CLOSING ECONOMIC MODELS BY INFORMATION ASSUMPTIONS[1]

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In this paper it is shown how interactive social systems, like microeconomic models and noncooperative games, can be closed by assumptions on the participants' information concerning the structure of the social system, the beliefs of other agents, and the information structure itself. A coherent framework for analyzing information assumptions in games as well as in economies that cannot be written in strategic form is developed. A variety of information assumptions is analyzed, and it is demonstrated that these assumptions are equivalent to certain solution concepts. Results are applied to a two–period exchange economy. A sufficient condition is derived for which the Walrasian equilibria of this economy describe the event that households are completely informed about the economy and about the beliefs of the other households.

Keywords: admissible priors, Arrow–Debreu economy, information, rational expectations, rationalizable expectations, self fulfilling prophecies, Walras equilibria.

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1 INTRODUCTION

The economy is a social system composed of individual economic actors who are uncertain about each other's behavior. This kind of uncertainty has been called market uncertainty by economists (Shell, 1987). To game theorists it is known as strategic uncertainty. The uncertain actions as well as prices relying on these activities are endogenous variables. They depend on individual goals, constraints, and expectations. The expectations, relevant for an agent's decision in a situation of market uncertainty, concern the realizations of exogenous variables, like the state of the world, but also endogenous variables, like prices or activities of other agents. Because the endogenous variables depend on expectations of other agents, expectations about these variables can be traced back to expectations on expectations.

Keynes (1936) exemplified the problem of predicting the actual behavior in situations of market uncertainty by a beauty contest, in which a prize is given to that participant who best predicts the average opinion of the others. In order to close a model with inherent market uncertainty, a solution concept endogenizing expectations is needed. Noncooperative game theory has developed a variety of solution concepts for strategic games. These concepts cannot be applied directly to the leading models of market economies because the latter dispense with strategic behavior.

In this paper a general framework for closing models of non–strategic market economies as well as strategic games is developed. Solution concepts, based on decision theoretic considerations, can be applied directly to both kinds of models. The solution concepts considered in this paper are defined by different assumptions on the agents' information about the structure of the social system, about the beliefs of the other agents, and about the information structure itself.

In this, our work is closely related to a series of papers analyzing the decision theoretic foundation of game theoretic solution concepts. This string of literature was started by Bernheim (1984) and Pearce (1984). They introduced the concept of rationalizable strategies. Those are the strategies that remain after an iterative elimination of strictly dominated strategies if one assumes that players believe that their opponents' strategies are independent of each other. The iterative elimination procedure has first been suggested by Morgenstern

(1935). He proposed it as a solution to overcome logical difficulties implied by the assumption that agents behave as though they know the predictions of a theory that tries to explain their own behavior.

For normal form games Bernheim (1986) and Tan and Werlang (1988) have shown formally that the iterative elimination of strictly dominated strategies amounts to assuming that the payoff matrix and the rationality of the players are common knowledge. Adding the assumptions that agents believe that their opponents' strategies are independent of each other and that this fact is also common knowledge yields the prediction of rationalizable strategies.

Correlated and Nash equilibria could not be justified by information assumptions alone. Although a pure–strategy Nash equilibrium is the solution prescribed by assuming that each player knows the strategies of his opponents (Tan and Werlang, 1988), this characterization is of little help, because the usual interpretation of a game situation is that players who move simultaneously do not know each other's choices. If common knowledge of rationality, payoff matrix, and solution concept is combined with the assumption that players have a common prior then the solution of the game is a set of strategy combinations that are played with positive probability in a correlated equilibrium. Furthermore, any set of strategy combinations, that are played with positive probability in one correlated equilibrium, is a solution consistent with these assumptions (Bernheim, 1986). Aumann (1987) showed that correlated equilibria can also be characterized by assuming a common prior and common knowledge of rationality, payoff matrix, and the common prior. Mixed Nash equilibria are characterized by the same assumptions plus the restriction to beliefs that regard the strategies of different players as stochastically independent (Bernheim, 1986).

The assumption of common priors could not be justified by any premises on information.[3] Aumann and Brandenburger (forthcoming) have shown by way of examples that the common prior assumption is indispensible for the characterization of Nash equilibria in games with more than two players. However, there are special cases in which Nash equilibria coincide with solutions that have a decision theoretic foundation. From Tan and Werlang (1988), Bran-

[3]For a discussion of the common prior assumption see Morris (1993).

denburger (1992), and Aumann and Brandenburger (forthcoming) we know that in a two–person game the players' beliefs form a Nash equilibrium if each player knows the payoff matrix, the rationality, and the beliefs of the other player.

McAllister (1988, 1990) analyzed the nonstrategic market economy of Radner (1979) by closing the model with information assumptions. He introduced a solution concept called "weakly admissible priors" that reflects common knowledge of the structure of the economy and common knowledge of rationality. Adding mutual knowledge of all households' prior beliefs defines his concept of "strongly admissible priors." While rational expectations equilibria generically reveal all private information, McAllister showed that equilibria for admissible priors do not have this property.

This paper deviates from the aforementioned literature in four respects: First, our modelling of information does not require any of the abstract knowledge operators[4] that have been used in the other papers. Here, any information about the structure of a social system is modelled by the same kind of formal objects that are used to describe the social system itself. The problems, usually associated with modelling common knowledge, are circumvented by considering information about the information structure. It turns out that complete information about the information structure is equivalent to the assumption that whatever the agents are informed about is common knowledge. This method allows for a better comprehension of the results.

Second, some of the solution concepts considered in this paper are *uniquely* defined by information assumptions. The assumed amount of information is necessary and sufficient for justifying these concepts. In this paper the solution concepts are defined by information assumption only. We do not require any additional restriction on prior beliefs, such as independence or common priors. The price, we have to pay for this pureness, is that predictions based on these concepts are less precise than some of those that make use of, e.g. the common prior assumption. However, our set–up allows for additional restrictions, so that information assumptions can be combined with other assumptions suitable

[4]For an exposition of knowledge operators and their relation to information see Osborne and Rubinstein (1994).

to reduce the number of equilibria.

Third, our model of a social system allows for a sequential decision structure. Thus, we are able to give a decision theoretic foundation to solution concepts of extensive form games. From Rosenthal (1981), Binmore (1987), and Reny (1992, 1993), among others, we know that the common knowledge assumption seems to contradict rational behavior in extensive form games. Here, we circumvent this problem by assuming that all information is true. From Milgrom (1981) we know that truth is one of the axioms characterizing common knowledge. The well known paradoxa stem from violations of this axiom. Why truth is so essential for common knowledge can be understood as follows:

Decision theoretic analysis assumes that an agent assigns probability one to any event whose occurrance she has been informed of. Rational agents will only do so if they believe that their information is true. Common knowledge amounts to complete information about the information structure. If the information structure would allow for false information and the information structure itself is known to the agent, then her belief in the truth of all information would be inconsistent with her information.[5]

Fourth, as mentioned above, our model of a social system allows for economies that cannot be written in strategic form. Hence, we provide a method to close models of non–strategic market economies by information assumptions only. This is demonstrated by the example of a two–period exchange economy in the spirit of Arrow (1983 [1953]), Debreu (1959), and Radner (1972). If this economy has a unique market clearing equilibrium for each relevant profile of subjective expectations then the set of Walrasian equilibria can be interpreted as the event that households are completely informed about the structure of the economy and about each other's expectations.

The paper is organized as follows: Section 2 introduces a social system in which the agents' decisions are based on beliefs about the realizations of *all* variables. In Section 3 the social system is extended by modelling information about the structure of the social system, about beliefs, exogenous probabilities,

[5]The impossibility to regard false information is closely related to the impossibility theorem of Basu (1990).

and about the information structure. Section 4 analyzes different information structures and shows their equivalence to certain solution concepts closing the social system. In Section 5 we demonstrate how these decision theoretic solution concepts can be applied to a non–strategic market economy. For this, we rewrite the two–period exchange economy using the notation introduced in Section 2.

2 SOCIAL SYSTEM

A social system consists of a list of variables and a description of dependencies between these variables. Variables can be arranged into exogenous variables, endogenous decision variables, and other endogenous variables (Radner, 1982).

Exogenous variables cannot be influenced by decisions of economic actors who are considered in the model. The state of the world, the result of a random process, parameters of technologies or utility functions, political decisions, and property rights are examples of variables that are considered as exogenous by some economic theories.

A variable is called endogenous if its realization depends on the activities of economic actors. A variable that is controlled by a single actor i is called decision variable of i. A decision variable may be a strategy, individual demand, or a signal. Other endogenous variables, like aggregate demand, prices, national product, or the unemployment rate, may depend jointly on decisions of several actors and on realizations of exogenous variables.

All variables are distinguished by the period in which they realize. Time is described by a nonempty set of periods, $T \subseteq \mathbb{Z}$. A period $t \in T$ is said to be "later" than period $\tau \in T$ if $t > \tau$.

Decision variables are further distinguished by the individuals who control them. Let I be the set of individuals. We assume that I is denumerable. $I_t \subseteq I$ is the set of individuals who are in control of some variable in period t.

2.1 *Notation of Variables*

A decision of individual $i \in I_t$ in period $t \in T$ is denoted by $z_{i,t} \in W_{i,t}$. A vector of decisions of all actors in period t is

$$z_{I,t} := (z_{i,t})_{i \in I_t} \in W_{I,t} := \prod_{i \in I_t} W_{i,t}.$$

Let $z_{Y,t} \in W_{Y,t}$ be a vector of the other endogenous variables that realize in period t. A realization of the exogenous variables in period t is denoted by $z_{E,t} \in W_{E,t}$. For short notation we define a realization of all variables in period t by

$$z_t := (z_{E,t}, z_{I,t}, z_{Y,t}) \in W_t := W_{E,t} \times W_{I,t} \times W_{Y,t}.$$

z_t may also be called "state of the economy in period t". Accordingly, a sequence of states of the periods $\tau \leq t$ is denoted by

$$z_{\tau \leq t} := (z_\tau)_{\tau \leq t} \in W_{\tau \leq t} := \prod_{\tau \leq t} W_\tau.$$

A sequence of states of all periods is a singular economic event or "path of the economy"

$$z \in (z_t)_{t \in T} \in W := \prod_{t \in T} W_t.$$

A sequence of exogenous variables of all periods is denoted by

$$z_E := (z_{E,t})_{t \in T} \in W_E := \prod_{t \in T} W_{E,t}.$$

W_t is assumed to be a nonempty topological space with denumerable basis, e.g. a subset of some $\mathbb{R}^n$. By $\mathcal{W}_{E,t}$ we denote a σ–algebra in $W_{E,t}$, e.g. the power set or the family of Borel subsets of $W_{E,t}$. Let $\mathcal{W}_{I,t}$ and $\mathcal{W}_{Y,t}$ be σ–algebras in $W_{I,t}$ and $W_{Y,t}$ respectively. By $\mathcal{W}_t$, $\mathcal{W}_{\tau \leq t}$, $\mathcal{W}$, and $\mathcal{W}_E$ we denote the according product–σ–algebras in W_t, $W_{\tau \leq t}$, W, and W_E. A subset $A \in \mathcal{W}$ is an economic event. A subset $B \in \mathcal{W}_E$ is called an exogenous event.

