UNIQUE EQUILIBRIUM
IN A DYNAMIC MODEL OF SPECULATIVE ATTACKS

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This version: December 2009

1. Introduction

One of the main challenges of modelling speculative attacks on currency pegs is to explain their puzzling timing. The "second generation" models, due to Obstfeld [23, 24], suggest that murky principles may govern the time of collapse of a currency peg. In these models, a self-fulfilling crisis is understood as a sudden, exogenous jump in market sentiment, from a "good" to a "bad" equilibrium—a shift on financial markets from euphoria to panic. While this perspective does justice to the mystifying timing of crises, it is of no help to explain it. In an article that argues for the merits of the second generation approach, Obstfeld and Rogoff suggest that "future research may succeed in pinning down more precisely the timing of speculative attacks without resorting to equilibrium indeterminacies" [25].

In this article, we take up this issue by embedding a bare-bones second generation currency crisis model in a dynamic context. In the model, time is continuous and agents continuously evaluate whether or not they want to speculate against a currency peg, while the fundamentals of the economy evolve. Like in second generation models, for a wide range of fundamentals, strategic interaction between speculators and a policy maker gives rise to two possible states: tranquillity or panic. Yet, small differences in the speed of agents' responses to the evolving fundamentals induce an unambiguous timing of the transition between the two states. Thus, the model endogenously accounts for sudden switches in the "equilibrium" that the market "selects" over time. The shift to a panic state occurs when economic fundamentals deteriorate beyond an endogenously determined threshold. We characterise this threshold for a natural class of stochastic processes driving economic fundamentals, and derive comparative statics.

The model draws on work by Burdzy et al. [6] and Frankel and Pauzner [11]. In Burdzy et al. [6], agents repeatedly play a stage game with strategic complementarities and multiple equilibria against randomly chosen opponents. Over time, the game slightly changes, while agents are locked into their action for a small period of time. Under these assumptions,

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agents follow a unique rationalisable strategy, and it is possible to predict theoretically which equilibrium of the game will be played at each point in time. Frankel and Pauzner [11] apply these results in the context of business cycles.

Strategic complementarities are also important during currency crises, but a model of a repeatedly played stage game makes less sense in this context. During a currency crisis, speculators interact over a longer period of time with a policy maker aiming to defend a currency peg. Complementarities arise because speculators need a critical mass to force the policy maker to devalue. Speculators may be locked into their choices, their short or long positions in a weak currency, for a (brief) period. Because taking short positions is costly, agents need to forecast evolving fundamentals, the future behaviour of the policy maker, and the speculative pressure resulting from the actions of other speculators to get the timing of an attack right. These are the main elements underlying the model in this article in a nutshell.

Guimarães [16] develops a currency crisis model based on similar elements, and it is closely related to the model in this article. He extends the first generation model (originally due to Krugman [19] and Flood and Garber [10]) with the assumption that agents may be locked into their positions due to substantial frictions on asset markets. In his model, an evolving fundamental follows a Wiener process with drift, and speculators attack when it crosses a critical threshold. In this article, we further extend this result, and improve its utility and generality by enhancing the realism of the model and giving a characterisation result for the threshold. In addition, we make a significant conceptual contribution by showing that the extended model is more closely related to a strand of "global game", second generation currency crises models (Morris and Shin [20]) than to the first generation model. Global game models have been prominent in recent currency crises literature (see e.g. Refs. [20], [14], [17]).

The technical and conceptual contributions of this article—as well as the differences with the article by Guimarães [16]—may be summed up more thoroughly in the following four points. First, we exploit the presence of strategic complementarities to solve our currency crisis model, using the method applied by Burdzy et al. and that is familiar from the global games literature. This approach gives the strong result that, in the model, speculators have a unique rationalisable strategy, in the sense of Bernheim [4] and Pearce [26]. Because of the form of the first generation model, Guimarães [16] cannot place strategic complementarities at the heart of the analysis and therefore he cannot apply the same argument. The method he uses to solve his model does not lead to the conclusion of a unique, rationalisable strategy.

Second, the Wiener process with drift considered in Ref. [16] describes continuous paths for economic fundamentals. But in reality, one observes shocks and discontinuities in data about economic fundamentals, of which the Wiener process may not give an adequate description. More to the point, such discontinuities are potentially important drivers of currency risk, since shocks may trigger large and sudden shifts on financial markets. To model the effects of shocks more clearly, here we consider the class of "jump diffusion processes" instead. These have been used more widely in the literature to take the discontinuities observed in reality into account. Jorion [18] applies them to analyse the fundamentals driving foreign exchange markets.

Third, we focus on the case where frictions are small, instead of when they are substantial. While *some* heterogeneity in agents' response times to shocks is natural, in reality this heterogeneity may well be small, perhaps in the order of minutes or even seconds. We account for this by deriving properties for the limit case of vanishing frictions. The focus on jump diffusion processes makes a key difference when frictions are small, because the risk of larger shocks does not vanish in the limit. In this case, important qualitative results do not depend on the assumption of large frictions. In particular, we characterise the critical threshold in the limit, and show that it obeys intuitive comparative statics.

Finally, the results in this article do not just show in detail how to apply the arguments of Burdzy et al. to a dynamic second generation model, but also how to generalise many of the insights of static global game currency crisis models to a dynamic context. Global game models perturb the information structure of static second generation models to achieve unequivocal equilibrium selection. The few existing articles on global games that attempt to apply this approach in a dynamic context (e.g. Angeletos et al. [1]) show that it is not obvious that this technique is helpful there. This is because a dynamic model gives rise to endogenous learning which complicates the subtle role of information in a static global game model. The approach used here, based on heterogeneity in response times to shocks instead of perturbing the information structure, circumvents these issues.

Although the information structure that is typical for global games is absent in the model in this article, the new results indicated above reveal a number of striking parallels with static global game models. Firstly, it shares the most acclaimed feature of global game models, the existence of a unique rationalisable strategy for speculators that reproduces the sudden shifts from tranquillity to speculative attacks observed in reality. Secondly, following the approach of Burdzy et al., the existence of this strategy is established using an infection argument in an underlying type space. And thirdly, agents' behaviour takes the form of a threshold strategy, which we can characterise in terms of the strategic uncertainty associated with speculation. These findings have almost exact counterparts in static global game models of currency crises.

2. The Model

Consider an economy with a pegged currency. There are an infinite number of identical, risk neutral speculators indexed on $N = [0, 1]$. Time is continuous and periodically each of the speculators reviews the decision whether or not to speculate against the pegged currency

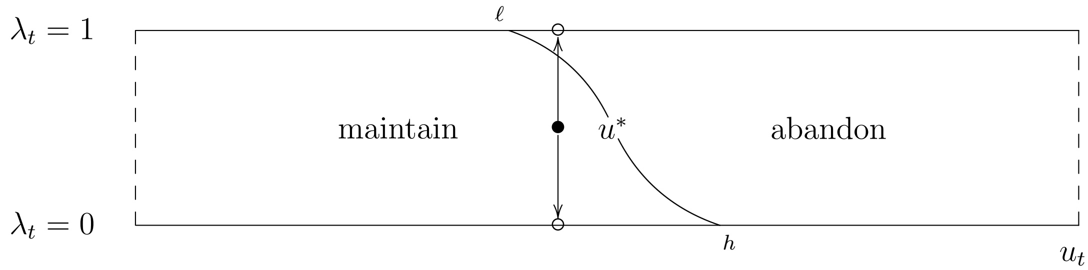

FIGURE 1. The optimal policy and the impact of speculative activity

by taking a short position against it. A speculator that participates in a successful speculative attack earns a positive payoff normalised to 1. The cost of holding a short position is equal to the length of time the position is held times a positive interest rate differential, which is r per unit of time. This payoff structure is often adopted in the global game literature.

