CHARACTERISING EQUILIBRIUM SELECTION IN GLOBAL GAMES WITH STRATEGIC COMPLEMENTARITIES

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Abstract. Global games are widely used to predict behaviour in games with strategic complementarities and multiple equilibria. We establish two results on the global game selection. First, we show that it is independent of the payoff functions of the global game embedding, though (as is well-known) it may depend on the noise distribution. Second, we give a simple sufficient criterion for noise independence in many-action games. A many-action game is noise independent if it can be suitably decomposed into smaller (say, binary action) games, for which there are simple criteria guaranteeing noise independence. We also delineate the games where noise independence may be established by counting the number of players or actions.

Keywords: global games, equilibrium selection, strategic complementarities.
JEL codes: C72, D82.

1. Introduction

Games with strategic complementarities often have multiple equilibria that give rise to coordination problems. Economic applications cover a wide range of topics, including poverty traps and underdevelopment (see for example Ray [25]) or financial crises (see for example Diamond and Dybvig [8], Obstfeld [21]). A widely used approach to predict behaviour in such games is by embedding them in a "global game". A global game extends a complete information game g by a payoff function u that depends on an additional state parameter θ not directly observable by agents. The payoff function u coincides with the payoff function g at the true state, say θ^*, but agents have to rely on noisy private signals about the true state. This leads to uncertainty not just about their own relevant payoff function, but also—and more importantly—about the beliefs of opponents.

Frankel, Morris and Pauzner [9] (henceforth "FMP") show that as the noise in private signals vanishes, agents coordinate on some action profile that is a Nash equilibrium of the complete information game g. This *global game selection* of g may be used as a prediction and to derive comparative statics results in games with multiple equilibria. Applications include Morris and Shin [18]; Cukierman, Goldstein and Spiegel [6]; Rochet and Vives [26]; Coresetti, Dasgupta, Morris and Shin [4]; Goldstein [10]; Corsetti, Guimarães and Roubini [5]; Guimarães and

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Morris [11], among others. The theory has also been corroborated by experimental evidence, see Heinemann, Nagel and Ockenfels [12] and [13].

Unfortunately, which equilibrium is selected may depend on the fine details of the global game. A well known problem is that it may depend on the signals' noise distribution. FMP discuss the robustness of the global game selection with respect to the signals' noise distribution and provide some conditions under which the global game selection is *noise independent*, that is, independent of this distribution.

In this paper, we establish two additional positive results on the robustness of global games. First, we show that the global game selection, though it may depend on the noise distribution, is always independent of the embedding payoff function u. This may come as a surprise, since the process of global game selection is often described informally as "infection" from high and low parameter regions of the payoff function. In fact we show that it does not even matter whether the embedding payoff function satisfies certain frequently made assumptions, such as continuity. This strengthens previous results on global game selection (FMP; Mathevet [16]).

Second, we provide new and simple conditions for noise independence in many-action games. We show that the global game selection in g may be noise independent if g can be suitably decomposed into smaller noise independent games. For example, we may split up a n-action game into many binary-action games for which there are simple criteria that guarantee noise independence and simple rules for deriving the global game selection.

This approach is useful, since the simplest known criteria to establish noise independence are through counting the number of players and actions. Carlsson and Van Damme [3] show that two-player binary-action games are noise independent. In this case, the global game selection is the risk dominant equilibrium. Morris and Shin [19] show how to find the global game selection in many-player symmetric binary-action games. Here, the global game selection is the best reply to the belief that the fraction of players choosing either action is uniformly distributed. Up to now, most applications of global games use these heuristics in binary-action models. Our result gives a simple tool to extend them to many action games.

But other criteria in terms of players and actions may also be applied. In this paper, we establish that all two-player $2 \times n$ action games are noise independent. Oyama and Takahashi [23] show that symmetric two-player 3×3 games are noise independent. Conversely, FMP show that symmetric 4×4 games may not be noise independent. An example by Carlsson [2] shows that noise independence may fail in three-player binary-action games. And in this paper, we establish that it may also fail in two-player 3×3 games and in symmetric three-player, three-action games. As far as we know, these two minimal counterexamples are novel contributions to the literature.

It completes the characterisation of games where noise independence can be established simply by counting the number of players or actions.

Another criterion that guarantees the noise independence of g is the existence of an equilibrium that is "robust to incomplete information", as defined by Kajii and Morris [14]. A heuristic argument may be found in Morris and Shin [19], but in this paper we give a formalisation. Our result is closely related to a similar theorem of Oury and Tercieux, who introduce a somewhat stronger, slightly non-standard, notion of robustness to exploit a link with so-called "contagious" equilibria [22]. Instead, we give a direct and elementary proof based on the standard notion of robustness. This allows the application of known conditions for robustness to incomplete information when trying to determine whether some game is noise independent.

Our paper proceeds as follows. Section 2 contains preliminary definitions and results. The rest of the paper is organised around a characterisation of the global game selection process given in section 3. Instead of analysing the limit of a series of global games with shrinking noise, we show that a single incomplete information game with a fixed noise structure allows one to determine the global game selection in g directly. Moreover, this incomplete information game does not incorporate the whole range of the payoff function u of the global game, but depends on the payoff structure of g alone. This establishes that the global game selection in g is independent of the embedding payoff function u. We also use the characterisation to prove generic uniqueness of the global game selection. In section 4 we use it to establish our results on noise independence in many-action games, and to prove that robustness implies noise independence. In section 5 we conclude. Most proofs are found in the appendix.

2. Setting and Definitions

In this paper we consider games with a finite set of players I, who have finite action sets $A_{i\in I} = \{0, 1, \ldots, m_i\}$ which we endow with the natural ordering inherited from $\mathbb{N}$. We define the joint action space A as $\prod_{i\in I} A_i$, and write A_{-i} for $\prod_{j\neq i} A_j$. We say that $a = (a_i)_{i\in I} \in A$ is weakly greater than $a' = (a'_i)_{i\in I}$ if $a_i \geq a'_i$ for all $i \in I$ and write $a \geq a'$. The greatest and least action profiles in A are denoted by m and 0. A complete information game g is fully specified by its real-valued payoff functions $g_{i\in I}(a_i, a_{-i})$, where a_i denotes i's action and a_{-i} denotes the opposing action profile. A game g is a game of *strategic complementarities*[1] if greater opposing action profiles make greater actions more appealing, or more precisely, if for all i, $a_i \geq a'_i$, $a_{-i} \geq a'_{-i}$,

$$g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i}) \geq g_i(a_i, a'_{-i}) - g_i(a'_i, a'_{-i}). \tag{1}$$

[1]Strictly speaking, it would be more correct to say that g is a supermodular game [28, 29]. However, FMP use the term strategic complementarities in this context, so we stick to it.

§2.1. Global Games

Following FMP, we now define a global game $G_v(u, \phi, f)$ as follows. It is an incomplete information game where payoffs depend on a real-valued random variable θ, called the state parameter, which is distributed according to a continuous density ϕ, called the prior distribution. The individual payoffs in the incomplete information game are given by $u_i(a_i, a_{-i}, \theta)$. f is a tuple of densities, whose support is a subset of $[-\frac{1}{2}, \frac{1}{2}]$, that we refer to as the *noise structure*. Each player $i \in I$ observes a private signal $x_i = \theta + v\eta_i$ about θ, where $v > 0$ is a scale factor and η_i an error that is distributed according to the density f_i. The random variables $\{\theta, \eta_1, \ldots, \eta_I\}$ are independently distributed.

Moreover, FMP define four conditions that the payoff function u needs to fulfil:

A1 *Strategic complementarities:* For every value of θ, the complete information game specified by $u_{i\in I}(\cdot, \theta)$ is a game of strategic complementarities.

A2 *Dominance regions:* Extreme values of θ make the extreme actions dominant choices. That is, there exist thresholds $\underline{\theta} < \overline{\theta}$ such that $[\underline{\theta} - v, \overline{\theta} + v]$ is contained in the interior of the support of ϕ and for all players i and all opposing action profiles a_{-i} we have

$$u_i(0, a_{-i}, \theta) > u_i(a_i, a_{-i}, \theta) \text{ for all } a_i > 0 \text{ and } \theta \leq \underline{\theta},$$

and

$$u_i(m_i, a_{-i}, \theta) > u_i(a_i, a_{-i}, \theta) \text{ for all } a_i < m_i \text{ and } \theta \geq \overline{\theta}.$$

A3 *State monotonicity:* Greater states make greater actions more appealing. More precisely, there exists $K > 0$ such that for all $a_i \geq a_i'$ and $\underline{\theta} \leq \theta' \leq \theta \leq \overline{\theta}$ we have

$$\begin{aligned}(u_i(a_i, a_{-i}, \theta) - u_i(a_i', a_{-i}, \theta)) - (u_i(a_i, a_{-i}, \theta') - u_i(a_i', a_{-i}, \theta'))\\ \geq K(a_i - a_i')(\theta - \theta') \geq 0.\end{aligned}$$

A4 *Payoff continuity:* Each $u_i(a_i, a_{-i}, \theta)$ is continuous in the state parameter.

While assumptions A3 and A4 are mathematically convenient, from an applied point of view they are not always desirable. Indeed, the pioneering speculative attack model of Morris and Shin [18] satisfies neither A3 nor A4. Further below, we will show that both assumptions may be weakened without forgoing the existence of a unique equilibrium prediction.

§2.2. Strategies in Global Games

A strategy for player i is a measurable function $s_i : \mathbb{R} \to A_i$ and s strategy profile s is a tuple of such strategies, $s = (s_i)_{i\in I}$. If x is a tuple of signals $(x_i)_{i\in I}$, then $s(x)$ denotes the action profile $(s_i(x_i))_{i\in I}$. Slightly abusing notation, for $x \in \mathbb{R}$ we also denote the action profile given by

$(s_i(x))_{i\in I}$ by $s(x)$. A strategy profile s is *increasing* if each component s_i is weakly increasing and *left (right) continuous* if each s_i is left (right) continuous. We say that the strategy profile s is weakly greater than the strategy profile s' if $s(x) \geq s'(x)$ for all $x \in \mathbb{R}$ and write $s \geq s'$.

Using Bayes' rule, agents can derive the conditional density of θ given one's signal x_i, $h_i(\theta|x_i)$, and the conditional density over opponents signals given θ, $\pi_{-i}(x_{-i}|\theta)$. The conditional density of x_{-i} given x_i can then be calculated as

$$\pi_i(x_{-i}|x_i) := \int_{\mathbb{R}} \pi_{-i}(x_{-i}|\theta) h_i(\theta|x_i) \ \ \mathrm{d}\theta.$$

To refer to probability of the event $x_{-i} \in E$, conditional on the signal x_i, we will sometimes use the notation

$$\mathbb{P}(x_{-i} \in E|x_i) := \int_E \pi_i(x_{-i}|x_i) \ \ \mathrm{d}x_{-i}.$$

Given x_i and assuming that opponents $j \neq i$ follow the strategies s_j given by some strategy profile s, the action $a_i \in A_i$ yields an expected payoff of

$$u_i(a_i, s_{-i}|x_i) := \int_{\mathbb{R}} \int_{x_{-i}\in\mathbb{R}^{|I|-1}} u_i(a_i, s_{-i}(x_{-i}), \theta)\pi_{-i}(x_{-i}|\theta) \ \ \mathrm{d}x_{-i} \ h_i(\theta|x_i) \ \ \mathrm{d}\theta.$$

against the induced opposing action distribution. Let $BR(s)_i(x_i)$ denote the set of best replies of player i conditional on the signal x_i, *viz* the set of actions that maximise the expected payoff. A strategy profile s is a (Bayes-Nash) *equilibrium strategy profile* if it is a best reply to itself, i.e.

$$\forall i \forall x_i, \quad s_i(x_i) \in BR(s)_i(x_i).$$

The upper-best reply strategy is defined as the strategy consisting of the *greatest* best replies:

$$\beta(s)_i(x_i) := \max BR(s)_i(x_i),$$

and determines the strategy profile $\beta(s)$. Strategic complementarities imply that if one opposing action distribution dominates another, the greatest best reply to the former is weakly greater than to the latter. In particular, β is monotonic, i.e., $\beta(s) \geq \beta(s')$ if $s \geq s'$ (see Topkis [28, 29] and Vives [30]). We can conduct upper-best reply iterations s, $\beta(s)$, $\beta(\beta(s))$, $\beta(\beta(\beta(s)))$, ... starting at some strategy profile s. If $\beta(s)$ is weakly greater (smaller) than s, the resulting sequence of strategy profiles will be monotonically increasing (decreasing). As the action space is bounded, the resulting sequence will then converge pointwise to an equilibrium strategy profile. If we choose the greatest strategy profile as the starting point for the iteration, the best reply to it can only be weakly smaller, so that the iteration will converge pointwise to the (necessarily) greatest equilibrium strategy profile.

As is usual when dealing with games of strategic complementarities, virtually all of our results are order-theoretic in nature. By standard order-theoretic duality, each result implies a dual result with all order-theoretic notions reversed (see Davey and Priestley [7], p. 15). We will invoke this duality throughout the text.