2.2 *Relative Probability Measures*

These σ–algebras are needed to define appropriate probability spaces. In this paper we use the concept of relative probability measures introduced by

Kohlberg and Reny (1992). Appendix A contains the axiomatic definition of relative probabilities and gives some properties of these objects. Here, we want to introduce only the notation: For any σ–algebra $\mathcal{W}$ let $\mathcal{P}(\mathcal{W})$ be the set of all relative probability measures on $\mathcal{W}$. A relative probability measure $\pi \in \mathcal{P}(\mathcal{W})$ assigns some nonnegative number $\pi(A, B)$ to any pair of events $A, B \in \mathcal{W}$. This number is an expression of the relative likelihood of the two events. For the impossible event $\emptyset$ by definition $\pi(\emptyset, A) \in \{0, 1\}$ for all $A \in \mathcal{W}$. An event A is interpreted as possible [impossible] if $\pi(\emptyset, A) = 0$ [1].

2.3 *Exogenous Variables*

Exogenous variables are not explained by the theory. Their realizations are viewed as being determined outside that part of the economy that is analyzed by the model. Here, they are treated as random variables underlying an exogenously given probability distribution. These so–called "objective probabilities" are given by a relative probability measure $\eta \in \mathcal{P}(\mathcal{W}_E)$ that assigns some nonnegative number $\eta(A, B)$ to any pair of exogenous events $A, B \in \mathcal{W}_E$. $\eta(A, B) = k$ should be interpreted as "event A is k–times as likely as event B". We assume that all events $A \in \mathcal{W}_E \backslash \{\emptyset\}$ are possible, i.e. $\eta(\emptyset, A) = 0$. A non–stochastic economy can be modeled either by omitting exogenous variables from explicit consideration or by choosing a single valued domain, $W_E = \{z_E\}$.

2.4 *Decisions and Expectations*

Consider an arbitrary period $t \in T$ and some individual $i \in I_t$. The agent has to select some action $z_{i,t} \in W_{i,t}$. This decision depends on the agent's beliefs about economic events. In principle, individuals might be uncertain about realizations of past, present, and future variables. An individual's beliefs or "subjective probabilities" are expressed by a relative probability measure $\pi^i \in \mathcal{P}(\mathcal{W})$. Here, we assume that there exists a number $\pi^i(A, B)$ for any pair of events $A, B \in \mathcal{W}$ that expresses agent i's belief that event A is $\pi^i(A, B)$–times as likely as event B. We say that agent i regards event A as possible [impossible], if $\pi^i(\emptyset, A) = 0$ [1].

Beliefs are not fixed for all periods. They may be changing with any information the agent receives. In each period individual i may get some information about past and present realizations. Each of these pieces of information is used to revise the individual's beliefs. The revision process is assumed to be a Bayesian update: Let $\pi^{i,t} \in \mathcal{P}(\mathcal{W})$ be agent i's beliefs in period t, and let us suppose that in period $t+1$ the agent gets the information that event $B \in \mathcal{W}$ occurs with certainty. The agent's beliefs in period $t+1$ are then expressed by the conditional relative probability measure $\pi^{i,t+1} = \pi_B^{i,t}$ which is defined by

$$\pi_B^{i,t}(A, C) := \pi^{i,t}(A \cap B,\, B \cap C).$$

The assumption, that agents revise beliefs by the method of Bayesian updating, basically states that they regard events as impossible after getting the information that these events are impossible, while relative probabilities, assigned to subsets of B, remain unchanged.

Bayesian updating as well as any other revision process for beliefs requires distinguishing between prior and posterior beliefs. Prior beliefs, or "priors", should be interpreted as beliefs of agents who are lacking any information about realized states. Posterior beliefs, or "posteriors", are revised priors, updated on the agents' information.

2.5 *Prior Beliefs*

Prior beliefs of agent i are denoted by π^i. A tupel of priors of all individuals is

$$\pi^I := (\pi^i)_{i \in I} \in \mathcal{P}^I \subseteq \prod_{i \in I} \mathcal{P}(\mathcal{W}).$$

The set $\mathcal{P}^I$ is the set of admissible priors. This may be a constrained subset of all tupels of relative probability measures. There are three reasons of why we might wish to constrain admissible priors:

1. In some models it is advantageous to restrict analysis to beliefs with special technical properties, e.g. beliefs that assign positive absolute probabilities to denumerable singular events.
2. It might be adequate to consider only relative probability measures that express the belief that individual decisions are independent of each other.

3. A model can be closed by restricting admissible beliefs to measures that are consistent with some information about the model and about beliefs. In chapter 4 we restrict ourselves to beliefs that are consistent with certain assumptions on individual information about the model and about the agents' beliefs. These consistency requirements may simultaneously restrict the whole vector of beliefs of all agents.

2.6 *Information Induced by Realized States*

An information is a statement. An information about an object A is modelled as a set of objects $B[A]$. It should be interpreted as the statement that "object A is contained in $B[A]$". This statement is true if $A \in B[A]$. The information about A is complete if $B[A] = \{A\}$.

In period t individual i may get some information about the realized states of the economy, $z_t, z_{t-1}, z_{t-2}, \ldots$ Direction of time forbids the agents to posess any information induced by realizations of the later variables z_τ, $\tau > t$, since those realizations do not yet exist. The information, induced by past and present realizations, is given by a subset $\beta_t^i(z_{\tau \le t}) \in \mathcal{W}_{\tau \le t}$. The mappings $\beta_t^i : W_{\tau \le t} \to \mathcal{W}_{\tau \le t}$ represent the structure of realization induced information in the economy. We assume that all information is true, i.e.

$$z_{\tau \le t} \in \beta_t^i(z_{\tau \le t}) \qquad \forall z \in W,\ \forall i \in I,\ \forall t \in T.$$

We also assume that the agents know that their information is true. From this, agent i can deduce in period t that the economy will take a path

$$z \in B_t^i(z_{\tau \le t}) := \left\{ \mathsf{z} \in W \;\middle|\; \mathsf{z}_{\tau \le t} \in \beta_t^i(z_{\tau \le t}) \right\}.$$

In addition, we assume that agents do not lose information, i.e.

$$B_t^i(z_{\tau \le t}) \subseteq B_{t-1}^i(z_{\tau < t}) \qquad \forall z \in W,\ \forall i \in I,\ \forall t \in T.$$

Note that this is the second point at which we use the direction of time. The agents are assumed to revise their beliefs by the method of Bayesian updating. Posterior beliefs of agent i in period t are given by her prior relative probabilities π^i conditioned on her present information $B_t^i(z_{\tau \le t})$. Thus, posterior relative probabilities are $\pi^{i,t} = \pi^i_{B_t^i(z_{\tau \le t})}$. These beliefs are the ones relevant for agent i's decision in period t.

2.7 *Decision Correspondences*

In period t each individual $i \in I_t$ carries out some action $z_{i,t} \in W_{i,t}$. The action, individual i decides for, may result from a conscious optimization of some goal function. The agent might have preferences on W and maximize expected utility or profits. We do not need any assumptions of why the agents act the way they do. We assume only that individual i's decision in period t depends on her posterior beliefs and thus, can be described by a decision correspondence, $\sigma_t^i : \bar{\mathcal{P}}(\mathcal{W}) \to \mathcal{W}_{i,t}$, that assigns a set of actions to each relative probability measure.[6]

The set $\sigma_t^i(\pi^{i,t})$ consists of all actions that individual i might carry out in period t when she has posterior beliefs $\pi^{i,t}$. This set may be single valued, e.g. if there is a unique strategy that maximizes expected payoff. We allow for $\sigma_t^i(\cdot)$ to be empty, because some models have the property that optimal decisions are not defined outside equilibrium. For technical reasons the domain of σ_t^i contains the function $\pi_\emptyset$ which cannot be interpreted as a measure of beliefs.[7] But, we assume $\sigma_t^i(\pi_\emptyset) = \emptyset$ for all i and t, so that posterior $\pi_\emptyset$ will be excluded by consistency requirements.

Putting together prior beliefs, information induced by realized states, Bayesian updating, and decision correspondences, we yield a consistency requirement demanding that chosen actions must conform with the posteriors which depend on the information that is induced by these actions, among other variables. This consistency requirement is formally expressed by

$$z_{i,t} \in \sigma_t^i \left(\pi^i_{B_t^i(z_{\tau \leq t})} \right) \qquad \forall i \in I,\ \forall t \in T. \tag{1}$$

Condition (1) constrains possible economic events. It comprises our assumptions on the information structure, on the revision process for beliefs, and on the individual decision processes. Only vectors $z \in W$, for which (1) holds, can arise when prior beliefs are π^I. An event $A \in \mathcal{W}$ is possible only if there is some $\pi^I \in \mathcal{P}^I$ and some $z \in A$ for which condition (1) holds.

[6] $\bar{\mathcal{P}}(\mathcal{W}) := \mathcal{P}(\mathcal{W}) \cup \{\pi_\emptyset\}$, where $\pi_\emptyset(A, B) := 1$ for all $A, B \in \mathcal{W}$.
[7] See Appendix A.

2.8 *Other Endogenous Variables*

The other endogenous variables, that realize in period t and are not controlled by a single agent, may depend on the exogenous variables of this period, $z_{E,t}$, on the decisions of all individuals in that period, $z_{I,t}$, and on the states of the economy in former periods, $z_{\tau<t}$. This dependence is expressed by a mapping

$$F_t : W_{\tau<t} \times W_{E,t} \times W_{I,t} \to \mathcal{W}_{Y,t}.$$

$F_t\left(z_{\tau<t}, z_{E,t}, z_{I,t}\right) \subseteq W_{Y,t}$ is the set of possible realizations of other endogenous variables in period t. We allow explicitly for $F_t(\cdot)$ being the empty set, because Walrasian equilibrium models have the property that prices are assigned only to market clearing net demands.[8] The correspondences F_t further restrict possible events. Only singular events

$$z \in F := \{z \in W \mid z_{Y,t} \in F_t\left(z_{\tau<t}, z_{E,t}, z_{I,t}\right) \quad \forall t \in T\}. \tag{2}$$

can occur. An event $A \subseteq W \backslash F$ is impossible, because it violates the basic assumptions on the dependencies between economic variables that are expressed by F.

2.9 *Solving a Social System*

A social system, or economy, is completely described by

$$\mathcal{M} = \left[I, T, W, \mathcal{W}, \eta, \mathcal{P}^I, \left(B^i_t, \sigma^i_t\right)^{i\in I}_{t\in T}, F\right].$$

Figure 1 illustrates the structure of a social system $\mathcal{M}$.

Insert Figure 1 here!

The assumptions, that are embodied in $\mathcal{M}$, restrict the possible realizations of all variables. These restrictions are expressed by the consistency requirements (1) and (2). By combining all elements of the social system $\mathcal{M}$, we yield a mapping $\zeta : \mathcal{P}^I \to \mathcal{W}$ which assigns a set of possible paths of the economy to any given vector of prior beliefs. The set of paths, that can arise in the social system for given priors π^I, is

$$\zeta(\pi^I) := \left\{z \in F \;\middle|\; z_{i,t} \in \sigma^i_t\left(\pi^i_{B^i_t(z_{\tau\leq t})}\right) \;\; \forall i \in I,\; \forall t \in T\right\}.$$

[8] For an example see Section 5.