There is a policy maker who, at each point in time $t \in \mathbf{R}_+$, faces the decision to abandon the currency peg or to keep it. She decides according to an instantaneous net benefit function that describes the trade-off between keeping and abandoning the peg. As in the static global game model of Morris and Shin [20] (and also their dynamic model [21]) the policy maker's net benefit depends on two factors: λ_t, the fraction of speculators attacking the currency, and u_t, a stochastic component that evolves over time. This component describes the economic fundamentals that affect the policy maker's trade-off between the benefits and costs of the peg (e.g., unemployment, or real overvaluation of the currency). An increase in u_t means a worsening of fundamentals; an increase of λ_t reflects more speculative pressure, and thus higher costs of maintaining the currency peg. Let net benefit at time t be given by:

$$\mathscr{B}_t := B(u_t, \lambda_t), \qquad B_{u_t} \leq c < 0, \;\; B_{\lambda_t} \leq c < 0, \text{ for some negative constant } c.$$

In this expression B is a time-invariant instantaneous benefit function that is continuously differentiable. Applying the implicit function theorem to B, we can characterise the policy maker's behaviour at each moment t as follows: the policy maker devalues if the stochastic variable u_t has crossed a certain threshold value for u_t that depends on λ_t. This threshold is given by a time-invariant, continuously differentiable, strictly decreasing and invertible function $u^* : [0, 1] \to \mathbf{R}$ with compact domain $[\ell, h]$. The policy maker's threshold is illustrated in figure 1 by the curve labelled u^*.

We assume that u_t evolves stochastically over time. Concretely, we let:

$$u_t = W_t + J_t, \tag{1}$$

where W_t is a the standard Wiener process, and J_t is a compound Poisson or "jump" process,[1] with jumps arriving at rate ϑ. The first component of the process, W_t, reflects small changes in u_t that occur continuously over time. The second component, the Poisson distributed jumps, reflects large shocks that only occasionally hit the fundamental u_t. They are independently and identically distributed and drawn according to a continuous density with support $\mathbf{R}_+$. (A more general interpretation of u_t would be that the variable represents *information* about some relevant economic fundamentals. In this case, the jumps can be interpreted as larger news announcements or the occurrence of important events.) As noted in the introduction, these kinds of jump diffusion processes are often adopted as models for fundamentals that drive asset prices. In fact, the process in equation (1) is well-studied in a literature on risk theory that originates with Dufresne and Gerber [9].

The stochastic process in equation (1) is more general than those considered by Burdzy et al. [6] and Frankel and Pauzner [11], and more general than the processes adopted by global game-like papers. Usually, the continuous process $u_t = W_t$ is investigated, notably in Morris and Shin [21] and Guimarães [16].[2] If speculators are able to revise portfolios quickly and the path of u_t is continuous, speculators face virtually no risk of u_t crossing u^* before they can revise. It is questionable whether this adequately models the risk and uncertainty associated with currency speculation. This observation motivates the more general approach in equation (1). We come back to the implications in detail in section 5.

Now suppose for the moment that all speculators are able to adjust their short positions instantaneously. In this case, the above setup will inherit the possibility of multiple equilibria from traditional second generation currency crisis models. To see this, consider the point indicated by the black point in figure 1, which denotes a designated state of the system, and now suppose that, suddenly, all speculators decide to attack. In this case, the state of the system will jump to the point indicated *above* the black point (so that $\lambda_t = 1$). This induces the policy maker to abandon the currency peg and thus will lead to a full-fledged currency crisis. Such a crisis may be labelled "self-fulfilling", since different behaviour of the speculators would have resulted in a different outcome. Specifically, if all speculators decide

[1]The (standard) *Wiener process* is the continuous time generalisation of the random walk. It satisfies the following two properties: (i) for all $t, \tau \in \mathbf{R}_+$, $W_t - W_{t+\tau}$ is normally distributed with variance τ; (ii) for $t_0, \tau_0, t_1, \tau_1, \ldots \in \mathbf{R}_+$ the increments $W_{t_i} - W_{t_i+\tau_i}$ and $W_{t_j} - W_{t_j+\tau_j}$ are independent when $[t_i, t_i + \tau_i)$ and $[t_j, t_j + \tau_j)$ are disjoint. The Wiener process is the mathematical model for Brownian motion.
A *compound Poisson process* is a process that has identically independently distributed increments ("jumps") at random moments and is otherwise constant; the inter-arrival or "waiting" time until the next jump follows an exponential distribution and has cumulative distribution function $1 - \exp[-\vartheta\tau]$, where τ is the waiting time and ϑ is a parameter. This process satisfies property (ii) above (with J substituted for W).
A process of the form $u_t = J_t + W_t$ is also called an (interlacing) *Lévy process* (see e.g. Applebaum [2]).

[2]Incorporating a drift term like is done in Ref. [16] would not substantially change the results in this article.

to refrain from attacking, the state of the system will jump to the point indicated *below* the black point, so that $\lambda_t = 0$, and this results in a tranquil situation.

Both jumps are *ex post* justifiable, since speculators have made a choice consistent with the actual outcome. Hence, the model exhibits an indeterminacy, that extends to any situation when u_t is in the region between ℓ and h. However, we will show that this ambiguity only arises if we allow for *instantaneous* adjustment of positions. Obviously, however liquid an asset, completely instantaneous portfolio adjustments of all speculators at the same time are an abstraction from reality, and in this case it is not an innocent abstraction.

3. Inertia

In reality it is not plausible that large groups of agents move at *exactly* the same point in time. Portfolio adjustments will exhibit some inertia, perhaps induced by small frictions in financial markets, such as monitoring costs or transaction costs, or simply by the time it takes to deliberate, place or execute an order. In this section, we introduce and discuss a straightforward way to model this inertia, based on ideas in Frankel and Pauzner [11]. In the next section, we show it removes the indeterminacy discussed above.

Of course, inertia may be very small in reality; on liquid financial markets agents respond to new information within minutes or even seconds. This is especially true during currency crises, when agents follow the market very carefully. But the main qualitative results in this article do not depend on the *degree* of inertia, only on the fact that there is *some* inertia, however small. Inertia introduces a specific feature into the model, namely that speculators respond somewhat heterogeneously to changes in u_t. This heterogeneity remains relevant even when the inertia becomes negligible in size. To substantiate this claim formally, section 5 investigates the properties of the model for the limit case of vanishing inertia.

To model a small degree of inertia in a simple way, we assume that speculators do not revise portfolios at every moment in time. Instead, each agent revises at specific, idiosyncratic moments in time that we refer to as her "revision opportunities". At each revision opportunity, a speculator can decide whether she wants to take a short position or not, and she sticks to this choice until she receives her next revision opportunity. For any given speculator, we assume that the revision opportunities arrive following a Poisson process with parameter $\zeta > 0$. That is, the time τ elapsed between revision decisions of any given agent is independent, identically distributed, and has cumulative density function $1 - \exp(-\zeta\tau)$.

Note that if ζ sufficiently large, a new revision opportunity is almost surely guaranteed to arrive within an extremely short time span. Hence a setup with small inertia does not preclude that actors on financial markets respond virtually instantaneously to changing conditions. Note also that the way inertia is modelled reflects a strategy used in other areas of economics. Particularly, under Calvo-style price setting, at every moment each price setter sets a new price with a certain probability; here, each speculator revises her portfolio with a certain probability. Our approach shares Calvo's "agnostic" attitude towards the origin of inertia, which is modelled in the simplest conceivable way to keep the model tractable. More involved setups that keep the presence of small inertia intact, but model its causes (or even the amount of inertia itself) endogenously, should in principle lead to similar results but will be much more difficult to handle.

An inert revision process helps to capture two elements of speculation that seem important in reality. First, because behaviour is slightly inert, there is the possibility that an individual speculator fails to take a short position before the peg collapses. Speculators have to weigh this risk against incurring the costs of taking short positions while the currency does not collapse. This captures a realistic aspect of speculation, which clearly is important even on very liquid financial markets.