§2.3. Summary of FMP's Results

Assuming (A1)–(A4) above, FMP show that in a global game the least and greatest equilibrium strategy profiles converge to each other as the noise vanishes.

Theorem. (Theorem 1 in FMP) *The global game $G_v(u, \phi, f)$ has an essentially unique equilibrium strategy profile in the limit as $v \to 0$. More precisely, there exists an increasing pure strategy profile s^f such that if, for each $v > 0$, s_v is an equilibrium strategy profile of $G_v(u, \phi, f)$, then $\lim_{v\to 0} s_{v,i}(x_i) = s_i^f(x_i)$ for all x_i, except possibly at the finitely many discontinuities of s^f.*

Since s^f is determined up to its points of discontinuity, we will work with the left and right continuous versions of s^f, which we denote by $\underline{s}^f$ and $\overline{s}^f$ respectively.

FMP's second result implies that the limit strategy profile s^f only selects Nash equilibria.

Theorem. (Theorem 2 in FMP) *Let $G_v(u, \phi, f)$ be a global game and s^f its essentially unique limit strategy profile. Then for any state parameter θ^*, $\overline{s}^f(\theta^*)$ and $\underline{s}^f(\theta^*)$ are Nash equilibria of the complete information game specified by the payoff function $u(\cdot, \theta^*)$.*

To prove these results, FMP introduce the notion of a simplified global game $G_v(u, f)$ that differs from $G_v(u, \phi, f)$ in that θ is uniformly distributed over a large interval containing $[\underline{\theta} - v, \overline{\theta} + v]$ and individual payoffs depend directly on the private signal x_i rather than on the true state θ. A simplified global game is much easier to analyse, as there is no uncertainty about payoff functions, and for signals x_i within $[\underline{\theta}, \overline{\theta}]$ the conditional densities of opponents' signals can be calculated straightforwardly from the noise structure f due to the uniform prior distribution.

Theorem. (Lemma A1, A3, and A4 in FMP) *The simplified global game $G_v(u, f)$ has an essentially unique, increasing equilibrium strategy profile s_v^f. In the limit as $v \to 0$, s_v^f converges towards s^f in horizontal distance, that is*

$$\textit{for all } \varepsilon > 0 \textit{ there is } \overline{v} > 0 \textit{ such that for } v \textit{ satisfying } 0 < v < \overline{v}, \textit{ we have:}$$

$$\forall i \forall x_i, \quad s_{v,i}^f(x_i + \varepsilon) \geq s_i^f(x_i) \geq s_{v,i}^f(x_i - \varepsilon).$$

Using this result, the simplified global game can be used to derive the global game selection. Figure 1 illustrates the notion of convergence. Note that this result also implies that the selection is independent of the prior distribution ϕ.

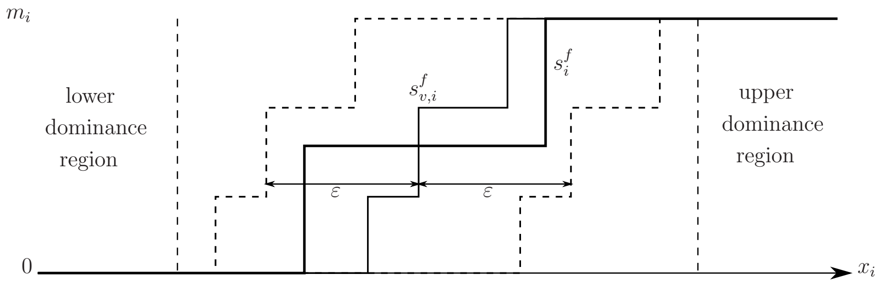

FIGURE 1. s_i^f and $s_{v,i}^f$ for $v < \overline{v}$

3. Equilibrium Selection in Global Games

For a given global game $G_v(u, \phi, f)$, we define the *embedded game* $g(\theta^*)$ as the complete information game that has the same set of players and actions as the global game, and which has the payoff function given by $u(\cdot, \theta^*)$. In this case we also say that $G_v(u, \phi, f)$ is a *global game embedding* of $g(\theta^*)$.

For $g(\theta^*)$, the global game can be viewed as an equilibrium refinement. The limit strategy profile of the global game, s^f, determines an action profile $s^f(\theta^*)$, which is in fact an equilibrium of the embedded game. Thus the global game approach generically selects a unique equilibrium of the game g, to which we refer as the *global game selection* (following Heinemann et al. [13]). In this section, we provide a conceptually simple characterisation of this selection process that implies that the selection depends only on the payoff structure of $g(\theta^*)$ and on the noise structure f.

§3.1. Global Games as an Equilibrium Refinement

Our first aim in this section is to show that this approach can by applied to any game with strategic complementarities.

Lemma 1. *For any game of strategic complementarities g, there exists a global game embedding of g.*

We prove this by constructing a global game $G_v(u, \phi, f)$ such that $g = g(0)$. Let u be given by $u_i(a_i, a_{-i}, \theta) := g_i(a_i, a_{-i}) + \theta a_i$, for $i \in I$, $a_i \in A_i$ and $a_{-i} \in A_{-i}$. Clearly, each u_i is continuous in θ, so u satisfies A4. For any fixed θ and $a_i \geq a_i', a_{-i} \geq a_{-i}'$ we have

$$\begin{aligned} u_i(a_i, a_{-i}, \theta) - u_i(a_i', a_{-i}, \theta) &= g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) + \theta(a_i - a_i') \\ &\geq g_i(a_i, a_{-i}') - g_i(a_i', a_{-i}') + \theta(a_i - a_i') \\ &= u_i(a_i, a_{-i}', \theta) - u_i(a_i', a_{-i}', \theta), \end{aligned}$$

as g is a game of strategic complementarities. So u exhibits strategic complementarities as well and satisfies A1. For $a_i \geq a_i'$ and $\theta \geq \theta'$ we have that

$$
\begin{aligned}
&(u_i(a_i, a_{-i}, \theta) - u_i(a_i', a_{-i}, \theta)) - (u_i(a_i, a_{-i}, \theta') - u_i(a_i', a_{-i}, \theta')) \\
&\quad = g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) + \theta(a_i - a_i') - g_i(a_i, a_{-i}) + g_i(a_i', a_{-i}) - \theta'(a_i - a_i') \\
&\quad = (\theta - \theta')(a_i - a_i'),
\end{aligned}
$$

that is, u satisfies the state monotonicity assumption A3. For all $a_i < m_i$ and sufficiently large $\overline{\theta}_i \geq 0$ we have

$$
u(m_i, a_{-i}, \overline{\theta}_i) - u(a_i, a_{-i}, \overline{\theta}_i) = g_i(m_i, a_{-i}) - g_i(a_i, a_{-i}) + \overline{\theta}_i(m_i - a_i) > 0,
$$

so m_i is the dominant action for $\theta \geq \overline{\theta}_i$. Analogously, there exists $\underline{\theta}_i \leq 0$ such that $a_i = 0$ is the dominant action for $\theta \leq \underline{\theta}_i$. Choosing $\overline{\theta} = \max\{\overline{\theta}_i\}$ and $\underline{\theta} = \min\{\underline{\theta}_i\}$, u satisfies A2. As we let θ be distributed over a large interval containing $[\underline{\theta} - \nu, \overline{\theta} + \nu]$ and choose f arbitrarily, we have constructed a global game $G_\nu(u, \phi, f)$ in which g is the embedded game $g(0)$.

This shows that, at least technically, global games can be used to derive an equilibrium refinement for all games of strategic complementarities. By constructing a global game embedding $G_\nu(u, \phi, f)$ of g, and taking $\nu \to 0$, we find two distinguished equilibria of $g = g(\theta^*)$, namely $\underline{s}^f(\theta^*)$ and $\overline{s}^f(\theta^*) \in A$, which generically coincide (proposition 4 below). FMP prove that this equilibrium selection is independent of the prior distribution ϕ, but may depend on the noise structure f. In the following subsection, we will show that it is also independent of the choice of payoff functions $u(\cdot, \theta)$ of the global game embedding of g.

§3.2. Attainability

Let g be a complete information game of strategic complementarities. The following incomplete information game, constructed around g, will be central to the rest of our results.[2]

Definition. A *lower-f-elaboration*, $\underline{e}(g, f)$, of g, is defined as the following incomplete information game. The state parameter θ is uniformly distributed over an interval $[-\frac{1}{2}, R]$, with $R \geq \overline{R} := \sum_{i \in I}(m_i + 1)$. All individuals receive a noisy signal $x_i = \theta + \eta_i$ about the true state, with each η_i drawn according to the density f_i, the support of which is a subset of $[-\frac{1}{2}, \frac{1}{2}]$. The random variables $\{\theta, \eta_1, ..., \eta_I\}$ are independently distributed. Players' payoffs u_i are given by

$$
u_i(a_i, a_{-i}, x_i) = \begin{cases} \tilde{u}_i(a_i, a_{-i}) & \text{if } x_i < 0, \\ g_i(a_i, a_{-i}) & \text{if } x_i \geq 0, \end{cases}
$$

[2]It is inspired by a construction in a proof of FMP [9].

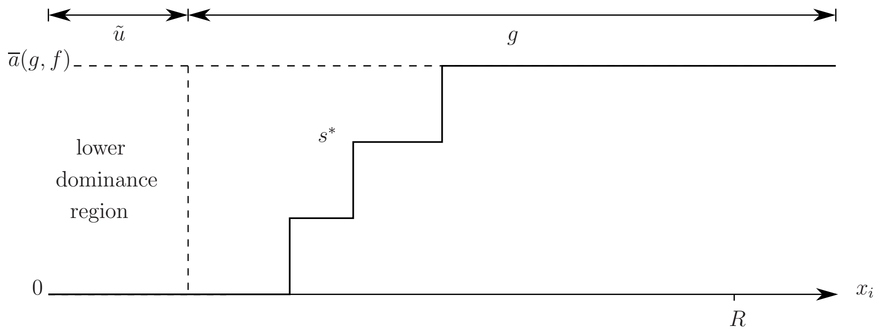

FIGURE 2. A lower-f-elaboration such that the greatest equilibrium strategy profile, s^*, attains $\overline{a}(g, f)$.

with $\tilde{u}_i$ being an arbitrary payoff function that makes the least action dominant, e.g. for all a_{-i}, $\tilde{u}_i(0, a_{-i}) = 1$ and $\tilde{u}_i(a_i, a_{-i}) = 0$ when $a_i \neq 0$. An *upper-f-elaboration*, $\overline{e}(g, f)$, is defined dually: θ is distributed over $[L, \frac{1}{2}]$, where $L \leq \overline{L} := -\overline{R}$, and for each player i the greatest action is dominant for all signals $x_i > 0$. □

Just as for global games, we define strategy profiles, expected payoffs $u_i(a_i|s, x_i)$, upper-best reply strategy profiles $\beta(s)$, and equilibrium strategy profiles in $\underline{e}(g, f)$. In any equilibrium strategy profile s of $\underline{e}(g, f)$ we must have $s(x) = 0$ for signals $x < 0$. In equilibrium, the behaviour of players receiving signals smaller than 0 affects the choices of players receiving signals greater than 0 by the usual "infection" argument (cf. Morris et al. [17]). As a consequence, even if the action profile $a \in A$ is a Nash equilibrium of the complete information game g, the action profile a is not necessarily played in any equilibrium strategy profile s of $\underline{e}(g, f)$. We say an equilibrium strategy profile s of the lower-f-elaboration $\underline{e}(g, f)$ *attains* a if $s(x) \geq a$ for some $x \in [-\frac{1}{2}, R]$ (and, dually, an equilibrium strategy profile s of an upper-f-elaboration $\overline{e}(g, f)$ attains a if $s(x) \leq a$ for some $x \in [L, \frac{1}{2}]$).

We can use standard results (e.g. Vives [30]) on games with strategic complementarities to analyse lower-f-elaborations. The upper-best reply operator $\beta(s)$ is monotonic, and hence there is a greatest equilibrium strategy profile s^*, which is increasing.

Definition. An action profile $a \in A$ is *attained from below under f* if in *some* lower-f-elaboration of g, the greatest equilibrium strategy profile attains a (see figure 2). In particular, we denote the greatest action profile that is attained from below under f by $\overline{a}(g, f)$. As the action space is finite, this notion is well defined. We define *attained from above under f* dually, and in particular we define $\underline{a}(g, f)$ as the least action profile used in the least equilibrium strategy profile of *some* upper-f-elaboration. □

We will prove that in order to determine the global game selection in g, it suffices to look at $\overline{a}(g, f)$ and $\underline{a}(g, f)$. A first easy but useful observation is that to determine $\overline{a}(g, f)$ it actually suffices to look at any one lower-f-elaboration. (Of course, a dual observation holds for $\underline{a}(g, f)$).