We refer to ζ as the solution correspondence of $\mathcal{M}$.

The domain of ζ is restricted by the set of admissible priors $\mathcal{P}^I$. A subset $A \in \mathcal{W}$ is a possible economic event if there is some admissible belief profile $\pi^I \in \mathcal{P}^I$ and some $z \in A$, such that $z \in \zeta(\pi^I)$. The solution of a social system is the set of all possible paths of the economy. It is denoted by

$$Z := \left\{ z \in W \;\middle|\; \exists \pi^I \in \mathcal{P}^I \text{ such that } z \in \zeta(\pi^I) \right\}.$$

We refer to Z as the solution of $\mathcal{M}$. Z is the prediction of the theory, and the size of Z is an indicator for the precision of the theory. The smaller the solution is, the more precise is the prediction of the theory.

The solution of a social system may be quite large, depending on the structural elements B_t^i, σ_t^i, F and $\mathcal{P}^I$. Of course, economists aim at giving predictions that are as precise as possible. There are different ways to yield precise predictions:

- The better informed the agents are, i.e. the smaller $B_t^i(z_{\tau \leq t})$ for given realizations $z_{\tau \leq t}$, the stricter is the consistency requirement (1), and the smaller is $\zeta(\pi^I)$ for given priors π^I.

- The more structure is given to the individual decision problems, i.e. the smaller $\sigma_t^i(\pi^{i,t})$ for given posteriors $\pi^{i,t}$, the less actions can be chosen by agents with given priors and given information, and the smaller is $\zeta(\pi^I)$ again.

- The more dependencies between the variables are embodied in the mappings F_t, the smaller are the sets F and $\zeta(\pi^I)$ for any given system of prior beliefs.

- The smaller the set of admissible priors $\mathcal{P}^I$ is, the smaller is the solution Z for a given solution correspondence ζ. Thus, restricting admissible priors can also make the prediction of a theory more precise.

Here, we will concentrate on the set of admissible priors. One reason to restrict admissible priors is the agents' information about the structure of the economy and about the beliefs of other agents. Any assumption about these kinds of information restricts the set of belief profiles that are consistent with this information. When admissible priors are then replaced by consistent priors we get a smaller solution.

3 INFORMATION ABOUT THE STRUCTURE OF THE ECONOMY AND ABOUT BELIEFS

In this section it will be demonstrated how information about the structure of an economy and about beliefs of other agents can be modelled, how this information is processed by rational individuals, and that certain assumptions based on this kind of information can be represented by proper consistency requirements on the set of admissible belief profiles.

3.1 *Information about Prior Beliefs*

The information, agent i has about the prior beliefs of all agents, is a set of belief profiles

$$\gamma^i(\pi^I) \subseteq \prod_{i \in I} \mathcal{P}(\mathcal{W}).$$

This information is interpreted as: "The prior beliefs of all individuals form a system of relative probability measures on $\mathcal{W}$ which is contained in $\gamma^i(\pi^I)$." We assume that every information is true, i.e. $\pi^I \in \gamma^i(\pi^I)$ for all $\pi^I \in \mathcal{P}^I$ and for all $i \in I$. Agent i is completely informed about prior beliefs if $\gamma^i(\pi^I) = \{\pi^I\}$. We say, that agent i is informed only about the set of admissible priors, if $\gamma^i(\pi^I) = \mathcal{P}^I$.

3.2 *Information about the Solution Correspondence*

The information of agent i about the solution correspondence is a set of functions assigning events to belief profiles

$$\gamma^i[\zeta] \subseteq \left\{ f : \mathcal{P}^I \to \mathcal{W} \right\}.$$

This information is interpreted as: "The solution correspondence is a function which is contained in $\gamma^i[\zeta]$." We assume again that the information is true, i.e.

$\zeta \in \gamma^i[\zeta]$. Agent i is completely informed about the solution correspondence if $\gamma^i[\zeta] = \{\zeta\}$.

3.3 *Information Processing*

Let us now assume that the agents know that only singular events, which are contained in $\zeta(\pi^I)$, can occur. But, the agents do not necessarily know the priors or the solution correspondence. Their information about these objects is given by $\gamma^i(\pi^I)$ and $\gamma^i[\zeta]$. All of this information is true by assumption, and we assume further that the agents know that their information is true.

Under these assumptions agent i can conclude that the realization of all variables is a path z for which there is some belief profile $\mathsf{p}^I \in \gamma^i(\pi^I)$ and some mapping $f \in \gamma^i[\zeta]$, such that $z \in f(\mathsf{p}^I)$. A rational agent with this information must regard all paths of the economy as impossible that are not contained in

$$(3) \quad C^i(\pi^I) := \left\{ z \in W \;\middle|\; \exists \mathsf{p}^I \in \gamma^i(\pi^I), \;\; \exists f \in \gamma^i[\zeta], \;\; \text{such that} \;\; z \in f(\mathsf{p}^I) \right\}.$$

The mapping $C^i : \mathcal{P}^I \to \mathcal{W}$ represents the structure of individual i's information about solution correspondence and beliefs. $C^i(\pi^I)$ is the set of singular events which individual i can regard as possible from her information about priors and about the solution correspondence when the actual priors are π^I.

Now, consider a rational agent whose information about priors and about the solution correspondence is expressed by a set $C^i(\pi^I)$. She knows that all possible events are contained in this set and that event $W \backslash C^i(\pi^I)$ is impossible. The agent's prior beliefs π^i are consistent with this information if

$$\pi^i \left(\emptyset, \, W \backslash C^i(\pi^I)\right) = 1$$

which is equivalent to $\pi^i = \pi^i_{C^i(\pi^I)}$.

3.4 *Information about Objective Probabilities*

Agent i's information about objective probabilities of exogenous events is a set of relative probability measures on $\mathcal{W}_E$

$$\gamma^i[\eta] \subseteq \mathcal{P}(\mathcal{W}_E).$$

This information is interpreted as: "The relative probabilities of exogenous events are expressed by some measure which is contained in $\gamma^i[\eta]$." We assume again that the information is true, i.e. $\eta \in \gamma^i[\eta]$. Agent i is completely informed about the probabilities of exogenous events if $\gamma^i[\eta] = \{\eta\}$. Agent i is said to have no information about objective probabilities if $\gamma^i[\eta] = \mathcal{P}(\mathcal{W}_E)$.

Note that an agent with beliefs $\pi^i \in \mathcal{P}(\mathcal{W})$ assigns marginal relative probabilities

$$h[\pi^i](A, B) := \pi^i(A \times W_I \times W_Y,\, B \times W_I \times W_Y)$$

to any pair of exogenous events $A, B \in \mathcal{W}_E$. Agent i's prior beliefs are consistent with her information about objective probabilities if the marginal probabilities for exogenous events, that are induced by π^i, are contained in $\gamma^i[\eta]$. That is $h[\pi^i] \in \gamma^i[\eta]$.

3.5 *Consistent Priors*

The set of all belief profiles, that are consistent with the individuals' information about beliefs, solution correspondence, and objective probabilities, is

$$\mathcal{P}^I_\gamma := \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{C^i(\pi^I)} \;\text{ and }\; h[\pi^i] \in \gamma^i[\eta] \quad \forall i \in I \right\}. \tag{4}$$

$\mathcal{P}^I_\gamma$ comprises the whole structure of the agents' information about the economy and about beliefs. We assume that agents process their information efficiently, so that their beliefs are consistent with their information.

3.6 *Information about the Information Structure*

The agents might not only posess information about the structure of the social system $\mathcal{M}$, they might also have some information about the information structure itself. We have seen that the information structure imposes a restriction on the set of prior beliefs. Let us assume that all agents know that prior beliefs are consistent with the information structure. Then, information about the information structure can be expressed by information about the set of consistent belief profiles. Agent i's information about the information structure is denoted by

$$\gamma^i[\mathcal{P}^I_\gamma] \subseteq \prod_{i \in I} \mathcal{P}(\mathcal{W}).$$

This information is true if $\mathcal{P}^I_\gamma \subseteq \gamma^i[\mathcal{P}^I_\gamma]$. We say, that agent i is completely informed about the information structure, if $\gamma^i[\mathcal{P}^I_\gamma] = \mathcal{P}^I_\gamma$.

Note however that the information about the information structure is itself a part of the information structure. Agents who know something about consistent priors will use this information to reduce the set of paths that they regard as possible. Let us now redefine the information correspondence by

$$(3') \quad C^i(\pi^I) := \left\{ z \in W \;\middle|\; \exists \mathsf{p}^I \in \gamma^i(\pi^I) \cap \gamma^i[\mathcal{P}^I\gamma],\;\; \exists f \in \gamma^i[\zeta],\;\; \text{such that}\;\; z \in f(\mathsf{p}^I) \right\}.$$

According to this definition $C^i : \mathcal{P}^I \to \mathcal{W}$ represents the structure of individual i's information about solution correspondence, beliefs *and information structure*. $C^i(\pi^I)$ is the set of singular events which individual i can regard as possible from these types of information.

Combining (3') and (4) shows that now, the set of consistent priors $\mathcal{P}^I_\gamma$ depends on the information about this set. For this reason we cannot simply assume that the information about the information structure is true. Here, truth becomes a consistency requirement, that has to be chequed, whenever it is not obviously fulfilled.

Note that any information about the information structure is redundant when the agent, say i, already knows the belief profile, i.e. $\gamma^i(\pi^I) = \{\pi^I\}$. Information about the information structure can at best restrict her knowledge about belief profiles to $\mathcal{P}^I_\gamma$. Since beliefs of rational agents are consistent with their information, we have $\pi^I \in \mathcal{P}^I_\gamma$. But then $\gamma^i(\pi^I) \subseteq \mathcal{P}^I_\gamma$, which shows that the agent already knows that the belief profile is contained in $\mathcal{P}^I_\gamma$. Hence, additional information about the information structure is redundant in this case.

3.7 *Solving a Social System with Information about the System*

A social system with information about the social system and about prior beliefs consists of a social system $\mathcal{M}$, sets $\gamma^i[\zeta]$, $\gamma^i[\eta]$, and $\gamma^i[\mathcal{P}^I_\gamma]$, and a correspondence $\gamma^i : \mathcal{P}^I \rightrightarrows \prod_{j\in I} \mathcal{P}(\mathcal{W})$ for each individual $i \in I$.

$\mathcal{P}^I_\gamma$ is the set of prior beliefs that are consistent with the information structure expressed by the various γ's. The solution of this extended social system consists of the paths of the economy that can arise if the individuals have consistent priors. Formally, the solution of the extended social system is denoted by

$$Z_\gamma := \left\{ z \in W \;\middle|\; \exists \pi^I \in \mathcal{P}^I_\gamma \text{ such that } z \in \zeta(\pi^I) \right\}. \tag{5}$$

Note that the solution of the extended social system coincides with the solution of $\mathcal{M}$ if the set of admissible priors is restricted to belief profiles that are consistent with the assumed information structure, i.e.

$$\mathcal{P}^I = \mathcal{P}^I_\gamma \quad \Rightarrow \quad Z = Z_\gamma.$$

4 DECISION THEORETIC FOUNDATION OF SOLUTION CONCEPTS

In the sequel it is shown that certain assumptions on the structure of information about the economy and about beliefs are equivalent to certain solution concepts.

