{j=1}^{N-1}(z-\beta_j)$$

are non-negative.

Lemma

Assume that $(\beta_{N-2}, \beta_{N-1})$ *is a conjugate zero pair unequal to zero, and define*

$$\sigma_n := (-1)^n S_n(\beta_1, \ldots, \beta_{N-3}).$$

Then Q *has non-negative coefficients if and only if* β_{N-1} *fulfils*

$$\sigma_{n-2}|\beta_{N-1}|^2 - 2\sigma_{n-1}\Re\beta_{N-1} + \sigma_n \geq 0$$

for $n = 0, \ldots, N-1$, *and if* σ_{N-3} *is non-zero.*

Definition (Elementary symmetric polynomials)

The *elementary symmetric polynomials* S_n are defined by

$$S_n(\beta_1, \ldots, \beta_{N-3}) := \sum_{1 \leq k_1 < \cdots < k_n \leq N-3} \beta_{k_1} \cdots \beta_{k_n} \qquad (n = 1, \ldots, N-3)$$

as well as $S_0 := 1$ and $S_n := 0$ for $n < 0$ and $n > N-3$.

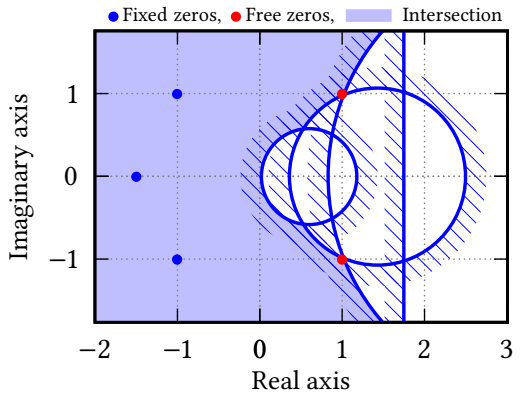
Corollary

Assume that the $\Re\beta_j < 0$ *for* $j = 1, \ldots, N-3$. *Then* Q *has non-negative coefficients if and only if*

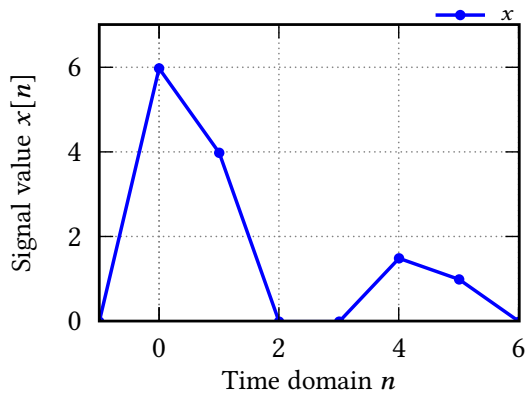
$$\Re\beta_{N-1} \leq \sigma_1/2 \qquad \text{and} \qquad \left|\beta_{N-1} - \frac{\sigma_{n-1}}{\sigma_{n-2}}\right| \geq \frac{\sqrt{\sigma_{n-1}^2 - \sigma_n\sigma_{n-2}}}{\sigma_{n-2}}$$