Lemma 2. *Let $\underline{e}(g, f)$ be any lower-f-elaboration, and s^* its greatest equilibrium strategy profile. Then*

$$s^*(\overline{R}) = \overline{a}(g, f).$$

Proof. By definition of $\overline{a}(g, f)$, there is a lower-f-elaboration $\underline{e}'(g, f)$ of g with a greatest equilibrium strategy profile s^{**} that attains $\overline{a}(g, f)$. Since s^{**} is increasing and the joint action space A is finite, we can identify s^{**} with a finite sequence $z_1, z_2, \ldots, z_k$, with $k \leq \overline{R}$, of jump points, at which players switch to greater action profiles. If players follow the strategy profile s^{**}, a small change in the jump point z_n would influence their expected payoffs compared with s^{**} only at signals in the interval $[z_{n-1}, z_{n+1}]$. Thus the maximum distance between any two adjacent jump points z_n and z_{n+1} must be less than 1. Otherwise, if s^{**} is an equilibrium strategy profile then, for sufficiently small ε, the similarly increasing strategy profile determined by the jump points $z_1, z_2, \ldots, z_{n-1}, z_n - \varepsilon, \ldots, z_k - \varepsilon$ would be an equilibrium strategy profile as well, contradicting the maximality of s^{**}. But if the distance between any two adjacent jump points is less than 1, then $z_k \leq \overline{R}$. Equivalently, $s^{**}(\overline{R}) = \overline{a}(g, f)$. Now we may easily verify that the strategy profile given by the jump points $z_1, z_2, \ldots, z_k$ can also be interpreted as the greatest equilibrium strategy profile s^* of $\underline{e}(g, f)$. ■

The following result is an immediate consequence of our characterisation of the global game selection process in terms of $\overline{a}(g, f)$ and $\underline{a}(g, f)$. In addition to providing a simple approach to compute the global game selection, it also allows us to use the concept with respect to a single game of strategic complementarities, thus making it analytically comparable to other equilibrium refinements.

Theorem 3. *Let $G_v(u, \phi, f)$ be any global game. The global game selection at any state parameter θ depends solely on the noise structure f and on the complete information game $g(\theta)$, and is independent of u and ϕ. More precisely, if s^f is the essentially unique limit strategy profile of $G_v(u, \phi, f)$, and $g = g(\theta^*)$ then $\overline{s}^f(\theta^*) = \overline{a}(g, f)$ and $\underline{s}^f(\theta^*) = \underline{a}(g, f)$.*

The irrelevance of the prior distribution ϕ for the global game selection was already shown by FMP. It may be surprising that the choice of payoff function of the global game embedding is irrelevant as well. After all, the global game selection process is often described as an infection process, starting from the high and low parameter regions. Thus, one might think that if the embedded game g is close to the lower dominance threshold $\underline{\theta}$, this may influence the global game selection so that it selects a lower equilibrium compared to a global game embedding of g in which g is close to $\overline{\theta}$. However, theorem 3 tells us that will not be the case.

Another way to think about theorem 3 is the following. In economic applications, the state parameter θ is typically interpreted as an economic fundamental affecting the decision problem of players. But there may be several economic variables that are candidates for the parameter θ. Theorem 3 tells us that the choice of the fundamental used to perturb the decision problem is irrelevant: the global game selection will be the same. It is determined by the payoff structure of the unperturbed game.

§3.3. The Global Game Selection is Generically Unique

Relative to the set of all games of strategic complementarities, how often does the global game approach give a sharp prediction? To answer this question, we will show that in a mathematically precise sense the global game selection is generically unique.

The set of games with a fixed player set I and fixed action sets $A_{i\in I}$ is naturally identified with Euclidean space $\mathbb{R}^{I\times A}$, endowed with the usual metric. Let $G \subseteq \mathbb{R}^{I\times A}$ be the set of games of strategic complementarities. The set G is closed and forms a proper convex cone[3] in $\mathbb{R}^{I\times A}$. For some fixed noise structure f, let us denote $G^{-f} := \{g \in G \mid \underline{a}(g, f) \neq \overline{a}(g, f)\}$ and let G^f be its complement in G. Then the set G^{-f} is small relative to G^f, both in a measure theoretic sense and in a topological sense.

Proposition 4. *G^{-f} is closed in $\mathbb{R}^{I\times A}$ and of zero Lebesgue measure, while G^f is of infinite measure. Moreover, G^f is open and dense in G, while G^{-f} is nowhere dense in G.*

Thus, intuitively, for any given game of strategic complementarities, it is extremely unlikely that the global game selection is not unique. Proposition 4 is our counterpart to FMP's observation that in any given global game embedding, the global game selection is unique at almost all values of the state parameter θ. Indeed, that must hold as long as changes in θ at least slightly perturb the game whenever the global game selection is not unique, even if assumptions A3 and A4 are not fully satisfied.

§3.4. Global Game Selection in Discontinuous Global Games

While assumptions A3 and A4 are mathematically convenient, from an applied point of view they are not always desirable. Consider the finite player three action game in figure 3. It is an exemplary regime change game, where players' payoffs depend on whether they reach critical mass. Intuitively, there are two ways to perturb the game: one can perturb the payoffs as in lemma 1, or one can perturb the critical threshold by setting $\xi = \theta$. In the latter case, which is often considered in the applied literature, the embedding payoff function u is no longer continuous in θ, and does not satisfy A3.

It does however satisfy the following, weaker form of state monotonicity.

[3]The set G is closed under addition and non-negative scalar multiplication of payoff functions, and has non-empty interior.

		$\alpha < \xi\lvert I\rvert$	$\xi\lvert I\rvert \leq \alpha$
	0	0	0
player i	1	$-\frac{1}{4}$	$\frac{1}{2}$
	2	-1	1

where α is the sum over all players actions, and $\xi \in [0, 2]$.

Figure 3. Finite-player three-action speculative attack game

A3* *Weak state monotonicity:* Greater states make greater actions weakly more appealing. More precisely, for all $\theta > \theta'$ and for all i, $a_i > a'_i$ and a_{-i} we find

$$u_i(a_i, a_{-i}, \theta) - u_i(a'_i, a_{-i}, \theta) \geq u_i(a_i, a_{-i}, \theta') - u_i(a'_i, a_{-i}, \theta').$$

Again, we find that the approach used to perturb the decision problem is irrelevant: the global game selection is determined by the payoff structure of the unperturbed game.

Let $G^*_\nu(u, \phi, f)$ be a *generalised global game*, differing from an ordinary global game in that u satisfies A3* but not necessarily A3 and A4. By standard results on games with strategic complementarities, for each $\nu > 0$ there exists a greatest equilibrium strategy profile $\hat{s}_\nu$ and a least equilibrium profile $\check{s}_\nu$. We define the pointwise limits $\hat{s} = \limsup_{\nu \to 0} \hat{s}_\nu$ and $\check{s} = \liminf_{\nu \to 0} \check{s}_\nu$. For any embedded game $g = g(\theta)$, we call $\check{s}(\theta)$ and $\hat{s}(\theta)$ the *generalised global game selections*.

Proposition 5. *Suppose the generalised global game $G^*_\nu(u, f, \phi)$ is a global game embedding of $g = g(\theta^*)$, such that u is continuous at θ^*. If $a^* = \overline{a}(g, f) = \underline{a}(g, f)$, then there is a unique generalised global game selection at θ^*, that is, $a^* = \hat{s}(\theta^*) = \check{s}(\theta^*)$.*

4. Noise Independence

For a global game $G_\nu(u, \phi, f)$, the embedded game $g(\theta^*)$ is called *noise independent* if the limit strategy profile s^f takes on the same values at θ^* regardless of the choice of f. Theorem 3 says that noise independence is a property of the complete information game g: a game of strategic complementarities is noise independent under one global game embedding if and only if it is noise independent under every other global game embedding.

In this section, we will use the notion of attainability to analyse how the global game selection may depend on the noise structure. For each player i, there must be thresholds $z_i^0, z_i^1, \ldots, z_i^k$ at which she is willing to switch to a greater action, given the action distribution of opponents' actions. The players' thresholds need to be mutually consistent under the noise structure f. This problem takes the simplest form—and may be solvable independent of f—if there are few players or few actions, or if some actions are very appealing for wide range of opposing action distributions.

The simplest non-trivial games with multiple equilibria are two-player, binary-action games. For such 2×2 games it is known that the global game selection is noise independent; it selects the risk dominant equilibrium (Carlsson and Van Damme [3]). In a symmetric 2×2 game this means it selects the best replies to the conjecture that the opponent mixes over both actions with equal probability. If both the least and greatest actions are best replies to this conjecture, then the least is prescribed by the left continuous version of the limit strategy s^f while the right continuous version prescribes the greatest.

But noise independence may fail quickly when the player set or the action sets of players are enlarged beyond size 2. An example of Carlsson [2] shows that noise independence may already fail in an (asymmetric) three-player, binary-action game.[4] Oyama and Takahashi [23] show that symmetric two-player 3×3 games are noise independent, but FMP present an example where noise independence fails in a symmetric two-player 4×4 game. Below, we provide three further results on how many players or actions it takes to violate noise independence. We show that any two-player game in which one player's action space is binary (i.e., every $2 \times n$ game) is noise independent. We will also give two other examples where noise independence fails in an asymmetric two-player 3×3 game and in a symmetric 3-player, 3-action game. This gives a full characterisation of games where noise independence can be established simply by counting the number of players or actions.

A quite restrictive criterion that guarantees noise dependence even in many-player and many-action games is the "p-dominance" criterion.

Definition. Let $p = (p_i)_{i \in I}$ and $\triangle(A_{-i})$ be the set of all probability distributions over A_{-i}. An action profile a^* in g is p-dominant if for each player i and any opposing action distribution $\mu \in \triangle(A_{-i})$ that assigns weight $\mu(a^*_{-i}) \geq p_i$ we find that

$$\forall a_i, \quad \sum_{a_{-i} \in A_{-i}} \mu(a_{-i}) g_i(a^*_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu(a_{-i}) g_i(a_i, a_{-i}),$$

i.e. a^*_i is a best response. □

If a^* is p-dominant for some p with $\sum_{i \in I} p_i < 1$, then a^* is the global game selection (see FMP). For example, it follows that independent of the noise structure, a^* is the global game selection in a two-player symmetric payoff game if it is a best reply on the conjecture that the opponent will play a^*_{-i} with probability less than $\frac{1}{2}$. The concept of attainability in a lower-f-elaboration enables us to see why. For any f we find thresholds $z_i \in [0, 1]$ such that $\mathbb{P}(x_{-i} > z_{-i} | x_i = z_i) = \frac{1}{2}$. Now, consider the strategy profile s^0 where each player i switches from her lowest action to a^*_i at z_i. As each player receiving the signal $x_i = z_i$ assigns a probability of $\frac{1}{2} \geq p_i$ to the event that his opponent plays a^*, p-dominance guarantees that the best reply to s^0 is weakly greater than s^0.

[4]Carlsson's example does not fit well with the usual definition of a global game, since in his setup the players' signals are not conditionally independent, which is required to apply the theory of FMP. In a previous version of our working paper we give a similar example where this assumption is satisfied [1].

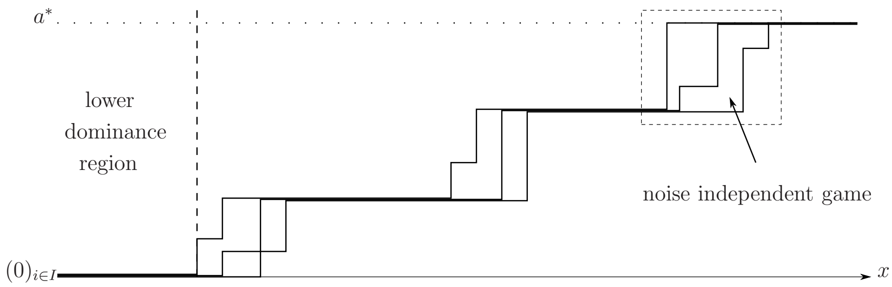

FIGURE 4. Attaining a^* by exploiting noise independence of restricted games.

Thus, an upper-best reply iteration from s^0 converges towards an equilibrium strategy profile s that attains a^*.

However, considering strategy profile s^0, it seems unnecessarily strict to require that players are willing to switch from 0 to a^* immediately. As the notion of attainability implies, it suffices if a^* is reached with some intermediate steps, considerably weakening the criterion.

§4.1. A Decomposition Approach to Noise Independence

Elaborating on this idea, we will show that if there exists a suitable decomposition of a game of strategic complementarities g into smaller, noise independent games, this may be a sufficient condition for noise independence of g itself. Figure 4 illustrates the idea. If we can show that certain action profiles are attained in certain lower-f-elaborations that we obtain by restricting the strategy space of g, we can "patch" these strategy profiles together to obtain a strategy profile for a lower-f-elaboration of the full game g. In this case, simple known criteria for games with small action spaces may prove to be extremely useful to analyse games with bigger action spaces.