4.1 *Weakly Admissible Priors*

Let us start by analyzing the assumption that agents are completely informed about solution correspondence and information structure, but do not have any additional information about prior beliefs nor about objective probabilities.

ASSUMPTION 1: *The individuals are completely informed about the solution correspondence and about the information structure. They have no further information about prior beliefs or objective probabilities. i.e.*

$$\gamma^i[\zeta] = \{\zeta\}, \quad \gamma^i[\mathcal{P}^I_\gamma] = \mathcal{P}^I_\gamma, \quad \gamma^i(\pi^I) = \mathcal{P}^I \quad \forall \pi^I, \quad \text{and} \quad \gamma^i[\eta] = \mathcal{P}(\mathcal{W}_E) \quad \forall i.$$

PROPOSITION 1.A: *Under Assumption 1 there exists a subset $\mathsf{Z} \in \mathcal{W}$, with*

$$\mathsf{Z} = \left\{ z \in W \;\middle|\; \exists \pi^I \in \mathcal{P}^I \text{ such that } \pi^i = \pi^i_{\mathsf{Z}} \;\forall i \text{ and } z \in \zeta(\pi^I) \right\} \tag{6}$$

and $Z_\gamma = \mathsf{Z}$.

PROPOSITION 1.B: *For any subset $\mathsf{Z} \in \mathcal{W}$, for which condition (6) holds, there exists an information structure $\gamma^i[\cdot]$ $\forall i$, such that Assumption 1 holds and $Z_\gamma = \mathsf{Z}$.*

The proof of Proposition 1, together with proofs of all subsequent results in the paper, can be found in Appendix B.

Proposition 1 demonstrates that any subset Z, for which condition (6) holds, is a solution of the social system with an information structure obeying to Assumption 1. Condition (6) describes a set–valued solution concept for the economy $\mathcal{M}$. In some social systems there may be exactly one subset of this kind, but in general there are multiple subsets fulfilling (6). This multiplicity is another expression of the self referencial status of complete information about the information structure.

A subset $\mathsf{Z} \in \mathcal{W}$, for which (6) holds, contains all paths of the economy that can arise when all individuals believe that only paths out of this set will occur. It is a set–valued self fulfilling prophecy.

In his analysis of Radner's (1979) model, McAllister (1988) calls a set of prior beliefs weakly admissible if these beliefs are consistent with common knowledge of the structure of the economy. This corresponds to our Assumption 1. In McAllister's terms a set of belief profiles $\mathsf{P} \subseteq \mathcal{P}^I$ is weakly admissible if there is a subset $\mathsf{Z} \in \mathcal{W}$, for which (6) holds, and

$$\mathsf{P} = \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{\mathsf{Z}} \;\; \forall i \in I \right\} .$$

Using McAllister's terminology, let us call a belief profile $\pi^I \in \mathcal{P}^I$ *weakly admissible* if it is contained in some weakly admissible set of priors.

The possible non–uniqueness of the solution of the extended system under complete information about solution correspondence and information structure gives rise to the question of which economic events are excluded by this information assumption. Since condition (6) is equivalent to Assumption 1, any path of the economy, that is contained in some subset Z for which (6) holds, can arise when rational agents posess the aforementioned information. On the other hand, Assumption 1 excludes every path of the economy that is not contained in any such subset. Thus, the union of all subsets Z, for which (6) holds, is the set of all paths of the economy that are consistent with Assumption 1. This union can be calculated by an algorithm which is closely related to the game theoretic procedure of iterative elimination of strictly dominated strategies.

Define $Z^0 := W$ and for all $k \in \mathbb{N}$

$$Z^k := \left\{ z \in W \;\middle|\; \exists \pi^I \in \mathcal{P}^I \text{ such that } \pi^i = \pi^i_{Z^{k-1}} \;\; \forall i \text{ and } z \in \zeta(\pi^I) \right\}. \tag{7}$$

The set Z^k contains all path of the economy that can arise when people believe that only paths contained in Z^{k-1} can occur. Obviously $Z^k \subseteq Z^{k-1}$. Hence, there exists a limit set

$$Z^\infty := \bigcap_{k=1}^{\infty} Z^k. \tag{8}$$

Lemma 1: *For the set Z^∞ condition (6) holds, and any set $\mathsf{Z} \in \mathcal{W}$, for which condition (6) holds, is a subset of Z^∞.*

Corollary 1: *Z^∞ is the union of all subsets $\mathsf{Z} \in \mathcal{W}$, for which condition (6) holds.*

Corollary 2: *The set of all weakly admissible belief profiles is*

$$\mathcal{P}^I_{WAP} := \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{Z^\infty} \quad \forall i \in I \right\}.$$

Z^∞ is another set–valued solution concept for the economy $\mathcal{M}$. It is unique and consists of all paths of the economy that can arise when the agents have complete information about solution correspondence and information structure. Proposition 1 and Corollary 1 show that Assumption 1 characterizes Z^∞, but this assumption does not define Z^∞ as the unique solution. This fact hints at some redundancy in the information described by Assumption 1. Next we want to explore the information assumption that defines Z^∞ as the unique solution of the extended social system.

Morgenstern (1935) suggested the iterative elimination procedure as a method to overcome logical difficulties implied by the assumption that individuals behave as though they know the predictions of the theory that explains their own behavior. Our Assumption 1 has not equalized the individuals' information with the predictions of the theory. While the agents have been assumed to know the set of consistent priors, the outside observer could only conclude that there exists a subset $\mathsf{Z} \in \mathcal{W}$, obeying to condition (6) and consistent prior beliefs are such that $\pi^i = \pi^i_{\mathsf{Z}}$ for all $i \in I$. Let us now assume that the agents have the same information about prior beliefs that we were able to deduce from Assumption 1.

Assumption 2: *The individuals have the same information as an outside observer, who assumes that everybody is completely informed about solution correspondence and information structure, except for that they have no information about objective probabilities. i.e.*

$$\gamma^i[\zeta] = \{\zeta\}, \quad \gamma^i(\pi^I) = \mathcal{P}^I \quad \forall \pi^I, \quad \gamma^i[\eta] = \mathcal{P}(\mathcal{W}_E), \quad \text{and}$$
$$\gamma^i[\mathcal{P}^I_\gamma] = \left\{\pi^I \in \mathcal{P}^I \;\middle|\; \exists \mathsf{Z} \in \mathcal{W} \text{ such that (6) holds and } \pi^j = \pi^j_{\mathsf{Z}} \quad \forall j \right\} \quad \forall i.$$

Note that Assumption 2 does neither state that the agents are completely informed about the information structure nor that this information is true. However, the logic of processing information, described above, requires that all information is true. Therefore, it is necessary to show that Assumption 2 implies the truth of the information about the information structure.

Proposition 2: *Assumption 2 implies*

$$\gamma^i[\mathcal{P}^I_\gamma] = \mathcal{P}^I_\gamma = \mathcal{P}^I_{WAP} \quad \forall i \in I, \qquad \text{and} \qquad Z_\gamma = Z^\infty.$$

Proposition 2 states that Assumption 2 fulfills the truth requirement and defines Z^∞ as the unique solution of the extended social system. Together with Corollary 2 it demonstrates that a prior belief profile is weakly admissible if, and only if, it is consistent with Assumption 2. In order to characterize Z^∞, it is not necessary to assume complete information about the information structure. It is sufficient to assume that all individuals *believe* that everybody knows the information structure. This belief is self fulfilling and can therefore be interpreted as true information. The fact that Assumption 2 defines Z^∞ as the unique solution of the extended social system demonstrates that it is also the weakest information assumption characterizing Z^∞.

There is yet another difference between the agent's information and that of an outside observer: While the outside observer knows the probabilities of exogenous events, the agents are assumed to have no information at all about these objective probabilities.

4.2 *Weakly Rationalizable Expectations*

Assumption 3: *The individuals are completely informed about solution correspondence, information structure, and about objective probabilities. They have* *no further information about prior beliefs. i.e.*

$$\gamma^i[\zeta] = \{\zeta\}, \quad \gamma^i[\mathcal{P}^I\gamma] = \mathcal{P}^I_\gamma, \quad \gamma^i[\eta] = \{\eta\} \quad \text{and} \quad \gamma^i(\pi^I) = \mathcal{P}^I \quad \forall \pi^i,\ \forall i \in I.$$

The consequences of Assumption 3 are very similar to those of Assumption 1. It also describes a set–valued solution concept. The only difference is that here, marginal probabilities for exogenous events, as induced by consistent beliefs, coincide with objective probabilities.

PROPOSITION 3.A: *Under Assumption 1 there exists a subset $\mathsf{Z} \in \mathcal{W}$, with*

$$\mathsf{Z} = \left\{ z \in W \;\middle|\; \begin{array}{c} \exists \pi^I \in \mathcal{P}^I \text{ such that } z \in \zeta(\pi^I) \\ \text{and } \pi^i = \pi^i_{\mathsf{Z}} \text{ and } h[\pi^i] = \eta \;\; \forall i \in I \end{array} \right\} \tag{9}$$

and $Z_\gamma = \mathsf{Z}$.

PROPOSITION 3.B: *For any subset $\mathsf{Z} \in \mathcal{W}$, for which condition (9) holds, there exists an information structure $\gamma^i[\cdot]$ $\forall i$, such that Assumption 1 holds and $Z_\gamma = \mathsf{Z}$.*

A subset $\mathsf{Z} \in \mathcal{W}$, for which (9) holds, contains all paths of the economy that can arise when all individuals know the objective probabilities and believe that only paths out of this set will occur. It is a set–valued self fulfilling prophecy again, but in difference to condition (6), beliefs are further restricted by objective probabilities.

In the context of non–stochastic economies, belief profiles, that survive the iterative elimination procedure, have been called *rationalizable expectations* by Bernheim (1984), Guesnerie (1992), and Hammond (1993). In order to distinguish two plausible extensions to stochastic economies, we use the term rationalizable expectations only for beliefs consistent with complete information about objective probabilities. McAllister distinguishes weakly and strongy admissible priors depending on the assumed knowledge about these priors. Here, we suggest the same distinction for rationalizable expectations. So, a profile of beliefs $\pi^I \in \mathcal{P}^I$ is called *weakly rationalizable* if there is a subset $\mathsf{Z} \in \mathcal{W}$, for which (9) holds, and $\pi^i = \pi^i_{\mathsf{Z}}$ and $h[\pi^i] = \eta$ $\forall i$.

Define $\hat{Z}^0 := W$ and for all $k \in \mathbb{N}$

$$\hat{Z}^k := \left\{ z \in W \;\middle|\; \begin{array}{c} \exists \pi^I \in \mathcal{P}^I \text{ such that } z \in \zeta(\pi^I) \text{ and} \\ \pi^i = \pi^i_{Z^{k-1}} \text{ and } h[\pi^i] = \eta \;\; \forall i \in I \end{array} \right\} \tag{10}$$

The set $\hat{Z}^k$ contains all paths of the economy that can arise when people, who know objective probabilities, believe that only paths contained in $\hat{Z}^{k-1}$ can occur. Since $\hat{Z}^k \subseteq \hat{Z}^{k-1}$, there exists a limit set $\hat{Z}^\infty := \bigcap_{k=1}^\infty \hat{Z}^k$.

