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# Currency Crises with the Threat of an Interest Rate Defence<sup>☆</sup>

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## Abstract

While virtually all currency crisis models recognise that the decision to abandon a peg depends on how tenaciously policy makers defend it, this is seldom modelled explicitly. We add the threat of an interest rate defence to the global game model of [Morris and Shin](#) (*American Economic Review* 88, 1998). With an endogenous defence, actions of speculators may become strategic substitutes instead of the usual complements. Nevertheless, our generalised model remains tractable and has a unique threshold equilibrium. It provides additional insights. For instance, the threat of an interest rate defence makes speculation riskier and this may be sufficient to keep speculators out when fundamentals are still relatively strong.

*Keywords:* Currency Crisis, Interest Rate Defence, Global Game, Strategic Substitutes.

*JEL Codes:* E58, F31, F33, G15.

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## 1. Introduction

Virtually all modern currency crisis models recognise that the decision to abandon a currency peg depends on how tenaciously the policy maker is willing to defend it. Yet this is seldom modelled explicitly. In the model in this paper, the policy maker is willing to increase the interest rate to offset the build up of speculative pressure on exchange markets. This increases the

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financing costs of speculators, and makes speculation riskier. We add this threat of an interest rate defence to [Morris and Shin's \(1998\)](#) global game model of the onset of a speculative attack, which has proved fruitful to investigate a wide variety of topics on crises (see e.g. [Heinemann and Illing, 2002](#); [Corsetti et al., 2004](#); [Cukierman et al., 2004](#); [Goldstein, 2005](#); [Corsetti et al., 2006](#); [Guimaraes and Morris, 2007](#); [Cornand and Heinemann, 2009](#)).

Including the threat of an interest rate defence leads to a crucial difference with standard global game models. There, a speculator's payoff from attacking a currency weakly increases when more speculators attack, no matter whether the attack is successful or not. Hence, the actions of speculators are strategic complements. In our model, a larger amount of speculators attacking the peg may lead to a harsher defence by the policy maker, which means that the costs of speculation are higher. In case the attack is unsuccessful, actions are strategic substitutes instead of complements.

Our model has a unique equilibrium in threshold strategies, as familiar from global games, but the threat of a policy response affects the equilibrium. As in standard models, strategic uncertainty about the odds of a successful attack plays an important role in our model. Yet the threat of an interest rate defence makes speculation even riskier and may be sufficient to keep speculators out when fundamentals are still relatively strong, even without actually setting a high interest rate. A relatively large deterioration of fundamentals is necessary to trigger a speculative attack, which may be accompanied by a spike in the interest rate due to the interest rate defence. This outcome of our model appears to be broadly consistent with a stylised fact about past currency crises, which suggests that a currency can remain overvalued during a protracted and by and large tranquil period before its peg finally collapses.

The robustness of global games to the inclusion of more realistic elements such as an interest rate defence has sometimes been questioned. For instance, [Chamley \(2003\)](#) challenges several aspects of global games, one of them being the role of small and fixed transaction costs. Indeed, although it is common in the global game literature to assume the costs of an unsuccessful speculative attack are fixed and known in advance, this is actually not the most realistic setting. Consider the following strategy that is widely used in reality. A speculator believes the weak currency will depreciate and instructs her broker to sell weak for strong currency spot.<sup>1</sup> Spot traded currency is due for delivery in two business days, yet the speculator has no interest in

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<sup>1</sup>Here, as in the rest of our paper, we focus on speculation in the spot market. However, the observation that the

actual delivery. Therefore, each night the broker will automatically roll-over the position via a so called tomorrow-next procedure, closing it out at the end of the day on her behalf and re-opening it the next day. The roll-over costs are determined by the prevailing overnight interest rate differential, and are debited from (or credited into) the speculator's margin account. Hence, for as long as the position remains open (typically a few weeks, and the exact time span is unknown to speculators when they start), the speculator is exposed to interest rate risk—besides exchange rate risk.