Definition. Consider a game of strategic complementarities g with joint action set A. For action profiles $a \le a'$, we define $[a, a'] := \{\tilde{a} \in A \mid a \le \tilde{a} \le a'\}$. The *restricted game* $g\restriction_{[a,a']}$ is defined by the restriction of the payoff functions of g to the set $[a, a']$. We write $a \xrightarrow{g} a'$ if and only if a' is the unique noise independent global game selection in $g\restriction_{[a,a']}$, and conversely $a \xleftarrow{g} a'$ if and only if a is the unique noise independent global game selection in $g\restriction_{[a,a']}$. □

We write $0 \overset{g}{\longmapsto} a^*$ if there exists an increasing sequence $0 < a^1 < \cdots < a^k < a^*$ in A such that $0 \xrightarrow{g} a^1 \xrightarrow{g} \ldots \xrightarrow{g} a^k \xrightarrow{g} a^*$ and $a^* \overset{g}{\longleftarrow\!\shortmid} m$ if there exists an increasing sequence $a^* < a^{k+1} < \cdots < m$ in A such that $a^* \xleftarrow{g} a^{k+1} \xleftarrow{g} \ldots \xleftarrow{g} m$.

Now we may state the following sufficient condition for noise independence.

Theorem 6. *If* $0 \overset{g}{\longmapsto} a^* \overset{g}{\longleftarrow\!\shortmid} m$*, then* a^* *is the unique noise independent global game selection in* g*. More precisely, let* $G_v(u, \phi, f)$ *be any global game embedding of* g*, and* s^f *its essentially* *unique limit strategy profile. Let* $g = g(\theta^*)$*. Then:*

$$\underline{s}^f(\theta^*) = a^* = \overline{s}^f(\theta^*).$$

Theorem 6 reveals a connection with the literature on robustness to incomplete information. Proposition 2.7 and 3.8 in Oyama and Tercieux [24] together imply that if a game with strategic complementarities can be decomposed—as above—into restricted games, each of which has a strict p-dominant equilibrium with sufficiently small p (rather than "just" a unique global game selection), then a^* is the unique equilibrium of g that is "robust to incomplete information". The formal link to our theorem runs via the observation that if an equilibrium is robust to incomplete information, then it is also the unique global game selection (at least generically), so that the conclusion of theorem 6 follows. This link is clarified below (cf. our proposition 8 and corollary 9).

However, theorem 6 allows application of wide range of known criteria for noise independence besides p-dominance, such as the fact that all symmetric 3×3 games, all symmetric n-player binary games, and (as we show shortly) all $2 \times n$ games are noise independent, or indeed the robustness to incomplete information of some equilibrium of the restricted game. None of these are equivalent to the p-dominance criterion—the conditions under which our theorem may be applied are strictly more general. Also, its conclusion does not hinge on the fact that a^* is a robust equilibrium. Thus theorem 6 establishes a more direct and more elementary result about noise independent global game selection.

§4.2. Applications

Consider the global game where payoffs depend on θ as in figure 5. The p-dominance criterion tells us that (c, c) is the unique noise independent selection for $\theta > 3$, as c is a best reply if one expects the opponent to chose c with probability one half. If $\theta < -1$, (a, a) is selected for the same reason, yet we cannot tell which action profile will be chosen if $\theta \in [-1, 3]$ or whether the selection will be noise independent at all. However, by looking at 2×2 restricted games, and applying the risk-dominance criterion, we find that $a \xrightarrow{g(\theta)} b$ and $b \xrightarrow{g(\theta)} c$ for $\theta > 0$ so (c, c) is the unique noise independent selection. If $\theta < 0$, (a, a) is uniquely selected as $c \xrightarrow{g(\theta)} b$ and $b \xrightarrow{g(\theta)} a$.

		player 2		
		a	b	c
player 1	a	$4 - \theta$	$-\theta$	$-4 - \theta$
	b	2	2	0
	c	$\theta - 6$	θ	$2 + \theta$

FIGURE 5. Symmetric two-player three-action game

As indicated, theorem 6 allows the application of more general criteria. For instance, FMP find that symmetric binary-action games are noise independent and give the following simple criterion to determine which action profile will be selected. Let $I = \{1, ..., I\}$, $A_{i\in I} = \{0, 1\}$ and suppose that $g_{i\in I}(a_i, a_{-i})$ depends only on a_i and the number of opponents that play 1 (this is always true if payoffs are symmetric). Furthermore, let Δ^n denote the payoff difference on playing 1 rather than zero if n opponents play 1. Then $(1)_{i\in I}$ is the unique noise independent selection if $\sum_{n=0}^{I-1} \Delta^n > 0$ and $(0)_{i\in I}$ is the uniquely noise independent selection if $\sum_{n=0}^{I-1} \Delta^n < 0$. In other words, the global game approach selects the best reply on the conjecture that the number of opponents using action 1 is uniformly distributed between 0 and $|I| - 1$. Theorem 6 allows us to apply this criterion to games with more than two actions.

		players $-i$					
		(a, a)	(a, b)	(b, b)	(a, c)	(b, c)	(c, c)
	a	2	−1	−2	−2	−3	−4
player i	b	0	0	0	0	0	0
	c	−15	−10	−2	−2	1	2

FIGURE 6. Symmetric three-player game

Consider the three-player symmetric payoff game g in figure 6. As b is a best reply if one expects his opponents to play (a, a), (a, b) or (b, b) with equal probability, we find that b is the unique noise independent selection in $g\restriction_{[a,b]}$, so $a \overset{g}{\to} b$. Analogously we find $b \overset{g}{\to} c$, so $a \overset{g}{\longmapsto} c$. Then theorem 6 implies that (c, c, c) is selected uniquely and noise independently by the global game approach. Note that this result holds irrespective of the payoffs against (a, c). Also, we could set the payoffs of playing c versus (a, a) or (a, b) arbitrarily low without deterring players to play c.

§4.3. Two player, $2 \times n$ action games

Obviously, in order to fruitfully apply theorem 6, we need as many basic conditions as possible that guarantee noise independent selection for restricted games. Just counting the number of players and actions is certainly one of the most simple conditions to check. One possible extension of a two-player 2×2 game is enlarging the action space of just one of the players. We will show that such games are always noise independent. Let g be any game of strategic complementarities with $I = \{1, 2\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, \ldots, m_2\}$. For example, figure 7 shows a 2×4 game. This game does not have a p-dominant action profile and is not concave, so standard results do not guarantee noise independence.[5]

[5]As Oyama and Takahashi [23] show, FMP's "local potential maximiser" condition is sufficient only in concave games.

		player 2			
		0	1	2	3
player 1	0	1, 6	0, 5	0, 4	0, 0
	1	0, 0	0, 3	0, 4	1, 7

FIGURE 7. A 2×4 game with no p-dominant action profile

For any $a_2^* \in A_2$, if $(0, a_2^*) = \overline{a}(g, f)$ for some noise structure, a_2^* is simply player 2's greatest best reply to player 1's action 0, and can be attained from below under any noise structure. Now suppose the action profile $(1, a_2^*) = \overline{a}(g, f)$. Then there exists a lower-f-elaboration $\underline{e}(g, f)$ of g with an equilibrium strategy profile s^* that attains $(1, a_2^*)$, and s^* may be identified with the thresholds z_1^0 and $z_2^0, z_2^1, \dots, z_2^k$, where players switch to greater actions. Without loss of generality, we may assume all thresholds are below $R - \frac{1}{2}$ (we can always increase R, extending the region of the signal space where players play the action profile $(1, a_2^*)$). The opposing action distribution faced by player 1 at $x_1 = z_1^0$ is determined by the probabilities

$$\mathbb{P}(x_2 < z_2^j | x_1 = z_1^0) = \int_{-\infty}^{z_2^j} \pi_1(x_2 | z_1^0) \, \mathrm{d}x_2, \quad j \in \{0, ..., k\}.$$

Now, the opposing action distribution that player 2 faces at each of her thresholds z_2^j is also given by these probabilities, since

$$\begin{aligned} \mathbb{P}(x_1 > z_1^0 | x_2 = z_2^j) &= \mathbb{P}(x_1 - x_2 > z_1^0 - z_2^j | x_2 = z_2^j) \\ &= \mathbb{P}(x_1 - x_2 > z_1^0 - z_2^j | x_1 = z_1^0) \\ &= \mathbb{P}(x_2 < z_2^j | x_1 = z_1^0) =: p_j, \qquad j \in \{0, ..., k\}, \end{aligned}$$

where the second equality follows from the uniform prior distribution of θ.

Let $\underline{z} = \min\{z_1^0, z_2^0, z_2^1, \dots, z_2^k\}$ be the smallest of the thresholds used in the strategy profile s^*, and let i^* be the associated player switching at $\underline{z}$ (that is, $\underline{z} = z_{i^*}^0$). Now, if we consider a different noise structure f', we can construct a similarly increasing strategy profile s^{**} by again putting i^*'s smallest threshold $\tilde{z}_{i^*}^0$ to $\underline{z}$, and then simply rearranging the $k + 1$ remaining thresholds $\{\tilde{z}_1^0, \tilde{z}_2^0, \tilde{z}_2^1, \dots, \tilde{z}_2^k\} - \{\tilde{z}_{i^*}^0\}$ such that the $k + 1$ equations

$$\mathbb{P}(x_2 < \tilde{z}_2^0 | x_1 = \tilde{z}_1^0) = p_j, \quad \text{for } j \in \{0, ..., k\}$$

hold under the new noise structure. In this way, the action distributions of both players at all of their thresholds remains unchanged. Hence, since s^* is an equilibrium strategy in $\underline{e}(g, f)$, it must be the case that s^{**} is an equilibrium strategy in $\underline{e}(g, f')$, and clearly s^{**} attains $(1, a_2^*)$. So $(1, a_2^*)$ is attained from below under f', and we may conclude that $\overline{a}(g, f') \geq \overline{a}(g, f)$. By a symmetric argument, $\overline{a}(g, f) \geq \overline{a}(g, f')$. Thus $\overline{a}(g, f) = \overline{a}(g, f')$, and by duality we may conclude that $\underline{a}(g, f) = \underline{a}(g, f')$. Since f and f' were arbitrary, g is noise independent. This establishes

Proposition 7. *Any $2 \times n$ game of strategic complementarities is noise independent.*

§4.4. Global Games and "Robustness to Incomplete Information"

A lower-f-elaboration $\underline{e}(g, f)$ of g is "close" to g in the sense that conditional payoffs in $\underline{e}(g, f)$ coincide with the payoffs in g with high *ex ante* probability. Kajii and Morris [14] examine incomplete information games that are close to some complete information game g in this sense. Specifically, they look for a Nash equilibrium a^* of g for which, in every incomplete information game sufficiently close to g, there exists an equilibrium strategy profile s in which players use the action profile a^* with high probability. Such equilibria are called *robust to incomplete information*. We will show that if a^* is robust to incomplete information, then a^* is attained both from below and from above under any noise structure f. Thus, we can make use of existing conditions on robustness to incomplete information (see for example Morris and Ui [20]) when trying to determine whether a given game of strategic complementarities is noise independent.

A conceptual problem that we need to overcome in order to connect robustness and noise independence, is that the former is defined in a discrete incomplete information framework.

Definition. A *discrete incomplete information game u* consist of a finite player set I, finite action sets $A_{i \in I}$, a countable probability space Ω and state dependent payoff functions $u_i : A \times \Omega \to \mathbb{R}$. Each player receives a measurable signal $P_i(\omega) = p_i$, where P_i can take on finitely many values and $\mathbb{P}(p_i) > 0$ for each $p_i \in P_i[\Omega]$. Under these assumptions, the conditional probabilities $\mathbb{P}(\cdot|p_i)$ are well defined, so that players have well defined posteriors over the true state ω and their payoff function $u_i(\cdot, \omega)$. □

Let $\triangle(A_i)$ denote the set of all probability measures on A_i. A (mixed) strategy for player i is a function $\sigma_i : P_i[\Omega] \to \triangle(A_i)$. When player i uses the strategy σ_i, the probability that she chooses action a_i given the signal p_i is denoted by $\sigma_i(a_i|p_i)$. A strategy profile $\sigma = (\sigma_i)_i$ is a tuple of mixed strategies. The probability that the action profile $a = (a_i)_{i \in I}$ is played given the strategy profile σ and given ω is denoted by $\sigma(a|\omega)$. The domain of u_i extends to mixed strategies as follows:

$$u_i(\sigma(\omega), \omega) = \sum_{a \in A} u_i(a, \omega)\sigma(a|\omega).$$

A strategy profile is a (Bayes-Nash) equilibrium strategy profile of a discrete incomplete information game u if for all $i \in I$, $p_i \in P_i(\Omega)$ and $a_i \in A_i$

$$\sum_{\omega \in P^{-1}[\{p_i\}]} u_i(\sigma(\omega), \omega)\mathbb{P}(\omega|p_i) \geq \sum_{\omega \in P^{-1}[\{p_i\}]} u_i(a_i, \sigma_{-i}(\omega), \omega)\mathbb{P}(\omega|p_i)$$

i.e. if it is a best reply to use σ_i at the signal p_i against the opposing action distribution induced by σ_{-i}.