The second element is that each speculator can condition her behaviour on what other speculators are doing. That is, a speculator can take into account the positions taken by others when making her decision to attack or not. Adding the possibility of "keeping an eye on what the market is doing" constitutes a step towards added realism over static currency crises models. For simplicity, in the model we assume that when an agent receives the opportunity to revise positions, she perfectly observes the current values of both λ_t and u_t.

A *strategy* for a speculator is a decision rule that describes the decision whether or not to invest, at each possible combination of λ_t and u_t. If an agent receives a revision opportunity at time t, and the next at $t+\tau$, we refer to the time span $[t, t+\tau)$ as her *lock-in period*. If the agent decides to speculate against the pegged currency at time t, the costs of speculation are thus τr, while the revenue from speculation is 0 if the peg survives the time span $[t, t+\tau)$ and 1 if it collapses during this spell. In expectation, the length of the lock-in period is given by $1/\zeta$. The expected opportunity cost associated with taking a short position is thus r/ζ.

4. Unique Equilibrium

Apart from the inertia induced by the revision process, the behaviour of agents in the model is fully rational and firmly forward looking. Speculators will try to determine what is the *optimal* decision, taking the possibility of a future collapse of the peg into account and also the possible behaviour of other speculators. In fact, the model exhibits a very straightforward and natural kind of strategic interaction, in which the behaviour of agents in "extreme" states of the system infects the behaviour of agents in other states. This is familiar from the static global games literature and closely resembles the argument given in Frankel and Pauzner [11], even though the currency crisis model setup is different from the pure coordination setup considered by these authors. The infection allows us to derive the optimal strategies for speculators.

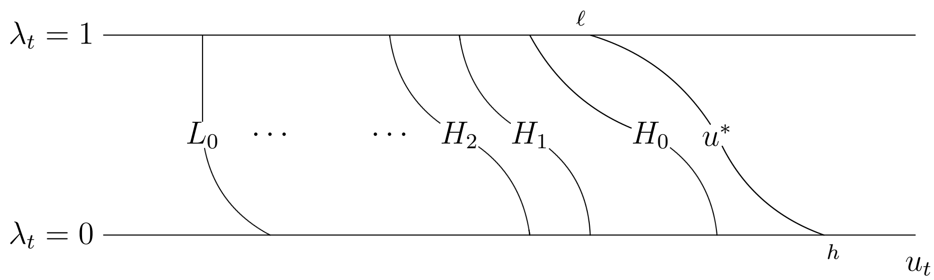

FIGURE 2. Infection in (u, λ)-space

Consider the situation of a speculator who has a revision opportunity at time t at some point (u_t, λ_t) just to the left of the curve u^* in figure 2. When deciding whether to speculate or not, she has to anticipate the possibility of a future policy discontinuity, namely the collapse of the peg that takes place if u crosses the curve u^* at some time $\hat{t} > t$. The likelihood of this event depends on two things. First, given that u_t evolves stochastically over time, it depends on the horizontal distance between u_t and the curve u^*. Second, because of the shape of the policy function u^*, for identical future paths of the variable u_t, higher paths of λ_t hasten a collapse. The dynamics of λ_t are driven by the decisions of speculators over time. Particularly, if in the near future more speculators are inclined to speculate against the pegged currency, speculation becomes more attractive at time t because λ_t increases, and for any given u_t, when λ_t increases the point (u_t, λ_t) always moves closer to the curve u^*. In sum, every individual speculator is confronted with two kinds of uncertainty: the first kind is due to the nature of the random process, and the second results from the strategic uncertainty about the decisions of other speculators.

Now, on the one hand, when u_t moves closer and closer to u^*, the probability of a quick collapse goes to one, and, if an agent would opt to refrain from speculation when u_t is close to u^*, it is likely that she will be stuck with this decision when an actual collapse takes place. Observe that this is true even if all other speculators refrain from attacking after time t; indeed, this holds *independently* of the decision rules followed by other speculators, so independent of the strategic uncertainty.

On the other hand, if the value of u_t is very far from u^*, a new revision opportunity will arrive almost certainly before u_t touches u^*. Since there are opportunity costs associated with speculation against the pegged currency, a speculator who receives a revision opportunity very far from u^* would prefer to refrain from speculation.

In sum, we can bound off a region of (u_t, λ_t)-space with a curve H_0, to the right of which we can be sure that, for any given speculator, to decision to speculate is optimal *independent* of the decision rules of other speculators; and similarly, there exists a region to the right of some curve L_0 to the left of which it is optimal to refrain from speculation, *independent* from the decisions of other speculators. These curves are indicated in figure 2. Following the literature on global games (see e.g. Frankel et al. [12]), we call these two regions *dominance regions*—because in these regions, one of the choices dominates the other (in the game-theoretic sense; see [13, Definition. 1.1]).

The dominance regions provide the starting points for an infection argument. Consider the situation of a speculator who has to decide at a point (u_t, λ_t), just slightly to the left of the curve H_0. She deduces that every speculator who will receive a revision opportunity to the right of H_0 will opt to speculate. Moreover, since u_t is close to H_0, she thinks it is very likely that the stochastic process spends some time to the right of this curve during her lock-in period. When given a revision opportunity to the right of the curve H_0, the speculator would be willing to speculate against the pegged currency *regardless* of what other agents would do. But now that our speculator can take into account that colleagues receiving revision opportunities while u_t is to the right of H_0 will *also* choose to speculate, she knows that when u_t is in this region λ_t will increase. Due to the shape of u^*, under the resulting dynamics of (u_t, λ_t) the probability of a collapse of the peg will increase. This makes speculation more attractive; indeed for some values of (u_t, λ_t) to the left of H_0, even if they are not in the original dominance region, the knowledge that others will speculate to the right of H_0 will be enough to tip the balance unequivocally in favour of speculation in this second step of reasoning at (u_t, λ_t). Thus, in what may be called a "second level" of reasoning (which conditions on the earlier observation that to the right of H_0 agents prefer to speculate), we can expand the region in which we can be sure that the safe asset will be chosen a bit to the left, choosing a new curve H_1 to the left of H_0 to bound it. A similar argument works with respect to the curve L_0, and here we find a new curve L_1 just to the right of L_0.

Repeating the argument, in a next step we can find curves H_2 and L_2, and so on. Eventually, iterative expansions of the areas both on the left and on the right leads to two curves $\overline{H}$ and $\overline{L}$ that can be thought of as the limits of the families $H_0, H_1, H_2, \ldots$ and $L_0, L_1, L_2 \ldots$ (a formal argument is given in lemma 4 in the appendix). Following the above reasoning, to the right of $\overline{H}$, agents always prefer to speculate, and to the left of $\overline{L}$, agents prefer to refrain from speculation. In fact, as we will argue below, the curves $\overline{H}$ and $\overline{L}$ coincide. Thus they give a complete characterisation of the optimal strategies of speculators. Formally:

Theorem 1. *There is a unique rationalisable strategy, described by a (Lipschitz continuous) function $Z^* : [0, 1] \to \mathbf{R}$, such that speculators refrain from speculation if $u_t < Z^*(\lambda_t)$ and attack if $Z^*(\lambda_t) < u_t$.*

The surviving strategy is the unique rationalisable strategy since in each of the steps that we take to derive the curves $\overline{L}$ and $\overline{H}$, we describe the optimal decision at time t for some combinations of λ_t and u_t—and deleted suboptimal, or "dominated", actions—conditional