Our model shows how global games can account for this. Its setup is close to that of [Morris and Shin \(1998\)](#). But rather than assuming that the costs of speculation are known *ex-ante*, it emphasises the additional uncertainty stemming from interest rate risk that results from the policy maker's readiness to defend the weak currency.

We thus contribute to a closely related literature that incorporates interest rates in global game models. This literature typically models the interplay between interest rates and the behaviour of speculators using a general equilibrium approach. [Angeletos and Werning \(2006\)](#), [Hellwig et al. \(2006\)](#) and [Tarashev \(2007\)](#) study global game models in which speculators can take positions in the weak currency conditional on the behaviour of the interest rate during the attack, and a Walrasian auctioneer then clears the market by setting the interest rate. Essentially, agents are able to condition their decision on the market clearing interest rate that is, in fact, the result of their joint behaviour. It is as if agents can observe the interest rate that will result from the attack before they commit to their decision.

This approach is fine for many economic applications, but less obviously so in the context of speculative attack models. First, in these models, the market clearing interest rate necessarily reflects the *outcome* of the speculative attack. If agents can predict the outcome using the interest rate, the element of strategic uncertainty associated with the speculation decision is substantially reduced. In reality, it is doubtful that the interest rate is informative enough as to substantially reduce strategic uncertainty. Second, in this approach there may be multiple market clearing interest rates, in each of which speculators conditionally demand the amount of weak currency consistent with the model's outcome. Multiple equilibria may reduce the predictive power of the model.

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costs of a speculative position can be volatile carries over to derivative positions, as they are typically priced on a mark-to-market basis.

In our paper, we adopt an opposite approach and model the interest rate as an uncertain policy response at the onset of the attack. The key difference between the models above and our own, discussed in detail in subsection 4.5, is the inability of speculators to condition decisions on the market clearing interest rate. Our approach abstracts from the informational role of the interest rate during the attack, and allows us to focus on the effect of interest rate risk on equilibrium instead. As we have indicated above, interest rate risk during the attack is a real-life phenomenon that affects the value of positions of speculators.

Another advantage of our model is its unique threshold equilibrium. Because of our endogenous interest rate, none of the usual equilibrium results for global games apply directly to our model. We extend the class of global game models for which a unique threshold equilibrium is known to exist. Our model's payoff structure is most closely related to a global game bank run model of [Goldstein and Pauzner \(2005\)](#) (and relatedly, [Dasgupta, 2004](#)), which also deals with strategic substitutes. Yet the threat of an interest rate defence makes the impact of changes in fundamentals on payoffs non-monotonic, which is not the case in their model. Under the additional assumption that information in the global game is characterised by a uniform noise distribution—the original assumption in [Morris and Shin's](#) article—the threshold equilibrium is the unique equilibrium of our model among all possible types of equilibria.

There are other studies about how an interest rate may help defend during a speculative attack. Some focus on how the interest rate affects the sustainability of government policies, in particular fiscal policy. Examples are [Flood and Jeanne \(2005\)](#) and [Lahir and Végh \(2003; 2007\)](#). This perspective suggests that a policy maker will be able and willing to go to greater lengths to defend the peg under stronger fundamentals. This trade-off is also captured in our model, where fundamentals affect the effectiveness of the interest rate defence.

Another strand of literature is based on the signalling model of [Drazen \(2000\)](#), in which the central bank sets the interest rate before speculators move. Speculators interpret the interest rate as a signal that transmits information about a policy maker's intentions, and this may lead to multiple equilibria. Global game variants of this model have been studied by [Angeletos et al. \(2006\)](#) and [\(2007\)](#). The question of signalling intentions in earlier stages is distinct from the issue considered in our paper. An important reason for governments to manipulate the interest rate in reality is to try to stabilise the spot exchange market. As an attack unfolds, the optimal level of the interest rate is determined endogenously by attempts to calm the market. It

is no longer set just to signal intentions.

The rest of the text is structured as follows. Section 2 explains how a defence policy affects the payoffs of speculators, and why this may lead to strategic substitutes. Section 3 develops a model that incorporates this feature in a global game model. We solve for the model's equilibrium and discuss its properties in section 4. In section 5 we conclude. Proofs are found in the appendix.