LEMMA 2: *For the set $\hat{Z}^\infty$ condition (9) holds, and any set $\mathsf{Z} \in \mathcal{W}$, for which condition (9) holds, is a subset of $\hat{Z}^\infty$.*

COROLLARY 3: *$\hat{Z}^\infty$ is the union of all subsets $\mathsf{Z} \in \mathcal{W}$, for which condition (9) holds.*

COROLLARY 4: *The set of all weakly rationalizable belief profiles is*

$$\mathcal{P}^I_{WRE} := \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{\hat{Z}^\infty} \text{ and } h[\pi^i] = \eta \quad \forall i \in I \right\}.$$

As the set of weakly admissible priors could be uniquely defined by an information assumption, the same is true for rationalizable expectations. For this we assume that the agents have the same information about prior beliefs that we were able to deduce from Assumption 3.

ASSUMPTION 4: *The individuals have the same information as an outside observer, who assumes that everybody is completely informed about solution correspondence, objective probabilities, and information structure. i.e. for all i*

$$\gamma^i[\zeta] = \{\zeta\}, \quad \gamma^i[\eta] = \{\eta\}, \quad \gamma^i(\pi^I) = \mathcal{P}^I \;\; \forall \pi^I, \quad \text{and}$$

$$\gamma^i[\mathcal{P}^I_\gamma] = \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \exists \mathsf{Z} \in \mathcal{W} \text{ such that (9) holds, } \pi^j = \pi^j_{\mathsf{Z}} \text{ and } h[\pi^j] = \eta \;\; \forall j \right\}.$$

Again, Assumption 4 does not state that the agents' information about the information structure is true, but the belief, that everybody knows the information structure, is self fulfilling.

PROPOSITION 4: *Assumption 4 implies*

$$\gamma^i[\mathcal{P}^I_\gamma] = \mathcal{P}^I_\gamma = \mathcal{P}^I_{WRE} \quad \forall \pi^I, \forall i, \qquad \text{and} \qquad Z_\gamma = \hat{Z}^\infty.$$

Proposition 4 demonstrates that Assumption 4 is equivalent to restricting beliefs to weakly rationalizable expectations, and that $\hat{Z}^\infty$ is the unique solution of the extended social system under this assumption.

4.3 *Strongly Admissible Priors*

Up to know we followed the view that individual expectations are private information. All information about other agents' beliefs were deduced by information about realizations, about the structure of the economy, or about the information structure itself. The was no direct link between the beliefs of two different individuals. Next, we analyze the extremly opposed assumption that all individuals are completely informed about everybody's prior beliefs. Remember that, with complete information about priors, any additional information about the information structure is redundant.

ASSUMPTION 5: *The individuals are completely informed about the solution correspondence and about prior beliefs. They have no information about objective probabilities. i.e.*

$$\gamma^i[\zeta] = \{\zeta\}, \quad \gamma^i(\pi^I) = \{\pi^I\}, \quad \text{and} \quad \gamma^i[\eta] = \mathcal{P}(\mathcal{W}_E).$$

Define

$$P^I_{SAP} := \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{\zeta(\pi^I)} \quad \forall i \in I \right\}.$$

PROPOSITION 5: *Assumption 5 implies $\mathcal{P}^I_\gamma = \mathcal{P}^I_{SAP}$ and*

$$Z_\gamma = \left\{ z \in W \;\middle|\; \exists \pi^I \in \mathcal{P}^I_{SAP} \;\text{ such that }\; z \in \zeta(\pi^I) \right\}.$$

In his studies of Radner's (1979) model, McAllister (1988, 1990) calls a prior belief profile *strongly admissible* if these beliefs are consistent with common knowledge of the structure of the economy and with mutual knowledge of beliefs. Since common knowledge corresponds with knowledge of the information structure in our model, but this additional information is redundant, McAllister's requirements amount to our Assumption 5. Thus, we call a belief profile $\pi^I \in \mathcal{P}^I_{SAP}$ strongly admissible.

4.4 *Strongly Rationalizable Expectations*

As a last point let us analyze the most extreme information assumption that can be studied in our framework:

ASSUMPTION 6: *The individuals are completely informed about the solution correspondence, about prior beliefs, and about objective probabilities. i.e.*

$$\gamma^i[\zeta] = \{\zeta\}, \quad \gamma^i(\pi^I) = \{\pi^I\}, \quad \text{and} \quad \gamma^i[\eta] = \{\eta\}.$$

Define

$$P^I_{SRE} := \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{\zeta(\pi^I)} \text{ and } h[\pi^i] = \eta \quad \forall i \in I \right\}.$$

PROPOSITION 6: *Assumption 6 implies $\mathcal{P}^I_\gamma = \mathcal{P}^I_{SRE}$ and*

$$Z_\gamma = \left\{ z \in W \;\middle|\; \exists \pi^I \in \mathcal{P}^I_{SRE} \text{ such that } z \in \zeta(\pi^I) \right\}.$$

To my knowledge the solution concept $\mathcal{P}^I_{SRE}$ has never been used before now. In non–stochastic economies it coincides with $\mathcal{P}^I_{SAP}$. However, in stochastic economies information about objective probabilities further restricts consistent priors. Following our previous terminology, let us call a belief profile $\pi^I \in \mathcal{P}^I_{SRE}$ *strongly rationalizable*. It should be noted that rational expectations in the definition of Lucas and Prescott (1971) are a subset of $\mathcal{P}^I_{SRE}$. If $\zeta(\pi^I)$ is single–valued for all $\pi^I \in \mathcal{P}^I_{SRE}$ then rational and strongly rationalizable expectations coincide (Heinemann, 1994).

The four solution concepts with decision theoretic foundations studied above are summerized in the following table.

Insert Table 1 here!

5 APPLICATION: TWO–PERIOD EXCHANGE ECONOMY

The purpose of this section is to demonstrate how market economies, that cannot be written in strategic form, can be formalized, in order to apply solution concepts with a decision theoretic foundation. For this purpose there is no need to construct games that mimick the economy nor to introduce generalized games.

The two–period exchange economy in the spirit of Arrow (1983 [1953]), Debreu (1959), and Radner (1972) is perhaps the paradigma of market economies that has found the widest approval and allows for more interpretations and applications than any other model in economic theory. We will take this model as an example to demonstrate the applicability of the discussed solution concepts. We introduce this economy by using the terminology developed in Section 2. In order to keep notation as simple as possible, we limit ourselves by regarding only assets with nominal payoffs. In this we follow Arrow (1983 [1953]) and Cass (1984), while Debreu (1959) and Radner (1972) concentrated on assets with real payoffs. The agents' uncertainty concerns both the exogenous state of the world *and* the price systems reigning in the different states. In this aspect our description is similar to that of Sondermann (1974).

Let us start by introducing the variables considered in the economy: There is a finite number of individuals, $I = \{1, 2, \ldots, |I|\}$, and there are two periods $T = \{1, 2\}$. There is one exogenous variable called "state of the world" and denoted by s. This is a random variable that realizes in the second period. There are finitely many states of the world, so that

$$z_E = z_{E,2} = s \in W_E = \{1, 2, \ldots, |W_E|\}.$$

By $\mathcal{W}_E$ we denote the power set of W_E. Objective probabilities for the different states are given by a relative probability measure $\eta \in \mathcal{P}(\mathcal{W}_E)$. Thus state s occurs with probability $\eta(\{s\}, W_E)$.

There are H consumption goods in the economy and N assets. Assets are traded during the first period. In the second period each asset is associated with a state dependent payoff in units of account. Let $y_{k,s} \in \mathbb{R}$ be the monetary payoff of asset k in state s. The agents have to decide about their net demand for consumption goods in both periods and for assets in the first period. Let x_t^i be agent i's net demand for commodities in period t and a^i her net demand for assets. The decision in the first period is a vector

$$z_{i,1} = (x_1^i, a^i) \in W_{i,1} = \mathbb{R}^{H+N}.$$

A decision in the second period is a vector

$$z_{i,2} = x_2^i \in W_{i,2} = \mathbb{R}^H.$$

Other endogenous variables are commodity and asset prices realizing in period 1,

$$z_{Y,1} = (p_1, q) \in W_{Y,1} = \mathbb{R}^H_+ \times \mathbb{R}^N,$$

and commodity prices of the second period,

$$z_{Y,2} = p_2 \in W_{Y,2} = \mathbb{R}^H_+.$$

Let $\mathcal{W}_{I,t}$ and $\mathcal{W}_{Y,t}$ be the families of Borel subsets of $W_{I,t}$ and $W_{Y,t}$ respectively. A path of the economy is a vector

$$z = \left(s, \left(x_1^i, a^i, x_2^i\right)_{i \in I}, p_1, q, p_2\right) \in W = W_E \times \mathbb{R}^{(H+N+H)|I|} \times \mathbb{R}^H_+ \times \mathbb{R}^N \times \mathbb{R}^H_+.$$

Let us consider an unrestricted set of admissible priors, so that any restriction of posterior beliefs is due to the information structure in this economy.

$$\mathcal{P}^I = \prod_{i \in I} \mathcal{P}(\mathcal{W}).$$

In period t the agents are able to observe the prices of this period. In the second period they can also observe the state of the world. So, the structure of information induced by realizations is described by

$$B_1^i(z_1) := \{\mathsf{z} \in W \mid \mathsf{z}_{Y,1} = (p_1, q)\}$$

and

$$B_2^i(z) := \{\mathsf{z} \in W \mid \mathsf{z}_Y = (p_1, q, p_2) \text{ and } \mathsf{z}_{E,2} = s\}.$$

Let $U^i : \mathbb{R}^{2H} \times W_E \to \mathbb{R}$ be the utility function of household i. $U^i(x_1^i, x_2^i, s)$ is agent i's utility from net trades (x_1^i, x_2^i) when the state of the world in the second period is s. We assume that U^i is strictly increasing and quasiconcave in (x_1^i, x_2^i) for each $s \in W_E$. The individuals are assumed to maximize expected utility with respect to budget constraints that they have to meet with certainty. Here, we demand that in period 1 each agent makes a plan for his future consumption contingent on the future state of the world and on the future price system. This plan must be such that the budget constraint holds in all states and under all future price systems that the agent regards as possible when making her plan. In the second period, when the agent is informed about the state of the world and about the price system, she carries out the plan made for this event.