Now let g be a complete information game of strategic complementarities. For an incomplete information game u, we define

$$\Omega_g = \{\omega | u_i(\cdot, \omega') = g_i(\cdot) \text{ for all } i \in I, \omega' \in P_i^{-1}(P_i(\omega))\}$$

as the set of states where each player i receives a signal p_i telling her that her payoff function is g_i. A discrete incomplete information game u said to be an ε-elaboration[6] of g if $\mathbb{P}(\Omega_g) \geq 1 - \varepsilon$, following Kajii and Morris [14]. Although in the event Ω_g each player i knows that payoffs are the same as in g, there will still be uncertainty about the signals that opponents receive, and this uncertainty affects players' higher order beliefs.

Definition. A Nash equilibrium a^* of g is said to be *robust to incomplete information* (Kajii and Morris [14]), or more succinctly, a *robust equilibrium* of g, if for every $\gamma > 0$, there exists $\varepsilon > 0$, such that in any ε-elaboration u of g, there exists an equilibrium strategy profile σ such that a^* is played with *ex ante* probability at least $1 - \gamma$, i.e. $\sum_{\omega \in \Omega} \sigma(a^* | \omega) \mathbb{P}(\omega) \geq 1 - \gamma$. □

The next proposition gives the formal link between robustness and noise independence, showing that every robust equilibrium is "sandwiched" between $\underline{a}(g, f)$ and $\overline{a}(g, f)$.

Proposition 8. *Let g be a game of strategic complementarities. If a^* is a robust equilibrium of g, then $\underline{a}(g, f) \leq a^* \leq \overline{a}(g, f)$, for any noise structure f.*

The proposition slightly generalises a similar result by Oury and Tercieux [22], who use a more restrictive notion of robustness to incomplete information. They require that a^* is robust not only in g itself but in all complete information games in a neighbourhood of g, and then exploit a link with so-called "contagious" equilibria to show that their notion implies that a^* is the unique noise independent global game selection. We can connect our theorem to their result as follows:

Corollary 9. *Let $G_\nu(u, \phi, f)$ be any global game and s^f its essentially unique limit strategy profile. Suppose $g(\theta_1) = g$ and $g(\theta_2) = g'$ for some $\theta_1 < \theta_2$. If a^* is a robust equilibrium of both g and g', then it is the unique noise independent global game selection at any state parameter $\theta^* \in (\theta_1, \theta_2)$. More precisely, $\underline{s}^f(\theta^*) = a^* = \overline{s}^f(\theta^*)$ for any noise structure f.*

Proof. Fix f and some $\theta^* \in (\theta_1, \theta_2)$. Since s^f is increasing, we find that $\overline{s}^f(\theta_1) \leq \underline{s}^f(\theta^*) \leq \overline{s}^f(\theta^*) \leq \underline{s}^f(\theta_2)$. From proposition 8 we infer $\underline{s}^f(\theta_2) \leq a^* \leq \overline{s}^f(\theta_1)$. Thus, $\underline{s}^f(\theta^*) = a^* = \overline{s}^f(\theta^*)$. ■

[6] ε-elaborations should not be confused with the notion of lower(upper)-f-elaborations, where f denotes the noise structure instead of the size of the event $\Omega - \Omega_g$.

§4.5. Examples of Noise Dependence

For an example where theorem 6 is of no help, turn to the game in figure 8. It is the two-player four-action counterexample to noise independence that was discovered by FMP. Using the attainability criterion, we find that $b \overset{g}{\longmapsto} a$ and $b \overset{g}{\longmapsto} d$, so we lack a unique focal point. All we can say is that either (a, a) or (d, d) will be selected.

player 1 \ player 2	a	b	c	d
a	2000	1936	1144	391
b	1656	2000	1600	1245
c	1056	1800	2000	1660
d	254	1000	2160	2000

FIGURE 8. FMP's counterexample

However, FMP's counterexample is not the only minimal counterexample to noise independence. We conclude this section with two other minimal examples. These examples show how the idea of attainability may be applied to establish the noise dependence of a game.

Let g be the two-player 3×3 game given by $I = \{1, 2\}$, $A_{i \in I} = \{a, b, c\}$ and payoffs as in figure 9. First suppose η_1 is distributed uniformly over $[-\frac{1}{2}, \frac{1}{2}]$ while η_2 is distributed uniformly

player 1 \ player 2	a	b	c
a	30, 10	−15, 0	−15, −15
b	0, 0	0, 0	0, 0
c	−10, −40	−10, 0	10, 10

FIGURE 9. Asymmetric two-player three-action game

over $[-\frac{1}{10}, \frac{1}{10}]$. Then (a, a) will be a global game selection. To see this, consider the upper-f-elaboration of g, in which the action c is dominant for strictly positive signals. The state parameter θ is distributed uniformly over some interval $[L, \frac{1}{2}]$, and given a signal $x_i \in [L + \frac{1}{2}, 0]$ the conditional distribution that each player holds over his opponent's signal is given by the density

$$\pi_i(x_{-i}|x_i) = \begin{cases} 1, & 0 \le |x_{-i} - x_i| \le 0.4, \\ 3 - 5\,|x_{-i} - x_i|\,, & 0.4 < |x_{-i} - x_i| \le 0.6. \end{cases}$$

Now, consider the strategy profile in which player 1 switches to action b at $z_1^b = 0$ and to the lowest action a at $z_1^a = -\frac{1}{6}$—see figure 10. Player 2 switches at thresholds $z_2^b = 0$ and $z_2^a = -\frac{1}{3}$.

If player 1 receives signal $x_1 = 0$, she assigns probabilities $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$ to the events that his opponent chooses actions a, b and c respectively. Her expected payoff of playing c is 0 and she is willing to switch from action c to action b. Similarly, we find that $\mathbb{P}(x_2 \le z_2^a | x_1 = z_1^a) = \mathbb{P}(z_2^a <$

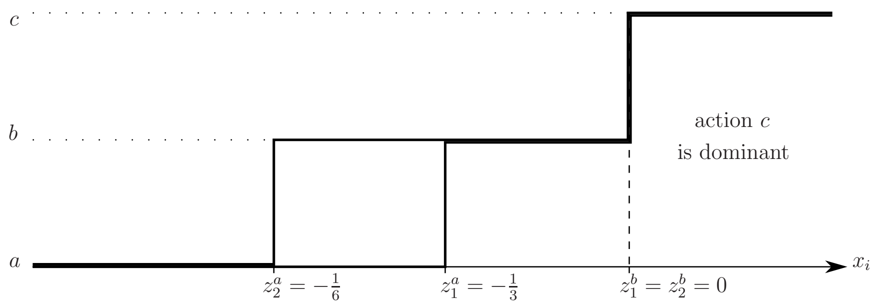

FIGURE 10. (a, a) is attained from above under f in $\overline{e}(g, f)$.

$x_2 \leq z_2^b | x_1 = z_1^a) = \mathbb{P}(x_2 > z_2^b | x_1 = z_1^a) = \frac{1}{3}$ and conclude that player 1 is willing to switch to the lowest action a given a signal $x_1 = z_1^a$.

For player 2, we find that $\mathbb{P}(x_1 \leq z_1^a | x_2 = z_2^b) = \frac{1}{3}$ and $\mathbb{P}(x_1 > z_1^b | x_2 = z_2^b) = \frac{1}{2}$, implying that the expected payoff of playing c is zero at z_2^b, hence equal to the payoff of playing b. Finally, we find that $\mathbb{P}(x_1 \leq z_1^a | x_2 = z_2^a) = \frac{2}{3}$ and $\mathbb{P}(x_1 > z_1^b | x_2 = z_2^a) = \frac{1}{6}$, implying that the expected payoff of playing a given the signal $x_2 = t_2^a$ is zero and equal to the payoff of playing b. Thus there exists an equilibrium profile in the upper-f-elaboration of g in which the action profile (a, a) is used, so the equilibrium (a, a) of g is selected by the left continuous version of the limit strategy profile.

However, if η_1 follows the density $f_1(x) = 1 - 2x$ with support $[-\frac{1}{2}, \frac{1}{2}]$, while η_2 is distributed uniformly over $[-\varepsilon, \varepsilon]$ the global game approach uniquely selects (c, c) for sufficiently small ε. Consider a lower-f-elaboration of g and a strategy profile specified by the four thresholds z_i^b, z_i^c $(i \in \{1, 2\})$ at which the players switch to greater actions. Let $\Delta_i^{a,b}(x_i)$ denote player i's difference in payoff of playing b rather than a when given a signal x_i, and define $\Delta_i^{b,c}(x_i)$ analogously. We first examine the limit $\varepsilon = 0$, so that player 2 is informed about the true state θ. We find that for $z_1^b = 0$, $z_1^c = 0.15$, $z_2^b = 0.07$, $z_2^c = 0.34$

$$\Delta_1^{a,b}(0) = \frac{759}{2000}, \qquad \Delta_1^{b,c}(0.15) = \frac{239}{500}, \qquad \Delta_2^{a,b}(0.07) = \frac{61}{200}, \qquad \Delta_2^{b,c}(0.34) = \frac{69}{200},$$

so players strictly prefer to switch to greater actions at the thresholds and (conducting an upper-best reply iteration) we see that (c, c) is attained from below under this noise structure. Indeed, since players strictly prefer to switch at the thresholds, the same must be true if we draw x_2 from $[\theta - \varepsilon, \theta + \varepsilon]$ and choose ε very small—this only slightly perturbs expected payoffs. And even as we very slightly perturb the payoffs of the game g, the expected payoff differences at the thresholds remain positive. Thus (c, c) is a global game selection for all games in some neighbourhood of the game g. This implies that, for any global game embedding of g, (c, c) is the unique noise independent global game selection for both the left and right continuous versions of the limit strategy profile s^f.

		players $-i$					
		(a,a)	(a,b)	(b,b)	(a,c)	(b,c)	(c,c)
	a	1	1	1	0	−1	−1
player i	b	0	0	0	0	0	0
	c	−1	−1	−1	−1	2	2

FIGURE 11. Symmetric three-player game

Finally, let us consider the symmetric three-player, three-action game g specified in figure 11. If all agents noise terms η_i are distributed identically, then (a, a) will be part of the global game selection. To see this, consider the upper-f-elaboration of g and the strategy profile s where all players choose to play a for negative signals and the dominant action c for positive signals. At $x_i = 0$, agents hold a Laplacian belief about opponents actions, i.e. they expect to face (a, a), (a, c), (c, c) with probability $\frac{1}{3}$. On that belief, each action yields an expected payoff of 0, and agents are indeed willing to switch at the threshold. That is to say, s is an equilibrium strategy profile in $\overline{e}(g, f)$ and attains (a, a, a) from above. Thus, for any global game embedding with $g = g(\theta^*)$ we find $\underline{s}^f(\theta^*) = (a, a, a)$.

However, if η_1 and η_2 are both uniformly distributed over $[-\frac{1}{2}, \frac{1}{2}]$ while the third agent receives a very precise signal, the global game approach uniquely selects (c, c, c). Consider a lower-f-elaboration of g and the strategy profile s where agents 1 and 2 switch from a to c on receiving a signal $x_i = 0$. The third agent switches to action b at the threshold $t_b = 0.01$ and to action c at $t_c = 0.1$. Let $\Delta_i^{a,b}(x_i)$ denote player i's difference in payoff of playing b rather than a when given a signal x_i, and define $\Delta_i^{b,c}(x_i)$ and $\Delta_i^{a,c}(x_i)$ analogously. We first examine the degenerate case $x_3 = \theta$, where player 3 is informed about the true state. We find that for $i \in \{1, 2\}$

$$\Delta_i^{a,c}(0) = \frac{299}{5000}, \qquad \Delta_i^{b,c}(0) = \frac{2197}{20000}, \qquad \Delta_3^{a,b}(0.01) = \frac{1}{50}, \qquad \Delta_3^{b,c}(0.1) = \frac{2}{25},$$

so agents strictly prefer to switch to greater actions at the thresholds and (using an upper-best reply iteration) we see that (c, c, c) is attained from below under this noise structure. Indeed, since players strictly prefer to switch at the thresholds, the same must be true if we draw x_3 from $[\theta - \varepsilon, \theta + \varepsilon]$ and choose ε very small—this only slightly perturbs expected payoffs. And even as we very slightly perturb the payoffs of the game g, the expected payoff differences at the thresholds remain positive. Thus (c, c, c) is a global game selection for all games in some neighbourhood of the game g. This implies that, in any global game embedding of g, (c, c, c) is the unique noise independent global game selection for both the left and right continuous versions of the limit strategy profile s^f.

Symmetric Games			
actions:	2 each	3 each	4 each
2 players	✓[a]	✓[b]	×[c]
3 players	✓[c]	×[*]	
n players	✓[c]		

Asymmetric Games			
actions:	2 each	$2 \times n$	3 each
2 players	✓[a]	✓[*]	×[*]
3 players	×[d]	n/a	
n players	×[e]	n/a	

✓ Noise independent × Counterexample to noise independence exists.
For empty cells noise dependence follows from a counterexample in a smaller game.
[*]This paper, Section 4. [a]Carlsson and Van Damme [3]. [b]Oyama and Takahashi [23].
[c]Frankel, Morris and Pauzner [9]. [d]Carlsson [2]. [e]Corsetti et al. [4]

TABLE 1. Noise (in)dependence

5. Conclusion

The theory of global games is a powerful tool for equilibrium selection in games of strategic complementarities with multiple equilibria. It is sometimes interpreted as a modelling device for the effects of strategic uncertainty on the stability of equilibria. Experiments support this view and indicate that, even though real players may deviate from a global game selection towards more efficient action profiles, comparative statics with respect to the parameters of the payoff function work precisely as predicted by the theory. In spite of these favourable properties, the theory of global games has almost exclusively been applied to binary-action games. The two main reasons that kept researchers from applying it to models with more than two actions seem to be (i) the tedious process required for deriving a global game selection and (ii) a lack of simple conditions under which the global game selection is independent of the chosen global game.