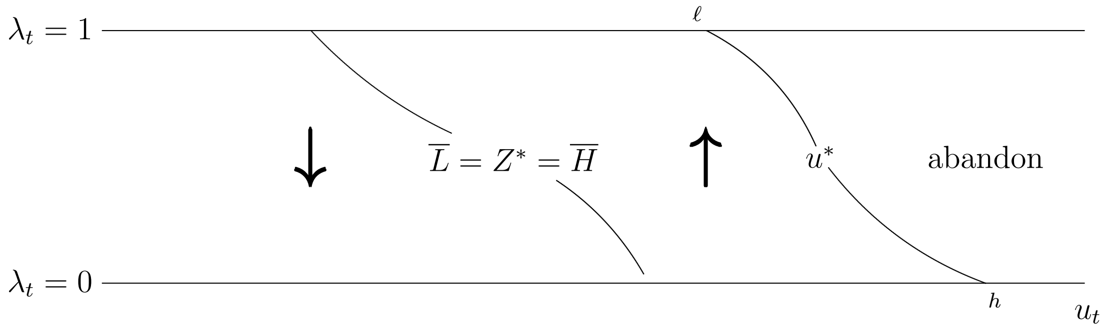

FIGURE 3. Dynamics

on the event that such combinations are reached at time t, but without reference to any other aspect of the history of the game. Formally, we apply iterative deletion of conditionally strictly dominated actions (Fudenberg and Tirole [13], definition 4.2), which is the continuous time counterpart of iteratively deleting dominated strategies. Since a strategy that specifies (iteratively) dominated actions at some point in time is never rationalisable, the surviving strategy must be the only rationalisable strategy in the model.

As a consequence of the theorem, the dynamics of λ_t are such that speculative activity is increasing whenever the state of the system (u_t, λ_t) is in the region to the right of the graph of Z^*, but decreasing to its left. The arrows in figure 3 show these dynamics. They bifurcate abruptly around the graph of Z^*, which we call the *equilibrium threshold*.

As in Burdzy et al. [6], and Frankel and Pauzner [11], theorem 1 may be proved using a "translation" argument. Since this argument is well-known from the literature on global games, we will only sketch it here. Consider the limiting curves $\overline{L}$ and $\overline{H}$. By lemma 4 in the appendix, these curves are well-defined, and $\overline{L}$ and $\overline{H}$ are continuous functions from $[0, 1]$ into $\mathbf{R}$. The lemma shows that agents are indifferent when choosing on the graph of $\overline{L}$ when they believe that all other speculators attack to the right of this graph and refrain to the left. Similarly, agents are indifferent when choosing on the graph of $\overline{H}$ and they believe that all other speculators attack to the right of this graph and refrain to its left.

Now suppose that $\overline{L} \neq \overline{H}$. Then we may assume that $\overline{L}$ lies everywhere to the left of $\overline{H}$. This means that by "shifting" $\overline{H}$ to the left we can construct a new curve $\overline{S}$ that (i) lies everywhere to the left of $\overline{L}$ and (ii) touches $\overline{L}$ in at least one point (u_t, λ_t). This kind of leftward shift of $\overline{H}$ is illustrated graphically by the curve $\overline{S}$ in figure 4. Now first, consider an agent that has to decide at the state (u'_t, λ_t) lying on the curve $\overline{H}$. This agent must be indifferent when others are using the strategy described by $\overline{H}$, so her expected payoff from attacking the pegged currency is 0. We will compare her payoff to an agent deciding at the state (u_t, λ_t) on the "shifted" curve $\overline{S}$ when agents are playing according to the strategy

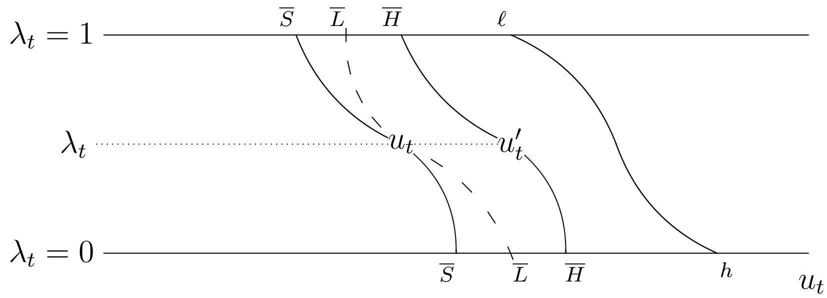

FIGURE 4. The Unique Equilibrium Threshold

described by $\overline{S}$. For this, we will consider the dynamics of the system depicted in figure 4 under identical relative dynamics for the stochastic process u_t, and compare a path starting from (u_t, λ_t) with one starting from (u'_t, λ_t). Under each of these paths, the induced dynamics for λ_t must be exactly the same, since $\overline{S}$ and $\overline{H}$ have the same shape. If the curves $\overline{L}$ and $\overline{H}$ do not coincide, then $u_{t'} < u'_{t'}$ for all $t' > t$, and a collapse of the currency peg on the path starting from (u_t, λ_t) implies an even earlier collapse on a path starting from (u'_t, λ_t). Indeed, since $\overline{S}$ is strictly to the left of $\overline{H}$, we conclude that expected payoff from attacking at the state (u'_t, λ_t) under the strategy $\overline{H}$ must be strictly greater than the expected payoff at (u_t, λ_t) under the strategy $\overline{S}$. Denoting the latter by $\pi(u_t, \lambda_t; \overline{S})$, we therefore have:

$$\pi(u_t, \lambda_t; \overline{S}) < 0.$$

Now compare $\overline{S}$ with $\overline{L}$. Since $\overline{S}$ lies to the left of $\overline{L}$ we see that for any given path of u_t, at each moment $t' > t$, weakly more agents attack when all of them use the strategy $\overline{S}$ instead of $\overline{L}$. This can only hasten a collapse of the currency peg. From this we conclude that the expected payoff $\pi(u_t, \lambda_t; \overline{S})$ from attacking at the state (u_t, λ_t) when agents follow the strategy described by $\overline{S}$ is higher than expected payoff at the same state (u_t, λ_t) when agents use the strategy described by $\overline{L}$. Denote this latter payoff by $\pi(u_t, \lambda_t; \overline{L})$. Now, an agent that has to decide at the state (u_t, λ_t), lying on the curve $\overline{L}$, is indifferent under strategy the described by $\overline{L}$, so that $\pi(u_t, \lambda_t; \overline{S}) \geq \pi(u_t, \lambda_t; \overline{L}) = 0$. Hence we have:

$$0 \leq \pi(u_t, \lambda_t; \overline{S}) < 0,$$

which is absurd. From this, we conclude that $\overline{L} = \overline{H}$ after all.

5. Limiting Dynamics and Strategic Uncertainty

In the previous section we showed that in the presence of inertia, that is, under the condition that $\frac{1}{\zeta} > 0$, the model has a unique equilibrium, characterised by a unique equilibrium threshold. Frictions on liquid financial markets are generally deemed to be of only small importance, suggesting that the most relevant scenario to study is the case when inertia is very small. Therefore, in this section, we study the properties of the model's equilibrium in the limiting case of vanishing inertia. We now index the equilibrium threshold by ζ and will show that as $\zeta \to \infty$ the shapes of the equilibrium thresholds Z^*_ζ converges to a solution that is determined by the risk associated with the speculation due to larger shocks.

To derive the limiting properties of the model, we draw on a number of highly non-trivial results in a mathematical companion paper to Frankel and Pauzner [11] on dynamical systems of the form in figure 3, Burdzy et al. [5]. As stated in section 2, a key difference between the model considered here and the model considered in Frankel and Pauzner [11] or Guimarães [16] is that we consider the possibility of large shocks hitting u_t (due to the jumps of the compound Poisson process). But these shocks have quite important implications in the currency crisis context. This is because, when inertia vanishes, during an agent's lock-in period, a crisis is most likely to be triggered by a shock which disturbs the dynamical system from the tranquil region in figure (3) to a the region where λ_t increases rapidly, and then crosses the threshold u^*. In order to apply the theory in [5] we will have to incorporate the possibility of large shocks in our arguments.