## 2. The Costs of Speculation and Strategic Substitutes

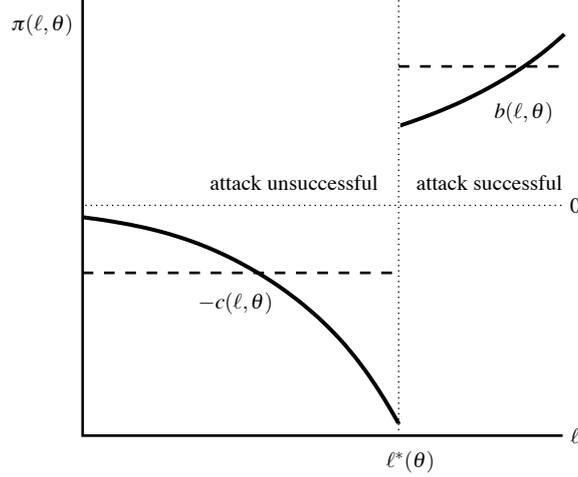
In a standard global game currency crisis model the speculator's payoff to attacking a currency weakly increases when more speculators attack, no matter whether the attack is successful or not. Hence, the actions of speculators are strategic complements.

However, the assumption of strategic complementarities misses an important dimension of currency crises in reality. During currency crises, policy makers try to defend the currency. Particularly, they raise the interest rate. In case the attack is unsuccessful, the more speculators attack, the harsher will be the interest rate defence. Since the interest rate influences the costs of speculation, a harsher defence implies higher costs. Thus, if an attack is unsuccessful, a speculator is *worse* off when more speculators attack, so their actions become strategic substitutes instead of complements.

In this paper, we analyse a global game model with a richer payoff structure that allows for the presence of strategic substitutes. For concreteness, we focus on the threat of an interest rate defence, though our approach will also apply to other defence policies that raise the costs of speculation.

To bring out the differences with standard models more clearly, let  $c(\ell, \theta)$  denote the speculator's costs of an unsuccessful attack, where  $\ell$  is the amount of speculative pressure, that is, the fraction of speculators who attack the currency, and  $\theta$  is the state of economic fundamentals. A higher  $\theta$  means stronger fundamentals. Similarly, write  $b(\ell, \theta)$  for the benefit from a successful attack, that is, from devaluation. The peg collapses if speculative pressure exceeds  $\ell^*(\theta)$ , which is strictly increasing in  $\theta$ , as usual. The payoff from attacking versus not attacking is then

$$\pi(\ell, \theta) := \begin{cases} -c(\ell, \theta) & \text{if } \ell \leq \ell^*(\theta), \\ b(\ell, \theta) & \text{if } \ell > \ell^*(\theta). \end{cases} \quad (1)$$



dashed: payoffs with strategic complementarities; solid: generalisation with strategic substitutes

Figure 1: Payoff functions in the global game currency crisis model

In the standard global game framework,  $\pi$  would be weakly increasing in both arguments. The increasing nature of  $\pi$  in  $\ell$  for given  $\theta$  implies strategic complementarities. This dependence is exemplified by the discontinuous dashed line in figure 1, using the step function that is commonly considered in currency crisis models, following [Morris and Shin \(1998\)](#). The costs part is typically interpreted as a fixed transaction cost or an exogenous interest rate differential.

Our benefit function  $b$  is rather standard. We assume that  $b > 0$ , so that speculation is profitable when the peg actually collapses, which appears consistent with how speculative attacks develop in reality.<sup>2</sup> Moreover,  $b$  is weakly increasing in  $\ell$  and strictly decreasing in  $\theta$ . This reflects that both a larger attack and worse fundamentals typically cause larger devaluations and thus increase profits.

We generalise the standard model regarding the costs function  $c$ . We allow  $c > 0$  to be weakly increasing in  $\ell$ , as motivated earlier by the endogenous interest rate defence. This is visualised by the left solid line in figure 1. It destroys the usual strategic complementarities. In addition, we allow  $c$  to be weakly decreasing in  $\theta$ . This captures that weaker fundamentals (smaller values of  $\theta$ ) are associated with reduced demand for the currency on the foreign exchange market. This necessitates a more vigorous defence, which entails higher costs of speculation (given  $\ell$ ).