Characterising the global game selection by attainable actions (theorem 3) simplifies the process of deriving it. It also shows that out of the three elements that define a global game embedding (the prior distribution, the generalised payoff function, and the conditional distribution of private signals) only the signals' distribution affects the global game selection. Generically, it does not matter if the generalised payoff function is continuous or not. If the number of players and actions is sufficiently small, the signal's distribution does not affect the global game either. In this paper, we have completed the characterisation of cases where noise independence can be judged simply by looking at the number of players and actions. Table 1 provides a summary of these results.

From an applied perspective, the most powerful result in this paper is theorem 6. It implies that the global game selection may be derived by decomposing a many-action game into smaller games, for which existing heuristics and noise independence results can be applied. For instance, in symmetric binary-action games, the noise independent selection is simply the best response to a uniform distribution of the share of other players choosing one or the other action. With our results, this rule can be applied to any game with finitely many actions to find the unique noise independent selection, provided that pair-wise comparisons of neighbouring action profiles all point towards the same equilibrium.

Simplified conditions for noise independence and a manageable heuristic to derive the global game selection in games with finitely many actions make it easier to apply the theory of global games. That should facilitate research on topics where strategic complementarities are crucial.

Appendix

Proof of Theorem 3

The theorem is an immediate consequence of the following three lemmas.

Lemma 10. *Let $G_v(u,\phi,f)$ be any global game embedding of g and s^f its essentially unique limit strategy profile. Let $g = g(\theta^*)$. We have:*

$$\overline{s}^f(\theta^*) \geq \overline{a}(g,f), \text{ and dually, } \underline{s}^f(\theta^*) \leq \underline{a}(g,f).$$

Proof. We prove the first inequality. Without loss of generality assume that $\theta^* = 0$. Also, for the moment, fix some $v \in (0,1)$.

Consider the simplified global game $G_v(u,f)$. We will want to assume that its random state parameter θ is distributed over an interval $[L,R]$ where $L \leq \min\{\underline{\theta}-1, -\frac{1}{2}\}$ and $R \geq \max\{\overline{\theta}+1, \overline{R}\}$. This will allow us to compare the simplified global game with a lower-f-elaboration of g later in the proof. For $G_v(u,f)$, this might mean extending the range over which θ is distributed further into the dominance regions. But that will not change its essentially unique equilibrium profile s_v^f, other than enlarging the range in which dominant actions are prescribed. So this is without loss of generality.

Consider the lower-f-elaboration $\underline{e}(g,f)$, with θ distributed on $[\frac{1}{v}L, \frac{1}{v}R]$. Its greatest equilibrium strategy profile s^* is increasing and satisfies $s^*(x) = \overline{a}(g,f)$ for $x \geq \overline{R}$. Define the profile s_v:

$$\text{for all i} \in I, \quad s_{v,i}(x) := \begin{cases} 0 & \text{if } x < 0, \\ s_i^*(x/v) & \text{if } 0 \leq x \leq v\overline{R}, \\ \overline{a}(g,f) & \text{if } v\overline{R} < x. \end{cases}$$

We will compare the simplified global game with a “compressed” version of $\underline{e}(g,f)$, where all individual noise variables η_i have been scaled by the factor v, and θ is distributed uniformly on $[L,R]$. We denote this compressed lower-f-elaboration of g by $\underline{e}_v(g,f)$. Compressing the elaboration amounts merely to a relabelling of signals. Therefore, the restriction of s_v to the signal space of $\underline{e}_v(g,f)$ is an equilibrium strategy profile of $\underline{e}_v(g,f)$. Note also that in the games

$G_v(u, f)$ and $\underline{e}_v(g, f)$ the distributions of opponents' signals conditional on a player's own signal are identical.

Now suppose that players follow the strategy profile s_v in the simplified global game $G_v(u, f)$. For any player i, and any signal $x_i < 0$, 0 is a dominant action in $\underline{e}_v(g, f)$, so that $\beta(s_v)_i(x_i) = 0 = s_{v,i}(x_i)$ in $G_v(u, f)$. For $x_i \geq 0, \beta(s_v)_i(x_i) \geq s_{v,i}(x_i)$, since $s_{v,i}(x_i)$ is the best reply to $s_{v,i}$ under the payoff function $u(\cdot, 0) = g$, and hence the greatest best reply under the payoff function $u(\cdot, x_i)$ must be weakly greater than by assumption A3.

In sum, $\beta(s_v) \geq s_v$, so an upper-best reply iteration starting at s_v yields a monotonically increasing sequence of strategy profiles that converges to the essentially unique, increasing equilibrium strategy profile s_v^f of $G_v(u, f)$. Thus,

$$s_v^f(v\overline{R}) \geq s_v(v\overline{R}) = s^*(\overline{R}) = \overline{a}(g, f).$$

Since s_v^f is increasing, it follows that for all $x \geq v\overline{R}$, $s_v^f(x) \geq \overline{a}(g, f)$.

Since the choice of v was arbitrary, the above argument shows that for all $\varepsilon > 0$, there is $\overline{v} > 0$ such that $s_v^f(\varepsilon) \geq s_v^f(v\overline{R}) \geq \overline{a}(g, f)$ for all $v \leq \overline{v}$ (just take $\overline{v} = \varepsilon/\overline{R}$). Hence:

$$\forall \varepsilon > 0 : \lim_{v \searrow 0} s_v^f(\varepsilon) \geq \overline{a}(g, f)$$

implying

$$\overline{s}^f(0) = \lim_{\varepsilon \searrow 0}(\lim_{v \searrow 0} \overline{s}_v^f(\varepsilon)) \geq \overline{a}(g, f)$$

as claimed. ∎

The next two lemmas establish that the converse of lemma 10 also holds.

Lemma 11. *Let $G_v(u, \phi, f)$ be any global game embedding of g and s^f its essentially unique limit strategy profile. Let $g = g(\theta^*)$, and assume that s^f is continuous at θ^*. Then we have:*

$$\overline{s}^f(\theta^*) \leq \overline{a}(g, f), \text{ and dually, } \underline{s}^f(\theta^*) \geq \underline{a}(g, f).$$

Proof. We will prove the first inequality. Consider again the simplified global game $G_v(u, f)$. Without loss of generality we will assume that $\underline{\theta} = 0$. Since the joint action space is finite, continuity of s^f at θ^* implies that, for some $\delta > 0$, $\overline{s}^f$ is constant, and equal to some $a^* \in A$, on the interval $[\theta^* - \delta, \theta^* + \delta]$. Since $\overline{s}_v^f$, the right continuous equilibrium strategy profile of the simplified global game $G_v(u, f)$, converges towards $\overline{s}^f$ in horizontal distance, there must be $\overline{v} > 0$ such that for $v < \overline{v}$, s^f equals $\overline{s}_v^f$ on the sub-interval $[\theta^* - \delta/2, \theta^* + \delta/2]$.

Fix some $v^* < \min\{\delta/2, \overline{v}\}$. Consider the "compressed" lower-f-elaboration $\underline{e}_{v^*}(g, f)$, where all individual noise variables η_i have been scaled by the factor v^*, and θ is distributed uniformly on the interval $[-\frac{1}{2}, R]$, with R the same as in the simplified global game. Assume that in this game players use the strategy profile s given by:

$$\text{for all } i \in I, \quad s_i(x) = \begin{cases} \overline{s}^f_{\nu^*,i}(x_i) & \text{if } x_i \leq \theta^*, \\ \overline{s}^f_{\nu^*,i}(\theta^*) & \text{if } x_i > \theta^*. \end{cases}$$

For any player i, and any signal $x_i < 0$, 0 is a dominant action for i both in $\underline{e}_{\nu^*}(g, f)$ and in $G_{\nu^*}(u, f)$. So in the game $\underline{e}_{\nu^*}(g, f)$, we have

$$\text{for } x_i < 0, \quad \beta(s)_i(x_i) = 0 = \overline{s}^f_{\nu^*,i}(x_i) = s_i(x_i).$$

For $x_i \in [0, \theta^*]$, player i's opponents receive signals smaller than $\theta^* + \frac{\delta}{2}$ and behave as if they were following $\overline{s}^f_{\nu^*,i}(x_i)$. Since the distributions of their signals are identical in $G_{\nu^*}(u, f)$ and $\underline{e}_{\nu^*}(g, f)$, but i's payoff function is given by $u_i(\cdot, \theta^*)$ in $\underline{e}_{\nu^*}(g, f)$ and by $u_i(\cdot, x_i)$ in $G_{\nu^*}(u, f)$, in the game $\underline{e}_{\nu^*}(g, f)$ we have

$$\text{for } 0 \leq x_i \leq \theta^*, \quad \beta(s)_i(x_i) = \beta(\overline{s}^f_{\nu^*})_i(x_i) \geq \overline{s}^f_{\nu^*,i}(x_i) = s_i(x_i),$$

where the inequality follows from the state monotonicity assumption A3 and the fact that $\overline{s}^f_{\nu^*}$ is the greatest equilibrium profile of $G_{\nu^*}(u, f)$.

For $x_i > \theta^*$, player i's opponents receive signals greater than $\theta^* - \frac{\delta}{2}$. Since for such signals s is constant and equal to $\overline{s}^f_{\nu^*}(\theta^*) = a^*$, player i faces action profile a^*_{-i}. Moreover, a^* is a Nash equilibrium under the payoff functions of the game g. This means that in $\underline{e}_{\nu^*}(g, f)$ we have

$$\text{for } x_i > \theta^*, \quad \beta(s)_i(x_i) \geq a^*_i = \overline{s}^f_{\nu^*,i}(x_i) = s_i(x_i).$$

In sum, $\beta(s) \geq s$. Therefore an upper-best reply iteration in the game $\underline{e}_{\nu^*}(g, f)$ starting from s yields a monotonically increasing sequence of strategy profiles that converge to an equilibrium profile $s^* \geq s$. It follows that $s^*(\theta^*) \geq s(\theta^*) = \overline{s}^f(\theta^*)$. Since compressing an elaboration amounts to a relabelling of signals, in the equivalent uncompressed lower-f-elaboration there is an equilibrium strategy profile s^{**} such that $s^{**}(\theta^*/\nu^*) \geq \overline{s}^f(\theta^*)$, Thus $\overline{s}^f(\theta^*)$ is attained from below under f, implying the inequality $\overline{s}^f(\theta^*) \leq \overline{a}(g, f)$. ∎

Lemma 12. *Let $G_\nu(u, \phi, f)$ be any global game embedding of g and s^f its essentially unique limit strategy profile. Let $g = g(\theta^*)$, and assume that s^f is not continuous at θ^*. Then we have:*

$$\overline{s}^f(\theta^*) \leq \overline{a}(g, f), \text{ and dually, } \underline{s}^f(\theta^*) \geq \underline{a}(g, f).$$

Proof. We will again prove the first inequality. Let $\{\theta_n\}_{n\in\mathbb{N}}$ be a sequence that converges to θ^* from above. By lemmas 11 and lemma 2, for any θ_n, the greatest equilibrium strategy profile s^{θ_n} of each lower-f-elaboration $\underline{e}(g(\theta_n), f)$ attains $\overline{s}^f(\theta_n) \geq \overline{s}^f(\theta^*)$ at $\overline{R}$, and hence attains $\overline{s}^f(\theta^*)$ at $\overline{R}$.