for $n = 2, \ldots, N-2$ *whenever the radius is real.*

Example

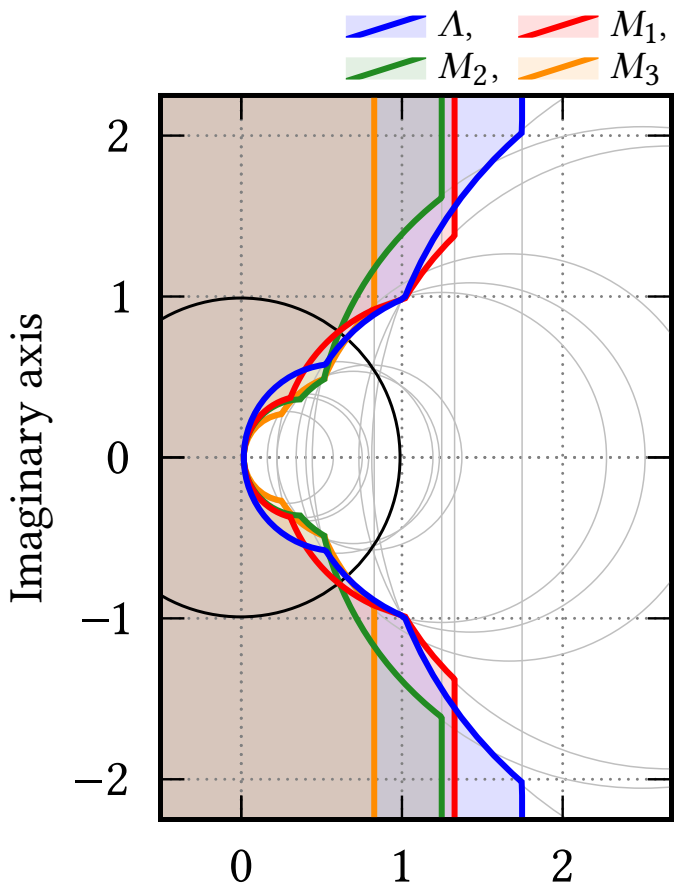


Restrictions on $(\beta_{N-2}, \beta_{N-1})$

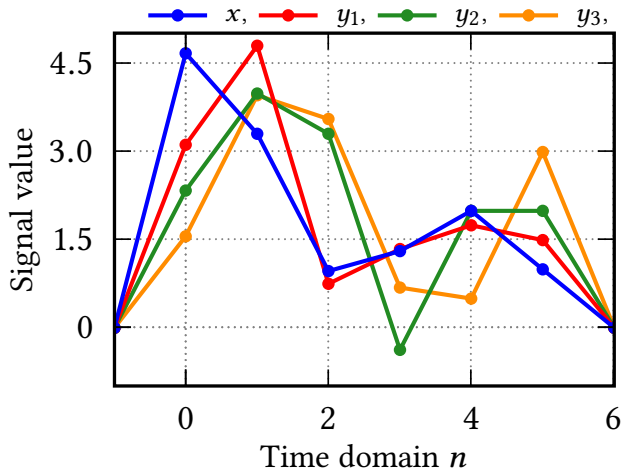


Corresponding signal

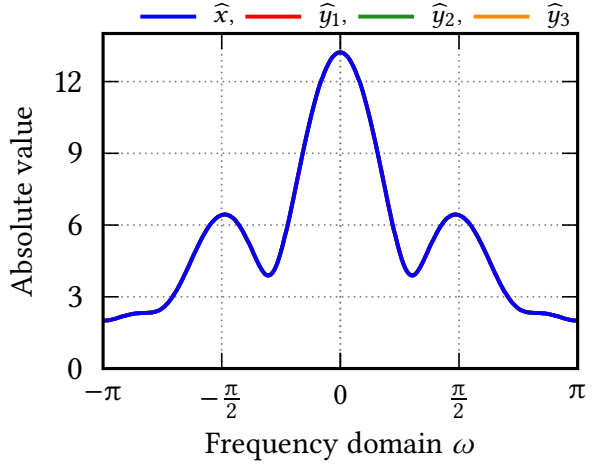
Example



Restriction on $(\beta_{N-2}, \beta_{N-1})$



Signals with $\beta_{N-1} = 3/4 + \mathrm{i}$



Fourier intensity for $\beta_{N-1} = 3/4 + \mathrm{i}$

Phase retrieval of non-negative signals

Theorem (Beinert [2015])

The sets of non-negative signals with support length $N > 3$ that

- *can be recovered uniquely up to reflection*
- *cannot be recovered uniquely up to reflection*

from their Fourier intensities are unbounded sets of infinite Lebesgue measure.

Knowledge of additional moduli

- Recover x from $|\widehat{x}|$ and $|x[N-1-\ell]|$ for an ℓ, where x has the support $\{0,\dots,N-1\}$.
- Assume that there exist two non-trivial solutions x and $\widetilde{x}$.
- The Fourier transform of x can be written as

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\alpha}\sqrt{|a[N-1]|\prod_{j=1}^{N-1}\left|\beta_j\right|^{-1}}\cdot\prod_{j=1}^{N-1}(\mathrm{e}^{-\mathrm{i}\omega}-\beta_j),$$

  and $\widehat{\widetilde{x}}$ has an analogous representation with $\widetilde{\beta}_j \in \{\beta_j, \overline{\beta}_j^{-1}\}$
- For $|x[N-1-\ell]| = |\widetilde{x}[N-1-\ell]|$, Vieta's formulae yield the condition

$$\prod_{j=1}^{N-1}\left|\beta_j\right|^{-\frac{1}{2}}\cdot\left|\sum_{1\le k_1<\cdots<k_\ell\le N-1}\beta_{k_1}\cdots\beta_{k_\ell}\right| = \prod_{j=1}^{N-1}\left|\widetilde{\beta}_j\right|^{-\frac{1}{2}}\cdot\left|\sum_{1\le k_1<\cdots<k_\ell\le N-1}\widetilde{\beta}_{k_1}\cdots\widetilde{\beta}_{k_\ell}\right|.$$