In sum, to capture the effects of the interest rate defence on the payoff  $\pi$ , we need to allow for a change of sign of the derivatives  $\pi_\ell$  and  $\pi_\theta$  at the point  $\ell^*(\theta)$ . These changes of signs

<sup>2</sup>This assumption may be dropped if  $b(\ell, \theta)$  is constant in  $\ell$ , as in [Morris and Shin \(1998\)](#) and many others.

prevent us from applying the standard theorems in the global game literature that prove the existence and uniqueness of equilibrium. For instance, the theorems in [Morris and Shin \(2003\)](#) require constant signs over the whole domain for  $\ell$  and  $\theta$ . An alternative would be to employ the theorem of [Goldstein and Pauzner \(2005\)](#), as it allows for a change in the effect of  $\ell$ . However, they focus on a different setting (bank runs) and in their model the effect of  $\theta$  does not change sign. Since their proofs depend on this feature, we cannot apply their theorem either. It is no longer obvious that a unique threshold equilibrium exists at all.

To deal with the presence of strategic substitutes, we impose some additional assumptions to solve our model. Let  $b$  and  $\ell^*$  be continuously differentiable,  $c$  be twice continuously differentiable and impose three additional conditions on  $c$ :

$$c(\ell^*(\theta), \theta) \text{ is weakly increasing in } \theta, \quad (2)$$

$$c_{\ell\ell} \geq 0, \quad (3)$$

$$\theta < \theta' \text{ and } c(\ell, \theta) = c(\ell', \theta') \implies c_{\ell}(\ell, \theta) \geq c_{\ell}(\ell', \theta'). \quad (4)$$

Condition (2) says that the costs of speculation when speculative pressure equals  $\ell^*(\theta)$  (so that speculative pressure is at the margin needed to bring down the peg) weakly increase when fundamentals become stronger. Intuitively, the policy maker is prepared to go to greater lengths to defend the peg under stronger fundamentals. Regarding condition (3), compare two unsuccessful attacks, such that speculative pressure is higher in the second situation. Convexity condition (3) says that for a marginal increase in pressure, a greater intensification of the defence ensues in the second case. For condition (4), compare two unsuccessful attacks with equal costs of speculation, but such that in the second situation fundamentals are stronger. The condition states that in the second case a marginal increase in speculative pressure causes a smaller intensification of the defence. It may be viewed, loosely, as a concavity condition.<sup>3</sup> If the inequality in the consequent of (4) holds strictly, condition (3) may be dropped.<sup>4</sup> Observe that the often-used step function trivially satisfies all three conditions.

How natural are these conditions? In the next section, we develop a concrete global game model of speculative attacks and show that all three follow naturally from the model structure. The main assumption driving this is that the policy maker uses the interest rate to maintain

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<sup>3</sup>Mathematically, the derivative  $c_{\ell}$  weakly decreases with  $\theta$ , and thus with  $\ell$ , along the isocost curves of  $c$ .

<sup>4</sup>See lemma A 1 for details.

equilibrium on the spot exchange market at a fixed exchange rate, so that the interest rate, and thus the costs of speculation, are determined endogenously. We will prove that our global game model indeed has a unique threshold equilibrium, using only the assumptions in this section on the payoff structure in addition to the standard global game assumptions such as the existence of dominance regions.

### 3. The Model

In this section we develop a stylised global game model of a speculative attack on a currency peg. We incorporate the main features of an endogenous interest rate defence and show this yields the payoff structure with strategic substitutes of section 2. In section 4 we solve the model.