By definition, for θ^n we have:

$$\forall i \forall x_i \forall a_i, \quad \int_{\mathbb{R}^{|I|-1}} \left(u_i(s_i^{\theta_n}(x_i), s_{-i}^{\theta_n}(x_{-i}), \theta^n) - u_i(a_i, s_{-i}^{\theta_n}(x_{-i}), \theta^n)\right) \pi_i(x_{-i}|x_i) \ \mathrm{d}x_{-i} \geq 0.$$

Moreover, by the state monotonicity assumption (A3), the sequence of profiles s^{θ_n} converges to $s^* = \inf\{s^{\theta_n} | n \in \mathbb{N}\}$ in monotonically decreasing fashion. By the dominated convergence theorem we find:

$$\begin{aligned}
\forall i \forall x_i \forall a_i, \quad & \int_{\mathbb{R}^{|I|-1}} (u_i(s_i^*(x_i), s_{-i}^*(x_{-i}), \theta^*) - u_i(a_i, s_{-i}^*(x_{-i}), \theta^*)) \, \pi_i(x_{-i}|x_i) \ \mathrm{d}x_{-i} \\
&= \int_{\mathbb{R}^{|I|-1}} \lim_{n\to\infty} \left(u_i(s_i^{\theta_n}(x_i), s_{-i}^{\theta_n}(x_{-i}), \theta_n) - u_i(a_i, s_{-i}^{\theta_n}(x_{-i}), \theta^n)\right) \pi_i(x_{-i}|x_i) \ \mathrm{d}x_{-i} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^{|I|-1}} \left(u_i(s_i^{\theta_n}(x_i), s_{-i}^{\theta_n}(x_{-i}), \theta^n) - u_i(a_i, s_{-i}^{\theta_n}(x_{-i}), \theta^n)\right) \pi_i(x_{-i}|x_i) \ \mathrm{d}x_{-i} \geq 0.
\end{aligned}$$

So s^* is an equilibrium strategy profile in $\underline{e}(g(\theta^*), f)$. Clearly s^* attains $\overline{s}^f(\theta^*)$ at $\overline{R}$. ■

Proof of Proposition 4

As in lemma 1, for any game $g \in G$ we define a global game embedding u^g by setting

$$u_i^g(a, \theta) := g_i(a) + \theta a_i \quad \text{for } i \in I, a \in A.$$

To prove that G^f is dense in G we need to show that whenever $\overline{a}(g, f) \neq \underline{a}(g, f)$ there is a game $g' \in G^f$ arbitrarily close to g. Suppose $\overline{a}(g, f) \neq \underline{a}(g, f)$. Since the joint action space is finite, there is $\varepsilon > 0$ such that in the global game embedding u^g we have $\underline{s}^f(\theta) = \overline{s}^f(\theta)$ for $\theta \in [-\varepsilon, 0) \cup (0, \varepsilon]$, i.e., the global game selection is unique. By theorem 3, if the global game selection is unique at θ, then $\overline{a}(u^g(\theta), f) = \underline{a}(u^g(\theta), f)$, so $u^g(\theta) \in G^f$, as required.

Next we will show that G^{-f} is closed, so that G^{-f} is closed in G and (hence) nowhere dense,[7] and G^f is open in G. For $r > 0$, let $B_r(g)$ denote the open ball in $\mathbb{R}^{A\times I}$ with radius r around g. First we prove:

Claim. If, for some g and $\varepsilon > 0$, we have $u^g(\theta) \in G^f$ for all $\theta \in [-\varepsilon, \varepsilon]$, then $g' \notin G^{-f}$ for any $g' \in B_{\frac{\varepsilon}{2}}(g)$.

Proof of Claim. By theorem 3, we find $a^* = \overline{a}(u^g(\theta), f) = \underline{a}(u^g(\theta), f)$ for all $\theta \in [-\varepsilon, \varepsilon]$. Now let $g' \in B_{\frac{\varepsilon}{2}}(g)$, and without loss of generality suppose $g' \in G$.[8] For all i, a_{-i}, and $a_i' \leq a_i$ we find

$$\begin{aligned}
u_i^g(a_i, a_{-i}, -\varepsilon) - u_i^g(a_i', a_{-i}, -\varepsilon) &= g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) - \varepsilon(a_i - a_i') \\
&\leq g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) - \varepsilon \\
&\leq g_i'(a_i, a_{-i}) - g_i'(a_i', a_{-i}).
\end{aligned}$$

[7] A set is nowhere dense if the complement of its closure is dense.

[8] If $g' \notin G$, then $g' \notin G^{-f}$ is trivially fulfilled.

For any opposing action distribution, this implies that the upper best reply under g' is weakly greater than under $u^g(-\varepsilon)$. Thus, the greatest equilibrium profile of $\underline{e}(g', f)$, derived in an upper best reply iteration, is weakly greater than the greatest equilibrium profile of $\underline{e}(u^g(-\varepsilon), f)$. So $a^* = \overline{a}(u^g(-\varepsilon), f) \leq \overline{a}(g', f)$. Using a symmetric argument, we establish that $\overline{a}(g', f) \leq \overline{a}(u^g(\varepsilon), f) = a^*$. In the same way, we show that $\underline{a}(g', f) = a^*$, proving our claim.

Now consider any $g \in \mathbb{R}^{A \times I} - G^{-f}$. If $g \notin G$, then one of the inequalities in (1) is violated, and that holds in an open ball around g. If $g \in G$, then we consider the global game embedding u^g. Surely $g \in G^f$, so by theorem 3 the limit strategy profile s^f is continuous at $\theta = 0$. Hence there is $\varepsilon > 0$ such that $u^g(\theta) \in G^f$ for all $\theta \in [-\varepsilon, \varepsilon]$. By our claim $B_{\frac{\varepsilon}{2}}(g) \subseteq \mathbb{R}^{A \times I} - G^{-f}$. Thus $\mathbb{R}^{A \times I} - G^{-f}$ is open, and G^{-f} is closed.

Finally, we establish genericity in a measure theoretic sense. A subset P of $\mathbb{R}^{A \times I}$ is called *porous* if there are $\lambda \in (0, 1)$ and $k > 0$ such that for any $g \in P$ and $\varepsilon \in (0, k)$, there exists some $y \in \mathbb{R}^{A \times I}$ such that $B_{\lambda\varepsilon}(y) \subseteq B_\varepsilon(g) - P$. Any porous subset of $\mathbb{R}^{A \times I}$ is a Lebesgue null set (see [15], p. 220–222). Define:

$$G_k^{-f} := \{g \in G \mid \overline{a}(g, f) \neq \underline{a}(g, f) \text{ and } u^g(\theta) \in G^f, \forall \theta \in [-k, 0) \cup (0, k]\}.$$

We will prove that G_k^{-f} is porous. Assume $g \in G_k^{-f}$ and choose $\varepsilon \in (0, k)$. Setting $g' := u^g(\frac{\varepsilon}{2})$, we know that $\{u^{g'}(\theta) \in G \mid \theta \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})\} \subseteq G^f$. Then our claim implies that $B_{\frac{\varepsilon}{4}}(g') \cap G_k^{-f} = \emptyset$. Setting $\lambda = \frac{1}{4}$, we have for $\varepsilon \in (0, k)$ that $B_{\lambda\varepsilon}(g') \subseteq B_\varepsilon(g) - G_k^{-f}$, i.e., G_k^{-f} is porous. This means that $G^{-f} = \bigcup_{\{k \in \mathbb{Q} \mid k > 0\}} G_k^{-f}$ is a countable union of Lebesgue null sets, so that G^{-f} is a null set itself. To see that, in contrast, G is of infinite Lebesgue measure, note that its interior is non-empty. In addition, for each ball B contained in G, we find another ball of greater measure, disjoint from B, by multiplying the payoffs of games in B with a sufficiently large constant. Thus G must be of infinite measure. ■

Proof of Proposition 5

Note that $\hat{s} \geq \check{s}$, so it suffices to show that $\check{s}(\theta^*) \geq a^* \geq \hat{s}(\theta^*)$. We will prove the first inequality; the second follows by duality. To do so, we will compare u to a payoff function u' that satisfies assumptions A1–A4 so that theorem 3 can be applied.

We say that g *strictly favours higher actions* than g' and write $g' \prec g$ if for all i, $a_i > a_i'$ and a_{-i} we find

$$g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) > g_i'(a_i, a_{-i}) - g_i'(a_i', a_{-i}).$$

Note that, given any opposing action distribution, the upper best reply under g will be weakly greater than the upper best reply under g'.

Write $\check{\theta}$ for the lowest possible state in the global game. Without loss of generality we may assume that $g(\check{\theta}) \prec g(\theta^*) \prec g(\overline{\theta})$.[9] Consider the game of strategic complementarities $g'_i(a_i, a_{-i}) = u_i(a_i, a_{-i}, \theta^*) - ka_i$. Choose k small enough, such that $a^* = \overline{a}(g', f) = \underline{a}(g', f)$ (G^f is open by proposition 4) and $g(\check{\theta}) \prec g'$. Also note that $g' \prec g(\theta^*)$. And since u is continuous in θ^*, there is some nearby game $g' \prec g(\theta^* - \varepsilon)$. Now, let us construct u':

$$u'_i(a_i, a_{-i}, \theta) = \begin{cases} u_i(a_i, a_{-i}, \check{\theta}) & \text{if } \theta < \theta^* - \varepsilon, \\ \frac{\theta^* - \theta}{\varepsilon} u_i(a_i, a_{-i}, \check{\theta}) + \frac{\theta - (\theta^* - \varepsilon)}{\varepsilon} g'_i(a_i, a_{-i}) & \text{if } \theta^* - \varepsilon \leq \theta < \theta^*, \\ \frac{\overline{\theta} - \theta}{\overline{\theta} - \theta^*} g'_i(a_i, a_{-i}) + \frac{\theta - \theta^*}{\overline{\theta} - \theta^*} u_i(a_i, a_{-i}, \theta^*) & \text{if } \theta^* < \theta < \overline{\theta}, \\ (\theta - (\overline{\theta} - 1)) u_i(a_i, a_{-i}, \theta^*) + (\theta - \overline{\theta}) u_i(a_i, a_{-i}, \overline{\theta}) & \text{if } \overline{\theta} \leq \theta < \overline{\theta} + 1, \\ u_i(a_i, a_{-i}, \overline{\theta}) & \text{if } \overline{\theta} + 1 \leq \theta, \end{cases}$$

Comparing u and u', we see that the dominance regions have been shifted to the right, $g(\check{\theta})$ is now embedded at $\theta^* - \varepsilon$, g' at θ^*, $g(\theta^*)$ at $\overline{\theta}$, $g(\overline{\theta})$ at $\overline{\theta} + 1$ and the remaining games are linear interpolations. Thus, u strictly favours higher actions than u' at each θ. Also, as payoffs are linearly interpolated between $g(\check{\theta}) \prec g' \prec g(\theta^*) \prec g(\overline{\theta})$, u' satisfies A3. Clearly, A4 is satisfied as well.

Finally, consider the global game $G_\nu(u', \phi, f)$. For any $\nu > 0$ there exists a lowest equilibrium strategy profile, denoted $\tilde{s}_\nu$. As u strictly favours higher actions than u', for the lowest equilibrium strategy profiles derived in an upper best reply iteration we find $\check{s}_\nu \geq \tilde{s}_\nu$. Thus,

$$\check{s}(\theta^*) = \liminf_{\nu \to 0} \check{s}_\nu(\theta^*) \geq \lim_{\nu \to 0} \tilde{s}_\nu(\theta^*) = a^*,$$

where the last equality follows by theorem 3 and the fact that $a^* = \overline{a}(g', f) = \underline{a}(g', f)$. ■

Proof of Theorem 6

In view of theorem 3, it suffices to prove that for any noise structure f, $\overline{a}(g, f) = a^* = \underline{a}(g, f)$. We will prove the first equality, the second follows by duality.

Fix some arbitrary noise structure f. By definition, there exists an increasing sequence $0 < a^1 < a^2 < \cdots < a^k < a^*$, such that for each adjacent pair a, a' in the sequence, the unique noise independent selection in the restricted game $g\restriction_{[a,a']}$ is a'. Consequently, for each adjacent pair a, a' there is a corresponding lower-f-elaboration $\underline{e}(g\restriction_{[a,a']}, f)$ with a corresponding equilibrium strategy profile s that attains a'.

We claim that if s^1 attains a^1 in $\underline{e}(g\restriction_{[0,a^1]}, f)$ and s^2 attains a^2 in $\underline{e}(g\restriction_{[a^1,a^2]}, f)$, then there is a lower-$f$-elaboration $\underline{e}(g\restriction_{[0,a^2]}, f)$ of $g\restriction_{[0,a^2]}$ with a corresponding equilibrium strategy profile

[9] We can choose $\underline{\theta}$ and $\overline{\theta}$ in the interior of the dominance regions so that these extend to states $\underline{\theta} + \varepsilon$ and $\overline{\theta} - \varepsilon$. But then, for noise $\nu < \frac{\varepsilon}{2}$ the exact payoff structure at $\theta \leq \underline{\theta}$ and $\theta \geq \overline{\theta}$ is irrelevant for the equilibrium strategy profiles in $G^*_\nu(u, \phi, f)$.

s^* such that s^* attains a^2. Consequently, there is a strictly *shorter* increasing sequence $0 < a^2 < \cdots < a^k < a^*$ with the property that for each adjacent pair a, a' in the sequence there is a corresponding lower elaboration $\underline{e}(g\restriction_{[a,a']}, f)$ and a corresponding equilibrium strategy profile s that attains a'. By induction, it follows that a^* is attained from below under f in $g\restriction_{[0,a^*]}$.