Knowledge of additional moduli

- Under the assumption $\widetilde{\beta}_j = \overline{\beta}_j^{-1}$ $(j = 1, \ldots, J)$ and $\widetilde{\beta}_j = \beta_j$ otherwise, we obtain

$$\left| \sum_{1 \le k_1 < \cdots < k_\ell \le N-1} \beta_{k_1} \cdots \beta_{k_\ell} \right|^2 = \prod_{j=1}^{J} \left|\overline{\beta}_j\right| \cdot \left| \sum_{1 \le k_1 < \cdots < k_\ell \le N-1} \widetilde{\beta}_{k_1} \cdots \widetilde{\beta}_{k_\ell} \right|^2 .$$

- Identify $\{\beta_1, \ldots, \beta_{N-1}\}$ with the real vector

$$(\Re\beta_1, \Im\beta_1, \ldots, \Re\beta_{N-1}, \Im\beta_{N-1})^T,$$

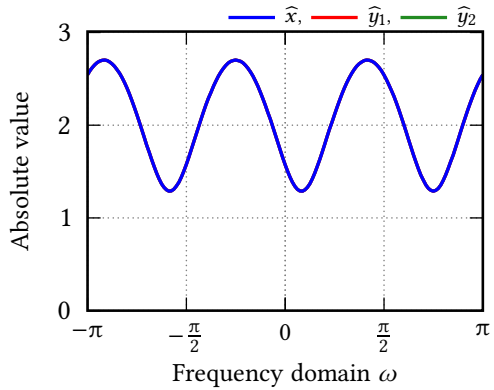
and consider the polynomial equation above.

Theorem (Beinert, Plonka [2015])

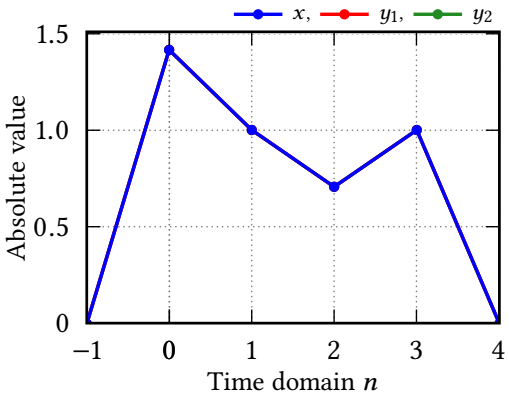
Almost every signal x can be recovered from $|\widehat{x}|$ and $|x[N-1-\ell]|$ for an arbitrary $\ell \neq (N-1)/2$ up to rotations, for $\ell = (N-1)/2$ up to reflection/conjugation and rotation.

Knowledge of additional moduli

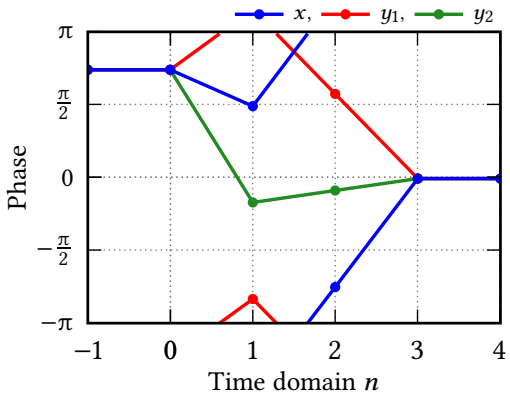
Example



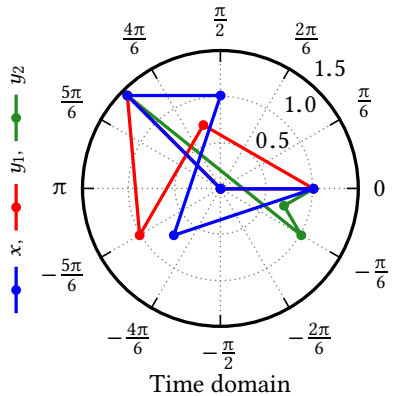
Fourier intensity: $|\widehat{x}(\omega)|$



Absolute value: $|x[n]|$



Phase: $\arg(x[n])$



Polar representation: $x[n]$

Knowledge of additional phases

- Recover x with the fixed support $\{0, \ldots, N-1\}$ from $|\widehat{x}|$, $\arg x[N-1-\ell_1]$, and $\arg x[N-1-\ell_2]$ for $\ell_1 \neq \ell_2$, where x has the support $\{0, \ldots, N-1\}$.
- Assume that there exist two non-trivial solutions x and $\widetilde{x}$.
- Rewrite phase conditions into the equation

$$\Re\left[S_{\ell_1}(B)\right]\Im\left[\overline{S_{\ell_2}(\widetilde{B})}\, S_{\ell_2}(B)\, S_{\ell_1}(\widetilde{B})\right] - \Im\left[S_{\ell_1}(B)\right]\Re\left[\overline{S_{\ell_2}(\widetilde{B})}\, S_{\ell_2}(B)\, S_{\ell_1}(\widetilde{B})\right] = 0$$

with

$$B := \{\beta_1, \ldots, \beta_{N-1}\} \qquad \text{and} \qquad \widetilde{B} := \{\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{N-1}\}.$$