Assume a weak currency has a fixed exchange rate *vis-à-vis* a stronger currency. There is a continuum of identical, risk-neutral speculators, indexed on the interval  $[0, 1]$ , each with wealth equal to one unit of the weak currency. They decide whether to speculate against the weak currency or not. The costs of speculation are given by the interest rate  $r$  (which may be interpreted as the interest differential *vis-à-vis* the stronger currency). A policy maker decides whether to maintain the peg by using  $r$ , taking account of the effects of  $r$  on the spot exchange market and of the costs of a high interest rate to the economy.

#### 3.1. The Time Structure

At the start of the game, the state of economic fundamentals  $\theta$  materialises. As in [Morris and Shin \(1998\)](#), the policy maker observes  $\theta$  perfectly, but speculators only observe it noisily. Next, the speculators and policy maker act.

The ordering of their decisions is where we deviate slightly from standard models. Usually, the speculators act before the policy maker, who then decides whether to devalue or not after observing speculative pressure. In our model this sequential setup would be problematic, since we focus on the threat of an interest rate defence instead of just a devaluation decision. By setting the interest rate, the policy maker decides whether the foreign exchange market clears at the fixed rate or not (similar to the devaluation decision), but it is also an instrument that directly affects the speculators' costs. This latter function is at odds with the ordering above, and reversing the ordering is at variance with the first function.

To capture both effects in our model, we let the speculators and policy maker act simultaneously. In equilibrium they take each others' decisions into account. This approach allows us to

integrate the threat of an interest rate defence into the global game model in a tractable way.

### 3.2. The Speculators' Problem

We start analysing the game by deriving the behaviour of speculators, taking the costs of speculation as given. In the next subsection, we then solve out for these costs by deriving the interest rate defence of the policy maker conditional on the behaviour of speculators.

Each speculator is endowed with some private information about the value of  $\theta$ . Yet the actual value of  $\theta$  remains unknown to speculators, so that there is no common knowledge of it. The signals are independently and identically distributed across speculators, and the distribution is commonly known to all agents. These are standard assumptions.

Concretely, each speculator  $i \in [0, 1]$  receives a signal  $x_i$ , distributed around  $\theta$ :

$$x_i = \theta + v \cdot \eta_i.$$

Each  $\eta_i$  is drawn from a noise distribution with continuous density function  $f > 0$  on a compact support  $S \subset \mathbb{R}$ . We normalise the standard deviation of  $f$  to 1, so that the standard deviation of the signals is equal to  $v > 0$ .<sup>5</sup> We also assume that  $f$  satisfies the monotone likelihood ratio property:

$$\text{for } \underline{x} < \bar{x}, \quad \frac{f(\underline{x} - \theta)}{f(\bar{x} - \theta)} \text{ is increasing in } \theta. \quad (5)$$

This property guarantees that a higher signal induces a higher belief about the value of  $\theta$ .<sup>6</sup> The uniform distribution, used by [Morris and Shin \(1998\)](#) and [Goldstein and Pauzner \(2005\)](#), satisfies this property, but  $f$  can also be a truncated normal or a number of other distributions; see [Karlin and Rubin \(1956a\)](#) for an overview.

A strategy for a speculator is a decision rule  $\sigma_i : x_i \mapsto d_i$ , where  $d_i$  is the short position of the speculator when her signal is equal to  $x_i$ . A strategy profile for all speculators,  $\sigma$ , describes the strategy of each individual speculator. Since speculators are risk neutral, speculator  $i$  should attack with her total wealth of unity if and only if the expected payoff from attacking the weak

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<sup>5</sup>For regularity, we impose  $v \cdot (\max S - \min S) < \bar{\theta} - \underline{\theta}$ , i.e. the support of the noise distribution is not too broad.