To prove the claim, assume that in $\underline{e}(g\restriction_{[a^0,a^1]}, f)$, θ is distributed on the interval $[-\frac{1}{2}, R^1]$ and that in $\underline{e}(g\restriction_{[a^1,a^2]}, f)$ θ is distributed on $[-\frac{1}{2}, R^2]$. Now consider the lower-f-elaboration $\underline{e}(g\restriction_{[0,a^2]}, f)$ such that θ takes values in the interval $[-\frac{1}{2}, R^1 + 1 + R^2]$, and consider the strategy profile defined by:

$$s_i(x_i) = \begin{cases} s_i^1(x_i) & \text{if } x \leq R^1, \\ a_i^1 & \text{if } R^1 < x < R^1 + 1, \\ s_i^2(x_i - (R^1 + 1)) & \text{if } R^1 + 1 \leq x. \end{cases}$$

For $x_i \leq R^1$, the opposing action distribution in $\underline{e}(g\restriction_{[a^0,a^2]}, f)$ conditional on x_i is just like that in $\underline{e}(g\restriction_{[0,a^1]}, f)$. Since s^1 is an equilibrium strategy profile of $\underline{e}(g\restriction_{[0,a^1]}, f)$, we know that $s_i^1(x_i)$ is the best reply to s_{-i}^1 among the actions $\{a_i \in A_i \mid a_i \leq a_i^1\}$. So it must be that $\beta(s)_i(x_i) \geq s_i^1(x_i) = s_i(x_i)$ in the game $\underline{e}(g\restriction_{[a^0,a^2]}, f)$.

For $x_i \in (R^1, R^1 + 1)$, the opposing action distribution in $\underline{e}(g\restriction_{[a^0,a^2]}, f)$ conditional on the signal x_i (weakly) dominates the opposing action distribution conditional on the signal R^1, since s is increasing. By strategic complementarities, $\beta(s)_i(x_i) \geq \beta(s)_i(R^1) = a_i^1 = s_i(x_i)$.

For $x_i \geq R^1 + 1$, the opposing action distribution in $\underline{e}(g\restriction_{[0,a^2]}, f)$ conditional on the signal x_i is just like in that $\underline{e}(g\restriction_{[a^1,a^2]}, f)$ conditional on the signal $x_i - (R^1 + 1)$. Moreover, the opposing action distribution given the signal x_i (weakly) dominates the opposing action distribution given the signal R^1, since s is increasing. This implies $\beta(s)_i(x_i) \geq a_i^1$. Furthermore, we know that $s_i^2(x_i)$ is the best reply to s_i^2 among the actions $\{a_i \in A_i \mid a_i^1 \leq a_i \leq a_i^2\}$, since s^2 is an equilibrium strategy profile of $\underline{e}(g\restriction_{[a^1,a^2]}, f)$. Combining, we must have $\beta(s)_i(x_i) \geq s_i^2(x_i - (R^1 + 1)) = s_i(x_i)$ in the game $\underline{e}(g\restriction_{[a^0,a^2]}, f)$.

In sum, $\beta(s) \geq s$. Hence an upper-best reply iteration converges monotonically to an equilibrium strategy profile $s^* \geq s$. Since, by construction, $s(R^1 + 1 + R^2) = a^2$, certainly s^* attains a^2. This proves the claim.

Conclude there is a lower-f-elaboration $\underline{e}(g\restriction_{[0,a^*]}, f)$ with an equilibrium strategy profile s that attains a^*. Since s is an equilibrium strategy profile, enlarging the joint action set from $[0, a^*]$ to $[0, m]$ cannot make players want to switch to smaller actions when they follow the strategy profile s. Hence a^* is attained from below under f given the original game g, implying $\overline{a}(g, f) \geq a^*$.

It remains to be shown that a^* is the *greatest* action profile that is attained from below under f. Towards a contradiction, suppose there is a lower-f-elaboration $\underline{e}(g, f)$ of g with a greatest equilibrium strategy profile s^* that attains some $a^{**} > a^*$. Let θ be distributed on $[-\frac{1}{2}, R]$ in

$\underline{e}(g, f)$ and, without loss of generality, let a^{**} be the *greatest* action profile that s^* attains. Recall that s^* is increasing.

By assumption, there is a restricted game $g\restriction_{[a,a']}$ with $a \overset{g}{\leftarrow} a'$ and such that $a < a^{**} \leq a'$. Consider the lower-f-elaboration of $g\restriction_{[a,a']}$ with θ be distributed on $[-\frac{1}{2}, R]$, and consider the strategy profile given by:

$$\text{for all } i \in I, \quad s_i(x_i) = \begin{cases} a_i & \text{if } s_i^*(x_i) \leq a_i, \\ s_i^*(x_i) & \text{if } s_i^*(x_i) > a_i. \end{cases}$$

For all signals $x_i < 0$, $s_i(x_i) = a_i$ is the dominant action in $\underline{e}(g\restriction_{[a,a']}, f)$. For all signals $x_i \geq 0$, the opposing action distribution in $\underline{e}(g\restriction_{[a,a']}, f)$, conditional on x_i and when players follow the strategy profile s, weakly dominates the opposing action distribution at x_i in $\underline{e}(g, f)$ when players follow s^*. Since s^* is an equilibrium strategy profile of $\underline{e}(g, f)$, it follows that $\beta(x_i)_i(s) \geq s_i(x_i)$ in $\underline{e}(g\restriction_{[a,a']}, f)$.

In sum, $\beta(s) \geq s$ in $\underline{e}(g\restriction_{[a,a']}, f)$. Thus an upper best-reply iteration converges to an equilibrium strategy profile s^{**} that attains a^{**}. Conclude that $\overline{a}(g\restriction_{[a,a']}, f) \geq a^{**} > a$. Yet this contradicts that $a \overset{g}{\leftarrow} a'$. So it must be that $\overline{a}(g, f) = a^*$ after all. As f was arbitrary, this proves the theorem. ■

Proof of Proposition 8

Fix some noise structure f. We will prove the proposition by showing that a^* is attained from below—and hence, dually, from above—under the noise structure f. First we make the information structures of lower-f-elaborations discrete, in order for them to fit the definition of a discrete incomplete information game. Let $\underline{e}(g, f)$ be a lower-f-elaboration of g and, for $\delta > 0$, let the signal space $(-1, R + \frac{1}{2})$ of $\underline{e}(g, f)$ be covered by a partition of intervals of length δ:

$$P^\delta = \{p^n \mid n \in \{\ell, \ell+1, ..., r\}\}, \quad \text{with } p_n = [n\delta, (n+1)\delta), \quad \ell, r \in \mathbb{Z}, \quad \ell < 0 < r,$$
$$\text{and the partition } P^\delta \text{ covering } (-1, R + \tfrac{1}{2}).$$

Now, for each $\delta > 0$, we may consider a discrete incomplete information game $\underline{e}^\delta(g, f)$ based on $\underline{e}(g, f)$, in which instead of receiving their signal x_i, players are only informed about the interval $p \in P^\delta$ that contains x_i and calculates a conditional density $\pi_p(x)$. A pure strategy profile s in $\underline{e}(g, f)$ is said to be an *equilibrium under δ-discretised information* if and only if it is constant on every $p \in P^\delta$ and maximises expected payoff under this constraint, assuming that opponents follow the same strategy. More precisely:

$$\forall i, a_i, p, \quad \int_{x \in p} u_i(s_i(p), s_{-i}|x)\pi_p(x) \, \mathrm{d}x \geq \int_{x \in p} u_i(a_i, s_{-i}|x)\pi_p(x) \, \mathrm{d}x,$$

where, as before, $u_i(a_i, s_{-i}|x)$ denotes the expected payoff of playing $a_i \in A_i$.

The main step in our argument is based on the following claim, which retrieves the robust equilibria of g in our continuous lower-f-elaborations via the discrete incomplete information games $\underline{e}^{\delta}(g, f)$.

Claim. If a^* is a robust equilibrium of g, then there exists a lower-f-elaboration $\underline{e}(g, f)$ such that for any $\delta > 0$ there is an increasing, pure strategy profile s^{δ} in $\underline{e}(g, f)$ that is an equilibrium under δ-discretised information and attains a^*.

Proof of Claim. Recall that in any lower-f-elaboration, the state parameter θ is distributed uniformly over some interval $[-\frac{1}{2}, R]$. For $\theta > \frac{1}{2}$, it is guaranteed that each player i will receive a positive signal $x_i = \theta + \eta_i$ that informs her that the relevant individual payoff function $u_i(\cdot, x_i)$ is given by $g_i(\cdot)$. Now fix some δ, $\delta > 0$. In any δ-discretised lower-f-elaboration $\underline{e}^{\delta}(g, f)$, a realisation $\theta > \frac{1}{2}$ guarantees that each player i knows that her payoff function is g_i, so

$$\mathbb{P}(\Omega_g) \geq \mathbb{P}(\theta > \tfrac{1}{2}) = \frac{R-\frac{1}{2}}{R+\frac{1}{2}}.$$

Since a^* is a robust equilibrium of g, if we choose R sufficiently large, there exists (by definition) a (mixed) equilibrium strategy profile σ in $\underline{e}^{\delta}(g, f)$, such that a^* is played in some interval $p^* \in P^{\delta}$ with strictly positive probability. Moreover, we can choose R independent of δ, thus fixing $\underline{e}(g, f)$. Conducting an upper-best reply iteration in $\underline{e}^{\delta}(g, f)$ starting at σ will give a pure equilibrium strategy profile s that prescribes actions weakly greater than a^* on the interval p^*. Similarly, if we conduct an upper-best reply iteration in $\underline{e}^{\delta}(g, f)$ starting at

$$\forall i, p, \quad s_i^0(p) = m_i,$$

we find the *greatest* (pure) strategy profile $s^{\delta} \geq s$ that is an equilibrium under δ-discretised information. Moreover, s^{δ} is increasing. This proves the claim.

Thus, if a^* is a robust equilibrium, then for some elaboration $\underline{e}(g, f)$ of g and arbitrarily small δ, there is a strategy profile s^{δ} that attains a^* and is an equilibrium under δ-discretised information. As we choose δ smaller and smaller, the discretised information structure resembles the continuous information structures ever more closely. Intuitively, there should be an *equilibrium* strategy profile in $\underline{e}(g, f)$ that attains a^*, which is all we need to show.

To do so, let $\delta_k = \frac{1}{k}$ and consider the sequence of increasing strategy profiles s^{δ_k}. As the ("Helly") space of increasing functions is sequentially compact (see [27], example 107), we may choose a subsequence s^n that converges pointwise towards some s^*. As each s^n attains a^* for signals below $R + \frac{1}{2}$, so does s^*. We need to show that s^* is a best reply to itself at any given signal. Assume the contrary, that is

$$\exists i \exists x_i \exists a_i \neq s^*(x_i), \quad u_i(a_i, s^*_{-i}|x_i) - u_i(s_i^*(x_i), s^*_{-i}|x_i) > 0.$$

Let us assume that $a_i > s^*(x_i)$ (a symmetric construction can be applied if $a_i < s^*(x_i)$). Define:

$$D_n = \{x \mid \exists m \geq n \text{ such that } s^m(x) \neq s^*(x)\}, \quad \text{and} \quad \tilde{s}^n(x) = \begin{cases} 0 & \text{if } x \in D_n, \\ s^*(x) & \text{if } x \notin D_n. \end{cases}$$

Note that $(\tilde{s}^n)_n$ converges towards s^* in pointwise and monotonic fashion, while $\tilde{s}^n \leq s^m$ for all $m \geq n$. Using the monotone convergence theorem, we conclude

$$\exists \overline{n} \text{ such that } u_i(a_i, \tilde{s}^{\overline{n}}_{-i}|x_i) - u_i(s^*_i(x_i), \tilde{s}^{\overline{n}}_{-i}|x_i) > 0.$$

Since, for a fixed opposing strategy profile, the opposing action distribution changes continuously in one's own signal, so do expected payoffs. Thus, there exists $\varepsilon > 0$ such that

$$\text{for all } x \in [x_i - \varepsilon, x_i + \varepsilon], \quad u_i(a_i, \tilde{s}^{\overline{n}}_{-i}|x) - u_i(s^*_i(x_i), \tilde{s}^{\overline{n}}_{-i}|x) > 0.$$

Now, denote by p_n the interval containing x_i under δ_{k_n}-discretised information and choose $n' > \overline{n}$ such that $s^{n'}(x_i) = s^*(x_i)$ and $p_{n'} \subset [x_i - \varepsilon, x_i + \varepsilon]$. Then

$$\begin{aligned} \int_{x \in p_{n'}} & [u_i(a_i, s^{n'}_{-i}|x) - u_i(s^{n'}_i(x_i), s^{n'}_{-i}|x)]\pi_{p_{n'}}(x) \ \mathrm{d}x \\ &= \int_{x \in p_{n'}} [u_i(a_i, s^{n'}_{-i}|x) - u_i(s^*_i(x_i), s^{n'}_{-i}|x)]\pi_{p_{n'}}(x) \ \mathrm{d}x \\ &\geq \int_{x \in p_{n'}} [u_i(a_i, \tilde{s}^{\overline{n}}_{-i}|x) - u_i(s^*_i(x_i), \tilde{s}^{\overline{n}}_{-i}|x)]\pi_{p_{n'}}(x) \ \mathrm{d}x > 0. \end{aligned}$$

where the first inequality is due to the fact that $s^{n'} \geq \tilde{s}^{\overline{n}}$, so switching to the higher action a_i yields a lower payoff when facing $\tilde{s}^{\overline{n}}_{-i}$. But this contradicts that $s^{n'}$ is an equilibrium under discretised information, so we conclude that s^* must be an equilibrium strategy profile after all. ■

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