We start by characterising the dynamics of λ_t when agents behave optimally, but away from the limit. Consider the dynamical system depicted in figure 3. Because of the strategies used by agents, the dynamics of the system bifurcate around the threshold Z^*_ζ, so that there are two regimes that may govern the dynamics of the system. As u_t crossed the equilibrium threshold from left to right, λ_t starts to increase. Initially, λ_t grows approximately exponentially. As more and more agents start to speculate, the rate at which λ_t is growing falls, proportionally to the declining fraction of agents that still have positions in the weak currency. When u_t crosses the equilibrium threshold from right to left, λ_t declines in a similar fashion. Hence, depending on the regime, the evolution of λ_t is governed by either

$$\dot{\lambda}^{\downarrow}_t = -\zeta\lambda_t, \quad \text{or} \quad \dot{\lambda}^{\uparrow}_t = \zeta(1-\lambda_t), \tag{2}$$

where $\dot{\lambda}^{\downarrow}_t$ denotes the time derivative of λ_t to the left of the threshold, and $\dot{\lambda}^{\uparrow}_t$ that to the right. An *upward bifurcation* of the dynamical system is a situation in which the evolution of λ_t is described by $\dot{\lambda}^{\uparrow}_t$ until λ_t almost reaches 1. A *downward bifurcation* is defined analogously.[3]

Now consider the behaviour of λ_t conditional on the event that no large jump arrives until λ_t reaches some small neighbourhood of 0 or 1. As shown in Burdzy et al. [5, theorem 2], close to the equilibrium threshold and for sufficiently large ζ, the probability of an upward bifurcation tends to $1-\lambda_t$, and the probability of a downward bifurcation tends to λ_t. Thus, around the threshold the system bifurcates almost surely, at a ratio which depends inversely

[3]This definition is still somewhat imprecise, because we have not specified what it means for λ_t to almost reach 0 or 1. A more precise definition of the concept of bifurcation, that takes care of this detail, emerges from the proof of theorem 3 below.

on the proportion of agents currently locked into either action. This bifurcation result depends only on the fact that the dynamical system is of the form in figure 3, so applies to our setting.

A consequence of this is that when ζ becomes large, the equilibrium threshold must converge to a curve that lies everywhere to the left of the line $u_t = \ell$. Why? Suppose $Z^*(\lambda_t) > \ell$ for some $\lambda_t < 1$. As $\zeta \to \infty$, the system bifurcates upwards with positive probability $1 - \lambda_t$. For large ζ, the variance of W_t is small relative to the speed at which speculative pressure λ_t changes. Then, under the induced dynamics, λ_t reaches the critical value determined by the curve u^* very quickly—refer to figure 1. Furthermore, there is always a positive probability that the speculator will not receive another revision opportunity before u^* is reached. While the speed at which λ_t changes depends on ζ, the probability of being locked into the current action when u^* is reached does *not* depend on ζ, because *all* speculators receive revision opportunities at rate ζ. If the speculator is still locked into her action when u^* is reached, she obtains a payoff of 1 if she chooses to speculate against the pegged currency, which—for small ζ—dwarfs the costs incurred during the short lock in period, and of 0 if she doesn't. The speculator will always choose to attack. (The same argument still applies if we add the possibility of large shocks, since they only lead to positive jumps in u_t.)

From the perspective of a speculator deciding to the left of ℓ, a collapse of the peg is almost surely triggered by a disturbance due to the compound Poisson process J_t, since otherwise u_t doesn't move much during her lock-in time. So we now explicitly consider the influence of such shocks on the dynamical system depicted in figure 3. Suppose agent i makes a decision at t. The waiting time between the larger shocks due to J_t is exponentially distributed, so that the probability that the next large shock arrives before time $t+\tau$ is given by $1-\exp(-\vartheta\tau)$ (recall that ϑ is the parameter of the compound Poisson process). Intuitively, at time t, the relative probability weight that the *next* shock arrives in an infinitesimal interval starting at $t+\tau$ is $\vartheta\exp(-\vartheta\tau)$. Now, conditional on the event that a large shock arrives during this interval and the agent is still locked in at time $t+\tau$, for any value ζ and any strategy characterised by a continuous and downward sloping curve Z, the probability that the fundamental will cross u^* during the agent's lock-in time after a large shock hits the dynamical system is a time-homogeneous function[4] $\rho(\lambda_t, u_t; Z)$. Given the shape of the curve u^*, the function $\rho(\cdot)$ must be continuous, increasing in its first two arguments, and positive.

Let $Z(\tilde{u})$ denote the "degenerate" strategy where speculators speculate against the weak currency if and only if $u_t \geq \tilde{u}$ independent of λ_t. First, observe that:

[4]The function must be time-homogeneous, given that directly *after* a shock that takes the dynamical system to (λ_t, u_t), the expected future dynamics of the system depend only on the new value of u_t and on λ_t. This is because the Poisson revision process, as well as W_t and J_t are stochastic processes without memory. In general no closed form expression may exist for the function ρ, though conceivably it may be approximated by numerical simulation.

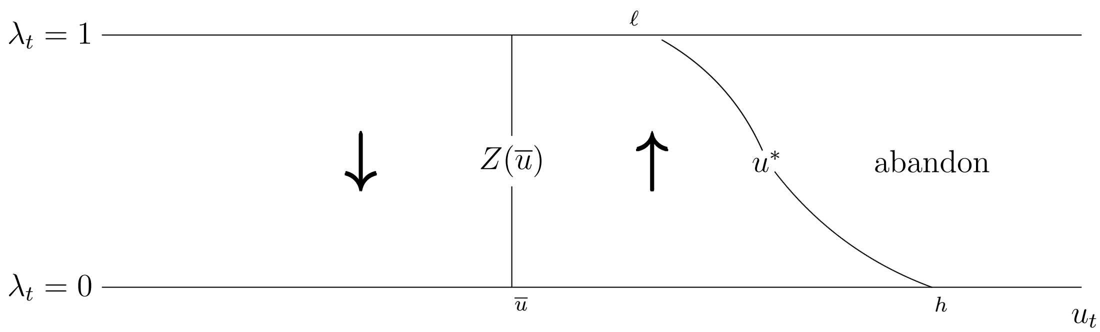

FIGURE 5. Dynamics when inertia is small

Lemma 2. *The equation* $\vartheta \int_0^1 \rho(\lambda, \tilde{u}; Z(\tilde{u}))\, d\lambda = r$ *has a unique solution for* $\tilde{u}$.

Denote this solution by $\overline{u}$. Now consider the function:

$$V(u_t) := \vartheta \int_0^1 \rho\left(\lambda, u_t; Z(u_t)\right)\, d\lambda. \tag{3}$$

(This function closely resembles the equation given in Frankel and Pauzner [11, p. 297]). We will say that speculation is a risk-dominant action at u_t if $V(u_t) \geq r$, and that refraining from speculation is a risk dominant action at u_t if $V(u_t) \leq r$. The following result then characterises the equilibrium of the model for $\zeta \to \infty$.

Theorem 3. *As* $\zeta \to \infty$*, the equilibrium threshold* Z^*_ζ *converges to the threshold* Z *characterised by the condition* $V(Z(\lambda_t)) = r$*. Therefore, as* $\zeta \to \infty$*, agents always choose a risk-dominant action.*

In the proof of the theorem we show that, for large ζ, at every point u_t lying on the curve $Z^*_\zeta(\lambda_t)$, we must approximately have $V(u_t) = r$. Clearly, to satisfy this requirement Z^*_ζ must be close to a vertical line and, as $\zeta \to \infty$, Z^*_ζ must therefore converge to the vertical line $Z(\overline{u})$ of lemma 2—see figure 5.