<sup>6</sup>We need this condition because the conditional profit function  $\pi$  is non-monotonic in both arguments. A similar condition is used in [Morris and Shin \(2003\)](#) to prove the existence of threshold equilibria under the assumption that  $\pi$  is non-monotonic in its first argument but still monotonic in its second argument.

currency  $\pi^e(x_i; \sigma)$ , conditional on the signal  $x_i$  and the strategy profile  $\sigma$ , is positive. Using payoff function (1) this expected payoff can be expressed as

$$\begin{aligned}\pi^e(x_i; \sigma) &= \mathbb{E}[\mathbb{E}[\pi(\ell, \theta) | \theta, \sigma] | x_i] \\ &= \mathbb{E}[\mathbb{E}[b(\ell, \theta) \cdot \mathbf{1}[\ell > \ell^*(\theta)] | \theta, \sigma] - \mathbb{E}[c(\ell, \theta) \cdot \mathbf{1}[\ell \leq \ell^*(\theta)] | \theta, \sigma] | x_i],\end{aligned}\tag{6}$$

where  $\mathbf{1}[\cdot]$  denotes the indicator function. Hence, the speculator's optimal decision depends on her signal  $x_i$ , the strategy profile  $\sigma$ , the benefit function  $b$ , and the costs function  $c$ . The costs of attacking are determined by the interest rate defence of the policy maker, as motivated in the introduction, which is uncertain. Hence, the threat of an interest rate defence will have an impact on the behaviour of speculators.

### 3.3. The Interest Rate Defence

We now turn to the optimal interest rate decision of the policy maker conditional on the behaviour of speculators. The policy maker chooses the interest rate to equilibrate the spot exchange market, but only if that rate is not too costly for the economy.

The spot market is in equilibrium at the fixed exchange rate if the amount of sales  $\ell$  of weak currency triggered by the speculators is offset by demand of other (non-speculative) traders in the market. Non-speculating agents may refuse to hold positions in a weak currency under bad fundamentals, unless compensated by a sufficiently high interest rate. Let their demand be given by  $D(r, \theta)$  under the fixed exchange rate regime. Hence, the spot market is in equilibrium if

$$D(r, \theta) = \ell.\tag{7}$$

To maintain the peg, the policy maker will have to set the interest rate to the value that solves equation (7). We assume  $D$  is twice continuously differentiable and

$$D_r \geq k > 0 \text{ (for some } k \in \mathbb{R}), \quad D_{rr} \leq 0, \quad D_\theta > 0, \quad D_{\theta\theta} \leq 0, \quad \text{and} \quad D_{r\theta} \geq 0.$$

The last of these conditions requires clarification. It says that the demand for the weak currency of non-speculative traders becomes less sensitive to raising the interest rate as fundamentals worsen. In other words, raising the interest rate to induce demand for domestic currency becomes a less effective defence policy as the economic situation deteriorates, which we think is natural.<sup>7</sup>

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<sup>7</sup>Our  $D(\cdot)$  function fulfils a similar role to the bond supply function in other global models with an endogenous interest rate, such as Hellwig et al. (2006). Differences, however, are the fact that we do not assume a specific closed form, and the additional dependency on the fundamental  $\theta$ . This dependency is the main reason that standard global game results do not apply to our model. In section 4.5 we make a detailed comparison.

Under our assumptions on  $D$ , the implicit function theorem implies that a unique solution to equation (7) exists. Let  $r^*(\ell, \theta)$  denote the interest rate that solves it for given  $\ell$  and  $\theta$ . This  $r^*$  is strictly increasing in  $\ell$  and strictly decreasing in  $\theta$ .

A high interest rate has a detrimental effect on the economy. Let  $\bar{r}(\theta)$  denote the *maximum* interest rate that the policy maker is prepared to use. We assume  $\bar{r}$  is continuously differentiable and  $\bar{r}_\theta > 0$ , which means that the policy maker is more willing to defend the peg under good fundamentals than under bad fundamentals.

This results in the following interest rate decision. If  $r^*(\ell, \theta) \leq \bar{r}(\theta)$ , the policy maker prefers to defend the currency peg, and it will set  $r = r^*(\ell, \theta)$ . However, if  $r^*(\ell, \theta) > \bar{r}(\theta)$ , then the policy maker prefers to abandon the peg, and it sets  $r$  at some other rate such that the weak currency devalues.<sup>8</sup> In the latter case  $\ell$  is apparently too high. Thus the behaviour of speculators directly influences the optimal interest rate decision.