Let $\pi \in \mathcal{P}(\mathcal{W})$. The term "$f(z) \leq 0 \quad \pi$–sure" is defined by

$$f(z) \leq 0 \qquad \forall z \in \operatorname{supp} \pi.$$

Now, the decision correspondences are given by

$\sigma_1^i(\pi) :=$

$$\operatorname{proj}_{W_{i,1}} \arg\max_{\mathsf{x}_1, \mathsf{a}, \mathsf{x}_2} \left\{ \mathrm{E}\big(U^i(\mathsf{x}_1, \mathsf{x}_2(s, p_2), s) \big| \ \pi\big) \left| \begin{array}{ll} p_1 \, \mathsf{x}_1 + q \, \mathsf{a} \leq 0 & \pi\text{–sure} \\ p_2 \, \mathsf{x}_2(s, p_2) \leq \sum_{k=1}^N \mathsf{a}_k \, y_{k,s} & \pi\text{–sure} \end{array} \right. \right\}$$

and

$$\sigma_2^i(\pi) := \arg\max_{\mathsf{x}_2} \left\{ \mathrm{E}\left(U^i(x_1^i, \mathsf{x}_2, s) \,\middle|\, \pi \right) \,\middle|\, p_2 \, \mathsf{x}_2 \leq \sum_{k=1}^N a_k^i \, y_{k,s} \quad \pi\text{–sure} \right\}.$$

Placing the realization induced information into the decision correspondences yields

$$\sigma_1^i \left(\pi^i_{B_1^i(\bar{z}_1)} \right) = \arg\max_{\mathsf{x}_1, \mathsf{a}} \left\{ \mathrm{E}\left(U^i(\mathsf{x}_1, X_2^i(\mathsf{x}_1, \mathsf{a}, p_2, s), s) \,\middle|\, \pi^i_{B_1^i(\bar{z}_1)} \right) \,\middle|\, \bar{p}_1 \, \mathsf{x}_1 + \bar{q} \, \mathsf{a} \leq 0 \right\},$$

with $X_2^i(\cdot)$ being uniquely[9] defined by

$$X_2^i(x_1^i, a^i, p_2, s) \in \sigma_2^i \left(\pi^i_{B_2^i(z)} \right) = \arg\max_{\mathsf{x}_2} \left\{ U^i(x_1^i, \mathsf{x}_2, s) \,\middle|\, p_2 \, \mathsf{x}_2 \leq \sum_{k=1}^N a_k^i \, y_{k,s} \right\}.$$

The restrictions of the other endogenous variables are such that consistency requirements guarantee market clearing. We define

$$F_1 \left(\left(x_1^i, a^i \right)_{i \in I} \right) := \begin{cases} \mathbb{R}_+^H \times \mathbb{R}^N & \text{if } \sum_{i \in I} (x_1^i, a^i) = 0 \\ \emptyset & \text{otherwise,} \end{cases}$$

$$F_2 \left(s, \left(x_1^i, a^i, x_2^i \right)_{i \in I}, p_1, q \right) := \begin{cases} \mathbb{R}_+^H & \text{if } \sum_{i \in I} x_2^i = 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus,

$$F = \left\{ z \in W \,\middle|\, \sum_{i \in I} (x_1^i, a^i, x_2^i) = 0 \right\}$$

is the economic event of market clearing net demands.

[9]The uniqueness follows from our assumptions on U^i.

Now the description of the economy $\mathcal{M}$ is completed. The solution correspondence of this social system is given by

$$\zeta(\pi^I) = \left\{ z \in W \;\middle|\; \begin{array}{l} (x_1^i, a^i) \in \sigma_1^i \left(\pi^i_{B_1^i(z_1)}\right) \;\; \forall i, \\ x_2^i = X_2^i(x_1^i, a^i, p_2, s) \;\; \forall i, \\ \text{and} \;\; \sum_{i \in I}(x_1^i, a^i, x_2^i) = 0 \end{array} \right\}. \tag{11}$$

$\zeta(\pi^I)$ is the set of all paths of the two–period exchange economy that can arise together with given priors π^I.

The solution concept, usually applied to market economies, is the rational expectations equilibrium (REE). An equilibrium is a function, assigning determinate values to endogenous variables for each state of the world. A REE is a function that is consistent with individual expectations reflecting the belief that the equilibrium is a certain event (Radner, 1979). Thus, the agents are assumed to behave as if they could give a correct forecast of the price systems reigning in each state of the world (Hirshleifer, 1979). In addition, Lucas and Prescott (1971) require that the agents' subjective probabilities for exogenous events are the same as the objective probabilities.

Definition 1: A *rational expectations equilibrium* (REE) of the two–period exchange economy consists of a price system $p_1, q \in \mathbb{R}^H_+ \times \mathbb{R}^N$, a function $\hat{p}_2 : W_E \to \mathbb{R}^H_+$, a system of net demands $(x_1^i, a^i)_{i \in I} \in \mathbb{R}^{|I|(H+N)}$, and a tupel of functions $\hat{x}_2^i : W_E \to \mathbb{R}^H, \; i \in I$, such that

$$(x_1^i, a^i, \hat{x}_2^i) \in \arg\max_{\mathsf{x}_1, \mathsf{a}, \mathsf{x}_2} \left\{ \mathrm{E}\left(U^i(\mathsf{x}_1, \mathsf{x}_2(s), s) \;\middle|\; \eta\right) \;\middle|\; \begin{array}{lcl} p_1 \, \mathsf{x}_1 + q \, \mathsf{a} & \leq & 0 \\ \hat{p}_2(s) \, \mathsf{x}_2(s) & \leq & \sum_{k=1}^N \mathsf{a}_k \, y_{k,s} \;\; \forall s \end{array} \right\}$$

$$\text{for all } i \text{ and} \qquad \sum_{i \in I}(x_1^i, a^i) = 0 \qquad \text{and} \qquad \sum_{i \in I} \hat{x}_2^i(s) = 0 \qquad \forall s \in W_E.$$

Note that a rational expectations equilibrium of this economy is a Walras equilibrium of the corresponding Arrow–Debreu economy for which the preferences over the extended commodity space $\mathbb{R}^H \times \mathbb{R}^{H|W_E|}$ are given by the utility function

$$\mathcal{U}^i\left(x_1^i, (\hat{x}_2^i(s))_{s \in W_E}\right) := \sum_{s \in W_E} \eta(\{s\}, W_E)\, U^i(x_1^i, \hat{x}_2^i(s), s).$$

It has often been claimed that rational expectations equilibria cannot be given a decision theoretic foundation unless there is a unique equilibrium. Here, we find a more moderate requirement:

PROPOSITION 7: *If for any strongly rationalizable belief profile $\pi^I \in \mathcal{P}^I_{SRE}$ there is a vector $z_1 \in W_1$ and a function $\varphi : W_E \to W_{I,2} \times W_{Y,2}$, such that*

$$\zeta(\pi^I) = \{\mathsf{z} \in W \mid \mathsf{z}_1 = z_1 \text{ and } \mathsf{z}_2 \in \operatorname{graph} \varphi\},$$

then Assumption 6 implies that Z_γ is the set of all paths, the economy can take in a rational expectations equilibrium, i.e.

$$\left\{z \in W \;\middle|\; \exists \pi^I \in \mathcal{P}^I_{SRE} \text{ such that } z \in \zeta(\pi^I)\right\}$$

$$= \left\{z \in W \;\middle|\; \exists REE \text{ such that } z = \left(s, (x^i_1, a^i, \hat{x}^i_2(s))_{i \in I}, p_1, q, \hat{p}_2(s)\right)\right\}.$$

Proposition 7 states that the set of rational expectations equilibria is a complete description of all economic events that can occur in the two–period exchange economy when the agents are completely informed about the economy and about prior beliefs, *provided that there is a unique market clearing equilibrium for each strongly rationalizable belief profile*. In this case strongly rationalizable expectations and rational expectations coincide. A unique rational expectations equilibrium is not necessary for this interpretation.

6 CONCLUSION

This paper has demonstrated how assumptions concerning the information, that economic actors have about the structure of the economy, about other actors' expectations, and about the information structure itself, can be embodied in the analysis of an economy. It has been shown that certain assumptions about these kinds of information are equivalent to certain solution concepts. Our analysis extends the literature on the decision theoretic foundation of solution concepts in two directions: The results in this paper apply to economies with a sequential decision structure, like extensive form games, and to economies that cannot be written in strategic form.

The application to the two–period exchange economy in the spirit of Arrow, Debreu, and Radner shows that Walrasian equilibria of this economy can be interpreted as solutions consistent with the assumption that each household is completely informed about the structure of the economy and about the expectations of the others, provided that there is a unique market clearing equilibrium for each strongly rationalizable belief profile. From Hildenbrand (1980, 1983, 1994) we know that uniqueness of equilibria for a given profile of preferences and beliefs can be guaranteed if there is enough diversity in the agents' preferences and that there is empirical evidence for this dispersion. However, the open question remains of how the agents should come to know each other's prior beliefs.

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APPENDIX A: RELATIVE PROBABILITY MEASURES

Let $\mathcal{W}$ be a σ–algebra in a topological space W. A relative probability measure on $\mathcal{W}$ is a function $\pi : \mathcal{W} \times \mathcal{W} \to \mathbb{R}_+ \cup \{\infty\}$, obeying to the following axioms for all $A, B, C \in \mathcal{W}$:

1. $$\pi(\emptyset, W) = 0 \quad \text{and} \quad \pi(\emptyset, A) \in \{0, 1\}.$$
2. $$\pi(A, B) = 0 \quad \Leftrightarrow \quad \pi(B, A) = \infty.$$
3. If $\pi(A, B) \in \mathbb{R}_+$ and $\pi(B, C) \in \mathbb{R}_+$ then $\pi(A, B)\,\pi(B, C) = \pi(A, C)$.
4. If $\pi(A \cup B, C) \in \mathbb{R}_+$ then $\pi(A \cup B, C) = \pi(A, C) + \pi(B, C) - \pi(A \cap B, C)$.
5. If $\pi(\emptyset, B) = 0$ and $\{A_n\}$ is a sequence of events with $A_n \subseteq B \;\; \forall n$ and $A_n \searrow \emptyset$ then $\pi(A_n, B) \searrow 0$.

A relative probability measure π on $\mathcal{W}$ assigns some nonnegative number $\pi(A, B)$ to any pair of events $A, B \in \mathcal{W}$. This number is an expression for the relative likelihood of the two events. $\pi(A, B) = k$ can be interpreted as "event A is k–times as likely as event B".

The axioms are relating relative probabilities to the usual description of probability spaces. With W being interpreted as the certain event and $\emptyset$ as an impossible event, axiom 1 states that the certain event is possible and each event is either possible or impossible. Axioms 2 and 3 are transitivity conditions, axiom 4 requires additivity and axiom 5 continuity of relative probabilities.

Let $\mathcal{P}(\mathcal{W})$ be the set of all relative probability measures on $\mathcal{W}$. The absolute probability of event $A \in \mathcal{W}$ under the measure $\pi \in \mathcal{P}(\mathcal{W})$ is the relative probability of events A and W, $\pi(A, W)$. The function $\mu[\pi] : \mathcal{W} \to [0, 1]$, defined by $\mu[\pi](A) := \pi(A, W)$, is a probability measure on $\mathcal{W}$.

Let $f : \mathcal{W} \to \mathbb{R}$ be some $\mathcal{W}$–measurable function. The expected value of $f(z)$ under the relative probability measure $\pi \in \mathcal{P}(\mathcal{W})$ depends only on absolute probabilities,

$$\mathrm{E}(f(z) \mid \pi) := \mathrm{E}(f(z) \mid \mu[\pi]),$$

where the second expression stands for the usual definition of the expected value of $f(z)$ given the probability measure $\mu[\pi]$.