Theorem 3 allows us to compare the solution of this dynamic currency crisis model to that of the static global game set-up of Morris and Shin [20], in which all speculators receive idiosyncratic noisy signals about a fundamental, and then all have to decide to attack or not, at the same time, and based on their private signal. In the equilibrium of the static global game model, as the variance of the noise vanishes, speculators choose the action that gives the highest payoff as if they are completely unsure about the choices of other agents and believe that the fraction of speculators that attack the currency can be anything from 0 to 1. Equation (3) expresses a rather similar idea. For small inertia, the value $V(u_t)$ describes the returns to attacking, normalised for lock in time, for a speculator who receives a revision opportunity when the fundamental equals u_t, but is unsure about how many others join the attack. She compares this with the expected costs of speculation, which equal r when normalised for lock in time. The proportional relation between $\vartheta \int_0^1 \rho(\lambda, u_t; Z_\zeta^*) d\lambda / \zeta$ and r/ζ remains intact as ζ tends to infinity and gives the characterisation in theorem 3.

The condition that $V(u_t) = r$ at the limiting equilibrium threshold immediately yields comparative statics, in this simple model with respect to the rate of return r. If r is increased, then the threshold will shift to the right, which indicates that the policy maker can reduce speculative pressure on the currency by increasing the interest rate differential—a result that conforms to intuition.[5] Note that the Poisson distributed large shocks to u_t are crucial for comparative statics results not to vanish in the limit.

To appreciate the subtle role played by the large shocks in this article's dynamic model, it is instructive to compare these comparative statics to results obtained by Guimarães [16]. In the model of Guimarães, u_t is driven solely by a Wiener process, so that there is no risk of large shocks hitting the fundamental. Guimarães shows that if the limit is $\zeta \to \infty$ is taken, the thresholds Z_ζ^* converge to a vertical line at ℓ, and from this he concludes that in the case of vanishingly small frictions his model takes on the well-known property of the first generation model that a speculative attack takes place exactly when it leads to an immediate collapse of the exchange rate. However, studying solely the implications of a Wiener process inside a currency crisis version of the model of Burdzy et al. [6] and Frankel and Pauzner [11] gives misleading results. This is because in such models the costs associated with speculation are necessarily proportional to the agents' lock-in time, so of order ζ, while for any fixed horizontal distance between u_t and u^*, the probability of the stochastic process crossing the threshold declines exponentially in ζ (see e.g. [15, section 13.4, theorem 5]). Because of this property, for any $r > 0$, however small, and large enough ζ, the opportunity cost associated with speculation will *always* dominate the payoffs associated with it to the left of the line ℓ, and *never* to the right of it, suggesting that interest rates do not influence the decisions of agents in the limit. In this case it is not possible to derive meaningful comparative statics results as $\zeta \to \infty$. The intuition for this is quite simple: under a Wiener process and with vanishing inertia, taking a position against the peg is not risky in this model. But if it is not risky, it cannot really be called speculation. Hence, Guimarães stresses the importance of considerable frictions on financial markets to understand how currency crises develop (but does not give a concrete explanation of why these frictions might be substantial in reality).

The above results show that the absence of meaningful comparative statics is actually a mathematical artifact of considering a Wiener process. For the more general class of stochastic

[5]This is only a partial comparative statics result, because in the current model the interest rate set by the policy maker is not determined endogenously. This issue is investigated in Daniëls et al. [7] in a simpler setting.

processes considered here, intuitive comparative statics can be obtained, even when inertia is small.

6. Concluding Remarks

We considered the issue of the timing of a currency crisis in a model that is inherently dynamic, but has a structure similar to second generation currency crises models. Concretely, we extended the results of Burdzy et al. [6] and Frankel and Pauzner [11] to the currency crisis setting. While in the articles by Burdzy et al. and Frankel and Pauzner agents engage directly in a coordination game, in the model in this article agents have to forecast a future policy decision, where the behaviour of the policy maker is influenced by the combined speculative pressure of all agents. This situation retains a number of features of a coordination game, foremost the presence of strategic complementarities, but the element of strategic forecasting adds additional complexity to the dynamic model.

As indicated in the introduction, many of the properties of the model that we derived in this article closely resemble those of static global game currency crisis models in the spirit of Morris and Shin [20]. In static global games, a noisy information structure forces each agent to adjust her strategy to take into account a global class of games, which leads to a unique equilibrium prediction. In the model in this article the noisy information structure of the static global game is absent, but possibility of briefly being locked into actions forces speculators to take into account a global state space which includes all possible combinations of λ_t and u_t. In this broader sense, the model is still a global game model. As discussed in section 3, the model can then be solved by an "infection" argument, almost identical to the argument that is applied in the static context. Similar to the static model, the dynamic model has a unique equilibrium in rationalisable strategies. The solution of the model takes the form of an equilibrium threshold that (in the limit) may be characterised in terms of a risk condition, again similar to the characterisation results obtained for the static model (see in particular Morris and Shin [22]). Thus, in more than one way, the model in this article can be regarded as a dynamical counterpart of the static global game currency crisis model.

Proofs

Lemma 4. *The limits $\overline{H}$ and $\overline{L}$ alluded to in the main text are well defined, Lipschitz continuous, decreasing functions. Agent choosing in the graph are of $\overline{H}$ (or $\overline{L}$) are indifferent between speculating or not, if they believe that all others follow a strategy where they attack if (u_t, λ_t) is to the right of $\overline{H}$ (respectively, $\overline{L}$), and refrain from this if (u_t, λ_t) is to its left.*

Proof. Let $\pi(u_t, \lambda_t; Z)$ denote the expected payoff for an individual speculator of choosing "attack" when the state of the system is (u_t, λ_t), and when others follow a strategy where they attack if (u_t, λ_t) is to the right of the function $Z : [0,1] \to \mathbf{R}$ (with Z Lipschitz continuous[6]), and refrain from this if (u_t, λ_t) is to its left. This expected payoff is composed of the expected costs associated with speculation r, versus the probability that u_t will cross the critical threshold u^* during the agent's lock in time if she chooses to attack at time t. Results by Burdzy et al. [5] (concretely, their lemma 2) imply that the function $\pi(u_t, \lambda_t; Z)$ is continuous for fixed Z. The initial curve H_0 may then be found as the graph of the relation $\pi(u_t, \lambda_t; Z) = 0$, where Z is chosen to lie everywhere to the right of u^* (implying that no other speculators choose to attack the currency).

Note that $u^* : [0,1] \to \mathbf{R}$ is continuously differentiable on a compact set and is therefore Lipschitz continuous (Dudley [8], p. 188). Let c be its Lipschitz constant. We will show that H_0, regarded as a function from $[0,1]$ into $\mathbf{R}$, is also Lipschitz continuous with constant c (the argument is a variation on that of Frankel and Pauzner [11], p. 301–302). Consider two arbitrarily chosen distinct points (u_t, λ_t) and (u'_t, λ'_t) in the graph of H_0, and suppose without loss of generality that $\lambda_t < \lambda'_t$ (the shape of u^* then implies that $H_0(\lambda_t) > H_0(\lambda'_t)$). We compare two paths of the process for u_t, one starting at u_t and one starting at u'_t under identical realisations of the stochastic process in (1). Recall that the curve H_0 is derived under the belief that all other agents refrain from attacking, and note that if all follow this strategy the difference between $\lambda_t \leq \lambda'_t$ on the two paths can only shrink. Now suppose:

(4) $H_0(\lambda) - H_0(\lambda') > u^*(\lambda) - u^*(\lambda')$, (where $u^*(\lambda) - u^*(\lambda') \leq c(\lambda' - \lambda)$, as u^* is Lipschitz).