Knowledge of additional phases

Theorem (Beinert [2015])

Let ℓ_1 and ℓ_2 two different integers in $\{0, \dots, N-1\}$. Then almost every signal x can be uniquely recovered from $|\widehat{x}|$ and the phases

$$\arg x[N-1-\ell_1] \quad \text{and} \quad \arg x[N-1-\ell_2] \qquad (\ell_1 + \ell_2 \neq N-1).$$

For $\ell_1 + \ell_2 = N-1$, the recovery of the unknown signal is only unique up to reflection/conjugation, except for the case where the phase of both end points is given.

Interference with reference signal

Theorem (Beinert, Plonka [2015])

Let x and h be complex-valued signals with finite support, and assume that the factorization of their symbols

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_1}\, x[N_1 - 1] \prod_{j=1}^{N_1-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \eta_j\right)$$

and

$$\widehat{h}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_2}\, h[N_2 - 1] \prod_{j=1}^{N_2-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)$$

have no common zeros. Then x and h can be uniquely recovered from $|\widehat{x}(\omega)|$, $|\widehat{h}(\omega)|$ and $|\widehat{x}(\omega) + \widehat{h}(\omega)|$ up to common trivial ambiguities.

Sketch of proof

- Assume there are two solutions $x[n]$, $h[n]$ and $\widetilde{x}[n]$, $\widetilde{h}[n]$.
- Use the factorization in the frequency domain:

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_1}\,\widehat{x}_1(\omega)\,\widehat{x}_2(\omega) \quad \text{and} \quad \widehat{\widetilde{x}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_1}\mathrm{e}^{\mathrm{i}\omega k_1}\,\widehat{x}_1(\omega)\,\overline{\widehat{x}_2(\omega)},$$

$$\widehat{h}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_2}\,\widehat{h}_1(\omega)\,\widehat{h}_2(\omega) \quad \text{and} \quad \widehat{\widetilde{h}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_2}\mathrm{e}^{\mathrm{i}\omega k_2}\,\widehat{h}_1(\omega)\,\overline{\widehat{h}_2(\omega)}.$$

- Consider the identity

$$\left|\widehat{x}(\omega) + \widehat{h}(\omega)\right|^2 = \left|\widehat{\widetilde{x}}(\omega) + \widehat{\widetilde{h}}(\omega)\right|^2.$$

Interference with reference signal

Theorem (Beinert [2015])

Let f and h be complex-valued continuous-time signals with compact support, and assume that the factorization of their Laplace transforms

$$F(\zeta) = C_1 \, \zeta^{m_1} \, \mathrm{e}^{\zeta \gamma_1} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\xi_j}\right) \mathrm{e}^{\frac{\zeta}{\xi_j}}$$

and

$$H(\zeta) = C_2 \, \zeta^{m_2} \, \mathrm{e}^{\zeta \gamma_2} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\eta_j}\right) \mathrm{e}^{\frac{\zeta}{\eta_j}}$$

have no common zeros. Then f and h can be uniquely recovered from $|\widehat{f}(\omega)|$, $|\widehat{h}(\omega)|$ and $|\widehat{f}(\omega) + \widehat{h}(\omega)|$ up to common trivial ambiguities.

Summary/Outlook

- Characterization of the ambiguities in the one-dimensional discrete-time phase retrieval problem.
- Investigation of the quality of different a priori conditions and additional data.

- Phase retrieval in higher dimensions.
- Transferring further results between the discrete-time and continuous-time problem.
- Investigation and development of numerical algorithms.

Thank you for the attention.

- R. Beinert. *Non-negativity constraints in the one-dimensional discrete-time phase retrieval problem.* Universität Göttingen (2016) – Preprint.
- R. Beinert. *One-dimensional phase retrieval with additional interference measurements.* Universität Göttingen (2016) – Preprint.
- R. Beinert, G. Plonka. *Enforcing uniqueness in one-dimensional phase retrieval by additional signal information in time domain.* Universität Göttingen (2016) – Preprint.
- R. Beinert, G. Plonka. Ambiguities in one-dimensional discrete phase retrieval from Fourier magnitudes. *J. Fourier Anal. Appl.* **21** (2015), No. 6, pp. 1169–1198.

Appendix

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