The support of a relative probability measure $\pi \in \mathcal{P}(\mathcal{W})$ is the certain event reduced by all impossible events:

$$\operatorname{supp} \pi := W \backslash \left(\bigcup B \;\middle|\; \pi(\emptyset, B) = 1 \right).$$

The reason, why we use relative probabilities in this paper, is their advantage in defining unique conditional probabilities for all events. Remember, for a given probability measure μ conditional probabilities are uniquely defined up to conditions of measure zero. For $\mu(A) = 0$ any probability measure μ_A with $\mu_A(A) = 1$ is a conditional probability measure for μ under condition A.

A relative probability measure $\pi \in \mathcal{P}(\mathcal{W})$ defines unique conditional probabilities for all conditions. The relative probability of events A and B under condition C is

$$\pi_C(A, B) := \pi(A \cap C, B \cap C).$$

If C is a possible event, i.e. $\pi(\emptyset, C) = 0$, then $\pi_C \in \mathcal{P}(\mathcal{W})$. Otherwise

$$\pi_C(A, B) = \pi_\emptyset(A, B) = 1 \qquad \forall A, B \in \mathcal{W}.$$

$\pi_\emptyset$ is closing $\mathcal{P}(\mathcal{W})$ with respect to the operation of Bayesian updating. It is a neutral element for this operation and obeys to all axioms except for $\pi_\emptyset(\emptyset, W) = 1$. For technical reasons we need $\bar{\mathcal{P}}(\mathcal{W}) := \mathcal{P}(\mathcal{W}) \cup \{\pi_\emptyset\}$.

The conditional expected value of some $\mathcal{W}$–measurable function $f : W \to \mathbb{R}$ for relative probabilities π and condition C is defined as

$$\mathrm{E}(f(z) \mid \pi_C) := \mathrm{E}(f(z) \mid \mu[\pi_C]).$$

This definition is unique even if C is a null–event. It is only required that event C is possible.

Relative probability measures are closely related to Rényi's (1976 [1955, 1956]) concept of conditional probability spaces, to Myerson's (1986) conditional probability systems, and to lexicographical probability systems, as introduced by Blume, Brandenburger, and Dekel (1991). For the relation between these concepts see Heinemann (1994). More details on relative probability measures and first applications can be found in Kohlberg and Reny (1992), Swinkels (1994), and Heinemann (1994).

APPENDIX B: PROOFS

Proof of Proposition 1.A: Using (3') and (5) Assumption 1 implies

$$C^i(\pi^I) = \left\{z \in W \;\middle|\; \exists \mathsf{p}^I \in \mathcal{P}^I_\gamma, \text{ such that } z \in \zeta(\mathsf{p}^I)\right\} = Z_\gamma \qquad \forall \pi^I,\ \forall i.$$

Using (4) we get $\mathcal{P}^I_\gamma = \left\{\pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{Z_\gamma} \quad \forall i \in I\right\}$. Now (5) shows that condition (6) holds for $Z_\gamma = \mathsf{Z}$. *Q.E.D.*

Proof of Proposition 1.B: Let $\mathsf{Z} \in \mathcal{W}$ be a subset for which condition (6) holds. Using (3), the information structure

$$\gamma^i[\zeta] = \{\zeta\}, \quad \gamma^i[\eta] = \mathcal{P}(\mathcal{W}_E), \quad \gamma^i[\mathcal{P}^I_\gamma] = \left\{\mathsf{p}^I \in \mathcal{P}^I \;\middle|\; \mathsf{p}^j = \mathsf{p}^j_{\mathsf{Z}} \quad \forall j \in I\right\} \text{ and } \gamma^i[\pi^i] = \mathcal{P}^I$$

for all π^I and i implies

$$C^i(\pi^I) = \left\{z \in W \;\middle|\; \exists \mathsf{p}^I \in \mathcal{P}^I \text{ such that } \mathsf{p}^j = \mathsf{p}^j_{\mathsf{Z}}\ \forall j \text{ and } z \in \zeta(\mathsf{p}^I)\right\} = \mathsf{Z}$$

for all $i \in I$ and for all $\pi^I \in \mathcal{P}^I$. Therefore

$$\mathcal{P}^I_\gamma = \left\{\pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{\mathsf{Z}} \quad \forall i \in I\right\} = \gamma^i[\mathcal{P}^I_\gamma] \qquad \forall i \in I.$$

Hence, assumption 1 holds. From the proof of part A we know already that $C^i(\pi^I) = Z_\gamma$ in this case. Therefore $Z_\gamma = \mathsf{Z}$. *Q.E.D.*

Proof of Lemma 1:

(i) Consider an arbitrary $z \in Z^\infty$. By definition (8), $z \in Z^k \quad \forall k$. Since $Z^k \subseteq Z^{k-1}$ for all k, there exists a belief profile $\pi^I \in \mathcal{P}^I$, such that $\pi^i = \pi^i_{Z^k} \quad \forall k, \forall i$ and $z \in \zeta(\pi^I)$. Now, $\pi^i = \pi^i_{Z^\infty}$ for all i, and therefore

$$Z^\infty \subseteq \left\{ z \in W \;\middle|\; \exists \pi^I \in \mathcal{P}^I \text{ such that } \pi^i = \pi^i_{Z^\infty} \;\; \forall i \text{ and } z \in \zeta(\pi^I) \right\}.$$

(ii) Next consider an arbitrary belief profile π^I with $\pi^i = \pi^i_{Z^\infty}$ for all i. It is true that $\pi^i = \pi^i_{Z^k}$ forall k and i. Hence, $\zeta(\pi^I) \subseteq Z^k$ for all k, and therefore, $\zeta(\pi^I) \subseteq Z^\infty$.

(iii) The Combination of (i) and (ii) shows that (6) holds for $\mathsf{Z} = Z^\infty$.

(iv) Let $\mathsf{Z} \in \mathcal{W}$ be an event fulfilling (6). If there is a k, such that $\mathsf{Z} \subseteq Z^k$, then for each $z \in \mathsf{Z}$ there is a belief profile $\pi^I \in \mathcal{P}^I$, with $\pi^i = \pi^i_{\mathsf{Z}} = \pi^i_{Z^k}$ for all i, and $z \in \zeta(\pi^I)$. Now (7) shows that $\mathsf{Z} \subseteq Z^{k+1}$. Since $\mathsf{Z} \subseteq Z^0 = W$, it is true that $\mathsf{Z} \subseteq Z^k$ for all k, hence $\mathsf{Z} \subseteq Z^\infty$. *Q.E.D.*

Proof of Corollary 1: Corollary 1 follows immediately from Lemma 1. *Q.E.D.*

Proof of Corollary 2: Corollary 2 follows immediately from Corollary 1. *Q.E.D.*

Proof of Proposition 2: By definition of C^i, given in (3'), and by Corollary 1, Assumption 2 implies

$$C^i(\pi^I) = \left\{ z \in W \;\middle|\; \begin{array}{l} \exists \mathsf{Z} \in \mathcal{W},\ \exists \mathsf{p}^I \in \mathcal{P}^I, \text{ such that} \\ \text{(6) holds, } \mathsf{p}^j = \mathsf{p}^j_{\mathsf{Z}} \;\; \forall j \in I, \text{ and } z \in \zeta(\mathsf{p}^I) \end{array} \right\}.$$

$$= \{ z \in W \mid \exists \mathsf{Z} \in \mathcal{W}, \text{ such that (6) holds and } z \in \mathsf{Z} \}.$$

$$= \bigcup \mathsf{Z} \mid \text{(6) holds } = Z^\infty.$$

Now the set of consistent priors, defined in (4), is

$$\mathcal{P}^I_\gamma = \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{Z^\infty} \quad \forall i \in I \right\} = \mathcal{P}^I_{WAP}. \tag{12}$$

By using Corollary 1 again, we yield

$$\gamma^i[\mathcal{P}^I_\gamma] = \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^j = \pi^j_{Z^\infty} \quad \forall j \in I \right\} \qquad \forall i \in I.$$

This shows that the information stated in Assumption 2 is true, i.e.

$$\gamma^i[\mathcal{P}^I\gamma] = \mathcal{P}^I_\gamma = \mathcal{P}^I_{WAP} \qquad \forall i \in I.$$

From Lemma 1 we know that (6) holds for $\mathsf{Z} = Z^\infty$, so that equations (5) and (12) imply $Z_\gamma = Z^\infty$. *Q.E.D.*

PROOF OF PROPOSITION 3: The structure of this proof is very similar to that of Proposition 1. Placing Assumption 3 into equations (3'), (4), and (5) yields part A. Using (3') the information structure

$$\gamma^i[\zeta] = \{\zeta\}, \quad \gamma^i[\eta] = \{\eta\}, \quad \gamma^i(\pi^I) = \mathcal{P}^I \ \ \forall \pi^I, \ \text{ and}$$

$$\gamma^i[\mathcal{P}^I\gamma] = \left\{\pi^I \in \mathcal{P}^I \;\middle|\; \pi^j = \pi^j_{\mathsf{Z}} \ \text{ and } \ h[\pi^j] = \eta \quad \forall j \right\} \quad \forall i,$$

where Z fulfills condition (9), implies $C^i(\pi^I) = \mathsf{Z}$ for all i and π^I. Then

$$\mathcal{P}^I_\gamma = \left\{\pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{\mathsf{Z}} \ \text{ and } \ h[\pi^i] = \eta \quad \forall i \in I \right\} = \gamma^i(\pi^I) \qquad \forall \pi^I, \ \forall i.$$

Hence, Assumption 3 holds. Now $C^i(\pi^I) = Z_\gamma$ for all i and π^I, and therefore $Z_\gamma = \mathsf{Z}$. *Q.E.D.*

PROOF OF LEMMA 2: This proof is closely related to that of Lemma 1. (i) For any $z \in \hat{Z}^\infty$ there exists a belief profile $\pi^I \in \mathcal{P}^I$, such that

$$z \in \zeta(\pi^I) \ \text{ and } \ \pi^i = \pi^i_{\hat{Z}^\infty} \ \text{ and } \ h[\pi^i] = \eta \quad \forall i.$$

(ii) On the other hand, for any belief profile π^I, with $\pi^i = \pi^i_{\hat{Z}^\infty}$ and $h[\pi^i] = \eta$ for all i, it is true that $\pi^i = \pi^i_{\hat{Z}^k}$ forall k and i. Hence, $\zeta(\pi^I) \subseteq Z^k$ for all k, and therefore, $\zeta(\pi^I) \subseteq \hat{Z}^\infty$.

(iii) Combining (i) and (ii) shows that (9) holds for $\mathsf{Z} = \hat{Z}^\infty$.