Then surely, if a path starting from (u'_t, λ'_t) has crosses u^* at some time $t^* > t$ then the corresponding path from (u_t, λ_t) must already have crossed u^* *before* time t^*. But the only way an agent choosing at time t can be *indifferent* both at (u_t, λ_t) and (u'_t, λ'_t) is if for every path of u that has crossed u^* at time $t^* > t$, the corresponding path starting from (u_t, λ_t), has also crossed u^* at time t^*. So the first inequality in (4) cannot hold, and we may conclude $H_0(\lambda) - H_0(\lambda') \leq c(\lambda' - \lambda)$. Therefore H_0 is Lipschitz with constant c.

For each $n \in \mathbf{N}$, $n > 0$, the curve H_n may be found recursively as the graph of the relation $\pi(u_t, \lambda_t, H_{n-1}) = 0$. By induction, following the reasoning by Frankel and Pauzner [11, p. 301–302], each H_n, regarded as a function, is again Lipschitz with constant c. The collection of all the H_ns is bounded (because of the existence of dominance regions) so converges pointwise to some limit $\overline{H}$. Moreover, since each H_n has the same Lipschitz constant, the collection of the H_ns is equicontinuous, which implies that the H_ns in fact converge uniformly to $\overline{H}$, and that $\overline{H}$ has Lipschitz constant c, and *a fortiori* is continuous (Dudley [8], p. 51–53). Finally, the operation of constructing H_{n+1} from H_n is continuous in leftward or rightward shifts of the curve H_n. Since continuous operations preserve limits, we may conclude that the graph

[6] A function $f : [0,1] \to \mathbf{R}$ is Lipschitz continuous (or "Lipschitz") if there exists a constant c such that $|f(\lambda) - f(\lambda')| \leq c|\lambda - \lambda'|$, for all $\lambda, \lambda' \in [0,1]$.

of $\overline{H}$ coincides with the relation $\pi(u_t, \lambda_t; \overline{H}) = 0$. The same property may be proved of the curve $\overline{L}$, by analogous arguments. □

Proof of Theorem 3. In Burdzy et al. [5], the mathematical companion paper to Frankel and Pauzner [11], the authors study the properties of a dynamical system governed by the equations in (2) when the variance of the stochastic process W_t become small and approaches 0, and the jump component J_t is absent (also in the model considered in [11], there is no compound Poisson process J_t). We can apply their theorem 2 by (a) choosing a rescaled unit of time (just like Frankel and Pauzner do in their proof of [11, theorem 3]), so that instead of taking $\zeta \to \infty$, the variance of W_t shrinks to 0 and then (b) incorporating the possibility of a discrete jump directly in our argument. To this end, we define a new time unit $\tilde{t} = t/\zeta$. In the new time units, the Poisson parameter for the revision process equals 1, the variance of $W_{\tilde{t}}$ is $1/\zeta$, the rate of arrival of the large shocks is $\tilde{\vartheta} = \vartheta/\zeta$, and costs of speculation are given by $\tilde{r} = r/\zeta$ per unit of time.

Let $Z : [0,1] \to \mathbf{R}$ be a continuously differentiable, decreasing, Lipschitz continuous function and suppose that agents follow a strategy where they attack if (u_t, λ_t) is to the right of the graph of Z and refrain from doing so if (u_t, λ_t) is to its left. By theorem 2 in Burdzy et al. [5] (see also Burdzy et al. [6]), as the variance of $W_{\tilde{t}}$ shrinks to 0, for any $\delta > 0$, and $\tau > 0$ and at any $(u_{\tilde{t}_0}, \lambda_{\tilde{t}_0})$ in the graph of Z, the *probability* that the process $\lambda_{\tilde{t}}$ bifurcates upwards before τ units of time have passed and subsequently reaches $1-\delta$ approaches $1 - \lambda_{\tilde{t}_0}$. Similarly, the probability that the process $\lambda_{\tilde{t}}$ bifurcates downwards before τ units of time have passed and subsequently reaches δ approaches the value of $\lambda_{\tilde{t}_0}$. The difference between Burdzy et al. [5] and the model in this article is that these authors do not consider the possibility of large shocks, so that in our setting δ is reached conditional on the fact that no large shock arrives. This means that in our setting, the bifurcation results in [5] still apply but become statements of the form: with probability $\lambda_{\tilde{t}}$, the process for $\lambda_{\tilde{t}}$ is described by $\dot{\lambda}^{\uparrow}_{\tilde{t}}$ until either $1 - \delta$ is reached *or* a large shock arrives; similarly for a downward bifurcation.

Now, for large ζ (or equivalently, small $\tilde{\vartheta}$), we will approximate expected payoffs of an agent choosing at time t by supposing that during agent i's lock-in time, the compound Poisson process $J_{\tilde{t}}$ jumps at most once. (As $\zeta \to \infty$, the probability that $J_{\tilde{t}}$ jumps twice during an agents lock-in period becomes negligible, as this probability is $\mathscr{O}\left(\frac{1}{\zeta^2}\right)$ as $\zeta \to \infty$.[7] The probability that a single jump arrives is of $\mathcal{O}\left(\frac{1}{\zeta}\right)$ as $\zeta \to \infty$ but so are a speculator's opportunity costs r/ζ. Hence the probability that a jump arrives stays relevant.) The probability a shock arrives in the infinitesimal interval $d\tilde{t}$ is then given by $\tilde{\vartheta} e^{-\tilde{\vartheta} t}$ in the new

[7]Recall that function $f(\zeta, \cdot)$ is called "$\mathscr{O}\left(g(\zeta)\right)$ as $\zeta \to \infty$" if there are ζ_0 and positive constant k such that $|f(x)| \leq k|g(x)|$ for all $\zeta > \zeta_0$. In equations (5), (6) and (7), the notation "$\mathscr{O}\left(g(\zeta)\right)$" refers to an error term that is of this order.

time unit. If—and only if, given the argument in the main text, section 5—$J_{\tilde{t}}$ indeed jumps during an agent's lock-in time, it is possible that $u_{\tilde{t}}$ crosses u^* at some time at which she is still locked into her action. Given that all stochastic processes considered in the model are memory-less, the *ex ante* probability of $u_{\tilde{t}}$ crossing u^* should depend solely on the position of $u_{\tilde{t}}$ and $\lambda_{\tilde{t}}$ just before the time of the jump, and is given by a time-invariant function $\rho(\cdot)$ that depends on $u_{\tilde{t}}$ and $\lambda_{\tilde{t}}$ just before the time of the jump.

Recall that each Z^*_ζ is Lipschitz continuous but not necessarily continuously differentiable. Because theorem 2 in Burdzy et al. [5] is stated for continuously differentiable functions Z, we will approximate each Z^*_ζ by smooth functions Z^n_ζ. Fortunately, Lipschitz functions with constant c can be approximated uniformly by functions that are infinitely differentiable (so-called C^∞-functions) with the same Lipschitz constant c. One way to do this is through a technique known as "mollification"; see e.g. Azagra *et alia* [3, p. 1370–1371] for details. Fixing ζ, lemma 2(iii) in [5] shows that as $Z^n_\zeta \to Z^*_\zeta$ the dynamics of $(u_{\tilde{t}}, \lambda_{\tilde{t}})$ induced by each Z^n_ζ converge to those induced by Z^*_ζ. This means that for each ζ we may choose n such that for each point on the graph of Z^*_ζ the expected difference between attacking and refraining under the dynamics induced by Z^n_ζ is within $\frac{1}{\zeta^2}$ of the expected difference between attacking and refraining under the dynamics induced by Z^*_ζ