(iv) Let $\mathsf{Z} \in \mathcal{W}$ be an event fulfilling (9). If there is a k, such that $\mathsf{Z} \subseteq \hat{Z}^k$, then for each $z \in \mathsf{Z}$ there is a belief profile $\pi^I \in \mathcal{P}^I$ with $\pi^i = \pi^i_{\mathsf{Z}} = \pi^i_{\hat{Z}^k}$ and $h[\pi^i] = \eta$ for all i and $z \in \zeta(\pi^I)$. Now (10) shows that $\mathsf{Z} \subseteq \hat{Z}^{k+1}$. Since $\mathsf{Z} \subseteq \hat{Z}^0 = W$, it is true that $\mathsf{Z} \subseteq \hat{Z}^k$ for all k, hence $\mathsf{Z} \subseteq \hat{Z}^\infty$. *Q.E.D.*

PROOF OF COROLLARY 3: Corollary 3 follows immediately from Lemma 2. *Q.E.D.*

PROOF OF COROLLARY 4: Corollary 4 follows immediately from Corollary 3. *Q.E.D.*

Proof of Proposition 4: This proof has the same structure as that of Proposition 2. Using (3') and Corollary 3, Assumption 4 implies

$$C^i(\pi^I) = \left\{ z \in W \;\middle|\; \begin{array}{l} \exists \mathsf{Z} \in \mathcal{W},\ \exists \mathsf{p}^I \in \mathcal{P}^I, \ \text{ such that (9) holds,} \\ \mathsf{p}^j = \mathsf{p}^j_{\mathsf{Z}} \ \text{ and } \ h[\mathsf{p}^j] = \eta \ \ \forall j \in I, \ \text{ and } \ z \in \zeta(\mathsf{p}^I) \end{array} \right\}.$$

$$= \{ z \in W \ \mid \exists \mathsf{Z} \in \mathcal{W}, \ \text{ such that (9) holds and } \ z \in \mathsf{Z} \}.$$

$$= \bigcup \mathsf{Z} \mid \text{(9) holds } = \hat{Z}^\infty.$$

Now the set of consistent priors, defined in (4), is

$$\text{(13)} \qquad \mathcal{P}^I_\gamma = \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{\hat{Z}^\infty} \ \text{ and } \ h[\pi^i] = \eta \quad \forall i \in I \right\} = \mathcal{P}^I_{WRE}.$$

By using Corollary 3 again, we yield

$$\gamma^i[\mathcal{P}^I\gamma] = \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^j = \pi^j_{\hat{Z}^\infty} \ \text{ and } \ h[\pi^j] = \eta \quad \forall j \in I \right\} \qquad \forall i \in I.$$

This shows that the information stated in Assumption 4 is true, i.e.

$$\gamma^i[\mathcal{P}^I\gamma]) = \mathcal{P}^I_\gamma = \mathcal{P}^I_{WRE} \qquad \forall i \in I.$$

From Lemma 2 we know that (9) holds for $\mathsf{Z} = Z^\infty$, so that equations (5) and (13) imply $Z_\gamma = \hat{Z}^\infty$. *Q.E.D.*

Proof of Proposition 5: Under Assumption 5 equation (3) implies $C^i(\pi^I) = \zeta(\pi^I)$ for all π^I and i. Then (4) implies

$$\mathcal{P}^I_\gamma = \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{\zeta(\pi^I)} \quad \forall i \in I \right\} = \mathcal{P}^I_{SAP}.$$

By using (5), we get

$$Z_\gamma = \left\{ z \in W \;\middle|\; \exists \pi^I \in \mathcal{P}^I \ \text{ such that } \ \pi^i = \pi^i_{\zeta(\pi^I)} \ \forall i \ \text{ and } \ z \in \zeta(\pi^I) \right\}.$$

Q.E.D.

Proof of Proposition 6: This proof is similar to the previous one. Under Assumption 6 equation (3) implies $C^i(\pi^I) = \zeta(\pi^I)$ for all π^I and i. Then (4) implies

$$\mathcal{P}^I_\gamma = \left\{ \pi^I \in \mathcal{P}^I \;\middle|\; \pi^i = \pi^i_{\zeta(\pi^I)} \ \text{ and } \ h[\pi^i] = \eta \quad \forall i \in I \right\} = \mathcal{P}^I_{SRE}.$$

Finally, by using (5), we get

$$Z_\gamma = \left\{ z \in W \;\middle|\; \exists \pi^I \in \mathcal{P}^I_{SRE} \;\text{ such that }\; z \in \zeta(\pi^I) \right\}.$$

Q.E.D.

Proof of Proposition 7: Consider a belief profile $\pi^I \in \mathcal{P}^I_{SRE}$, for which there exists a vector $z_1 \in W_1$ and a function $\varphi : W_E \to W_{I,2} \times W_{Y,2}$, such that

$$\zeta(\pi^I) = \{\mathsf{z} \in W \mid \mathsf{z}_1 = z_1 \;\text{ and }\; \mathsf{z}_2 \in \operatorname{graph} \varphi\}. \tag{14}$$

It has to be shown that $(z_1; \varphi) = ((x_1^i, a^i)_{i \in I}, p_1, q; (\hat{x}_2^i)_{i \in I}, \hat{p})$ is a rational expectations equilibrium. From (11) and (14) we get

$$(x_1^i, a^i) \in \sigma_1^i\left(\pi^i_{B_1^i(z_1)}\right) \quad \text{ and } \quad \hat{x}_2^i(s) = X_2^i(x_1^i, a^i, \hat{p}_2(s), s) \quad \forall i,\ \forall s,$$

$$\sum_{i \in I} (x_1^i, a^i) = 0 \quad \text{ and } \quad \sum_{i \in I} \hat{x}_2^i(s) = 0 \quad \forall s.$$

Note that $\pi^i = \pi^i_{\zeta(\pi^I)}$ for all i. Therefore

$$\pi^i(\{z_1\} \times \operatorname{graph} \varphi, W) = 1 \qquad \forall i.$$

This implies

$$(x_1^i, a^i) \in \arg\max_{\mathsf{x}_1, \mathsf{a}} \left\{ E(U^i(\mathsf{x}_1, \hat{x}_2^i(s), s) \mid h[\pi^i]) \;\middle|\; p_1 \mathsf{x}_1 + q\, \mathsf{a} \le 0 \right\} \tag{15}$$

and

$$\hat{x}_2^i(s) \in \arg\max_{\mathsf{x}_2} \left\{ U^i(x_1^i, \mathsf{x}_2, s) \;\middle|\; p_2 \mathsf{x}_2 \le \sum_{k=1}^{N} a_k^i\, y_{k,s} \right\} \tag{16}$$

for all i. Combining (15) and (16) and using $h[\pi^i] = \eta$ shows that (z_1, φ) is indeed a rational expectations equilibrium. *Q.E.D.*

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Footnotes

1 This paper is based on my PhD dissertation (Heinemann, 1994).

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3 For a discussion of the common prior assumption see Morris (1993).

4 For an exposition of knowledge operators and their relation to information see Osborne and Rubinstein (1994).

5 The impossibility to regard false information is closely related to the impossibility theorem of Basu (1990).

6 $\bar{\mathcal{P}}(\mathcal{W}) := \mathcal{P}(\mathcal{W}) \cup \{\pi_\emptyset\}$, where $\pi_\emptyset(A, B) := 1$ for all $A, B \in \mathcal{W}$.

7 See Appendix A.

8 For an example see Section 5.

9 The uniqueness follows from our assumptions on U^i.

	complete information about economy and information structure	
	unknown objective probabilities	known objective probabilities
unknown prior beliefs	weakly admissible priors $\pi^i = \pi^i_{Z^\infty} \quad \forall i$	weakly rationalizable expectations $\pi^i = \pi^i_{\hat{Z}^\infty}$ and $h[\pi^i] = \eta \ \forall i$
known prior beliefs	strongly admissible priors $\pi^i = \pi^i_{\zeta(\pi^I)} \quad \forall i$	strongly rationalizable expectations $\pi^i = \pi^i_{\zeta(\pi^I)}$ and $h[\pi^i] = \eta \ \forall i$

TABLE 1 *Consistency requirements for beliefs under different information assumptions.*

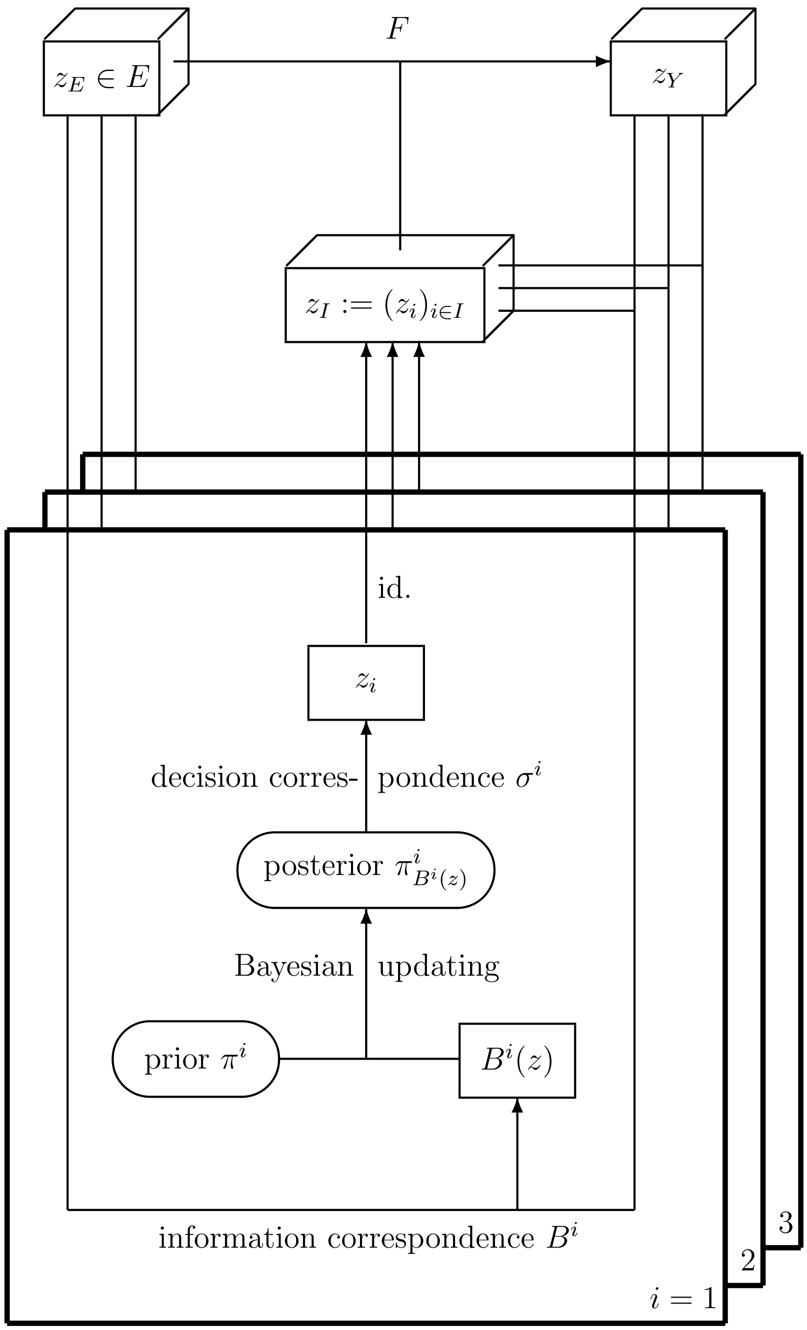

FIGURE 1 *Illustration of the different types of variables and the dependencies between them for a static economy with three agents.*