Normalise $\tilde{t} = 0$ and choose $\lambda_0 \in (0, 1)$. Let $u_0 = Z^*_\vartheta(\lambda_t)$. As shown in [11, proof of theorem 2], for large enough ζ and n, in equilibrium the expected payoff of choosing to speculate at $(u_0, \lambda_0) \in Z^*_{\tilde{\vartheta}}$ can be approximated by the following equation:

$$(5) \quad (1-\lambda) \int_0^\infty \tilde{\vartheta} e^{-(\tilde{\vartheta}+1)\tilde{t}} \rho(\lambda^\uparrow_{\tilde{t}}, u_0; Z^*_\zeta)\, d\tilde{t} + \lambda \int_0^\infty \tilde{\vartheta} e^{-(\vartheta+1)\tilde{t}} \rho(\lambda^\downarrow_{\tilde{t}}, u_0; Z^*_\zeta)\, d\tilde{t} + \mathscr{O}\left(\frac{1}{\zeta^2}\right)$$

which may be rewritten as:

$$(6) \quad \frac{\vartheta}{\zeta}\left[(1-\lambda) \int_0^\infty e^{-(\tilde{\vartheta}+1)\tilde{t}} \rho(\lambda^\uparrow_{\tilde{t}}, u_0; Z^*_\zeta)\, d\tilde{t} + \lambda \int_0^\infty e^{-(\vartheta+1)\tilde{t}} \rho(\lambda^\downarrow_{\tilde{t}}, u_0; Z^*_\zeta)\, d\tilde{t}\right] + \mathscr{O}\left(\frac{1}{\zeta^2}\right),$$

where $\lambda^\downarrow_{\tilde{t}} = \lambda_0 \exp(-\zeta\tilde{t})$ and $\lambda^\uparrow_{\tilde{t}} = 1 - (1 - \lambda_0)\exp(-\zeta\tilde{t})$. Here, the part between brackets in expression (6) corresponds exactly to the equation in [11, proof of theorem 2]. The authors show by change of variables that this part approximately equals the expression:

$$\int_0^1 w(l)[\rho(l, u_0; Z^*_\zeta)]\, dl,$$

with weights $w(l)$ which are given by:

$$w(l) = \begin{cases} \left(\frac{l}{\lambda_0}\right)^{\tilde{\vartheta}} & \text{if } l \le \lambda_0 \\ \left(\frac{1-l}{1-\lambda_0}\right)^{\tilde{\vartheta}} & \text{if } l > \lambda_0 \end{cases}$$

As $\tilde{\vartheta} = \vartheta/\zeta \to 0$, the weights converge to 1. If Z^*_ζ is the equilibrium threshold, it must be that an agent who receives a revision opportunity in the graph of Z^*_ζ is indifferent. The opportunity costs associated with speculation are given by r/ζ, so that for large ζ the indifference condition converges to:

$$\vartheta \int_0^1 \rho(l, Z^*_\zeta(\lambda_0); Z^*_\zeta)\, dl + \mathscr{O}\left(\frac{1}{\zeta}\right) = r, \tag{7}$$

or, in other words, the returns r compensate for a "weighted" expected gross return from attacking at $u_0 = Z^*_\zeta(\lambda_0)$. Since ρ is increasing in u_t, agents attack for sure to right of Z^*_ζ and refrain from attacking for sure to its left. It follows that for sufficiently large ζ at every point u_t lying on the curve $Z^*_\zeta(\lambda_t)$, we must approximately have $V(u_t) = r$. This condition corresponds to that in the statement of theorem 3. Clearly, to satisfy this requirement Z^*_ζ must be close to a vertical line and, as $\zeta \to \infty$, Z^*_ζ must therefore converge to the vertical line $Z(\overline{u})$ of lemma 2. Agents attack to the right of $Z(\overline{u})$ (i.e., when $V(u_t) > r$), and refrain from attacking its left ($V(u_t) < r$). □

References

[1] Angeletos, G.-M., Hellwig, C., and Pavan, A. Dynamic global games of regime change: Learning, multiplicity, and the timing of attacks. *Econometrica 75*, 3 (2007), 711–56.

[2] Applebaum, D. Lévy processes. from probability to finance and quantum groups. *Notices of the AMS 51*, 11 (2003), 1336–47.

[3] Azagra, D., Ferrera, J., López-Mesas, F., and Rangel, Y. Smooth approximation of Lipschitz functions on Riemannian manifolds. *Journal of Mathematical Analysis and Applications 326*, 2 (2007), 1370–8.

[4] Bernheim, P. D. Rationalizable strategic behavior. *Econometrica 52*, 4 (1984), 1007–28.

[5] Burdzy, K., Frankel, D. M., and Pauzner, A. On the time and direction of stochastic bifurcation. In *Asymptotic Methods in Probability and Statistics: A Volume in Honour of Miklos Csorgo*, B. Szyszkowicz, Ed. Amsterdam: Elsevier, 1998.

[6] Burdzy, K., Frankel, D. M., and Pauzner, A. Fast equilibrium selection by rational players living in a changing world. *Econometrica 69*, 1 (2001), 163–89.

[7] Daniëls, T. R., Jager, H., and Klaassen, F. Defending against speculative attacks: A hybrid model of exchange market pressure and crises. SFB-649 Discussion Paper 2009-011, Humboldt University, 2009.

[8] Dudley, R. *Real Analysis and Probability.* Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2002.

[9] Dufresne, F., and Gerber, H. U. Risk theory for the compound poisson process that is perturbed by diffusion. *Insurance: mathematics and economics 10*, 1 (1991), 51–9.

[10] Flood, R. P., and Garber, P. M. Collapsing exchange rate regimes: Some linear examples. *Journal of International Economics 17*, 1–2 (1984), 1–13.

[11] Frankel, D., and Pauzner, A. Resolving indeterminacy in dynamic settings: The role of shocks. *The Quarterly Journal of Economics 115*, 1 (2000), 285–304.

[12] Frankel, D. M., Morris, S., and Pauzner, A. Equilibrium selection in global games with strategic complementarities. *Journal of Economic Theory 108*, 1 (2003), 1–44.

[13] Fudenberg, D., and Tirole, J. *Game Theory.* Cambridge, London: The MIT Press, 1991.

[14] Goldstein, I. Strategic complementarities and the twin crises. *The Economic Journal 115*, 503 (2005), 368–90.

[15] Grimmitt, G. R., and Stirzaker, D. R. *Probability and Random Processes*, 3rd ed. Oxford etc: Oxford University Press, 2005.

[16] Guimarães, B. Dynamics of currency crises with asset market frictions. *Journal of International Economics 68*, 1 (2006), 141–58.

[17] Guimaraes, B., and Morris, S. Risk and wealth in a model of self-fulfilling currency attacks. *Journal of Monetary Economics 54*, 8 (2007), 2205–30.

[18] Jorion, P. On jump processes in the foreign exchange and stock markets. *The Review of Financial Studies 1*, 4 (1988), 427–45.

[19] Krugman, P. A model of balance-of-payments crises. *Journal of Money, Credit and Banking 11*, 3 (1979), 311–25.

[20] Morris, S., and Shin, H. S. Unique equilibrium in a model of self-fulfilling currency attacks. *American Economic Review 88*, 3 (1998), 587–97.

[21] Morris, S., and Shin, H. S. A theory of the onset of speculative attacks. In *The Asian Financial Crisis: Causes, Contagion and Consequences.* Cambridge etc.: Cambridge University Press, 1999, pp. 230–55.

[22] Morris, S., and Shin, H. S. Global games: Theory and applications. In *Advances in Economics and Econometrics: Eight World Congress*, M. Dewatripont, L. P. Hansen, and S. J. Turnovsky, Eds. Cambridge: Cambridge University Press, 2003, pp. 56–114.

[23] Obstfeld, M. Rational and self-fulfilling balance-of-payments crises. *American Economic Review 76*, 1 (1986), 72–81.

[24] Obstfeld, M. Models of currency crises with self-fulfilling features. *European Economic Review 40*, 3–5 (1996), 1037–47.

[25] Obstfeld, M., and Rogoff, K. The mirage of fixed exchange rates. *Journal of Economic Perspectives 9*, 4 (1995), 73–96.

[26] Pearce, D. Rationalizable strategic behavior and the problem of perfection. *Econometrica 52*, 4 (1984), 1029–50.