### 3.4. A Payoff Structure with Strategic Substitutes

Finally, we show that the model described above fulfils the requirements of the more general theory in section 2. In particular, we show that our model exhibits strategic substitutes. As our model does not restrict the speculators' benefits from a successful attack any further, we only have to verify the assumptions on the speculators' costs  $c$  of an unsuccessful attack. In our model these costs are determined by the defence policy. In fact, we have  $c(\ell, \theta) = r^*(\ell, \theta)$ , provided the policy maker defends optimally, so the costs are indeed increasing in  $\ell$  and decreasing in  $\theta$ . Thus the model gives rise to strategic substitutes among speculators.

The remaining conditions (2)–(4) are also satisfied, as the following lemma shows.

**Lemma 1.** *Under the optimal interest rate defence, the conditions on  $D$  and  $\bar{r}$  imply that  $c(\ell, \theta)$  satisfies conditions (2)–(4).*

Regarding condition (2), the critical amount of speculation that brings down the peg, denoted by  $\ell^*(\theta)$  in section 2, is endogenous in our model, as it is the solution to  $r^*(\ell, \theta) = \bar{r}(\theta)$ . Since  $\bar{r}(\theta)$  is weakly increasing in  $\theta$ , condition (2) holds. Implications for  $r^*$  of the assumptions on  $D$  translate into conditions (3) and (4). Intuitively, the concavity of demand for the currency in  $r$

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<sup>8</sup>For instance, she might set it to  $\bar{r}(\theta)$  to limit the devaluation or (as in our working paper Daniëls et al. (2009)) to an interest rate target based on domestic policy goals. The precise specification does not matter in our model.

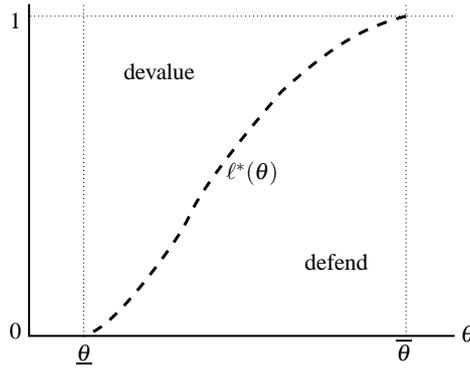


Figure 2: The decision to devalue or defend

implies condition (3), and (4) holds because the defence of the policy maker is supported by increased demand for the currency if fundamentals are better.

The implicit function theorem implies that  $\ell^*$  is a continuously differentiable and strictly increasing function, with compact domain, say,  $[\underline{\theta}, \bar{\theta}]$ . A consequence of the increasing nature of  $\ell^*$  is that the model satisfies the standard tripartite classification for economic fundamentals, familiar from the literature on second generation currency crisis models (and emphasised by e.g. [Obstfeld, 1996](#); [Jeanne, 1997](#); and [Morris and Shin, 1998](#)). If  $\theta < \underline{\theta}$  the policy maker never defends the peg, whatever  $\ell$ . If  $\theta \geq \bar{\theta}$ , the fundamentals are sound enough that the policy maker does not abandon the peg for any value of  $\ell$ . The regions  $\{\theta \leq \underline{\theta}\}$  and  $\{\theta \geq \bar{\theta}\}$  are called “dominance regions” in the global games literature.

The third and most interesting region is  $\{\underline{\theta} \leq \theta \leq \bar{\theta}\}$ . Here the decision to abandon the peg or not depends on the amount of speculative pressure  $\ell$ , as indicated in figure 2. The weak currency is “ripe for attack”, and a collapse of the peg can be triggered if more than  $\ell^*(\theta)$  speculators attack it. To solve the model, we examine for which values of  $\theta \in [\underline{\theta}, \bar{\theta}]$  this is the case.

#### 4. Equilibrium

An equilibrium of the model is a combination of strategies for the policy maker and for the speculators such that none of them has an incentive to deviate. More concretely, the equilibrium conditions for the respective agents are found as follows. In equilibrium, the policy maker must defend optimally, taking into account the behaviour of speculators. In section 3.3 we showed that the policy maker devalues if and only if  $\ell > \ell^*(\theta)$ . For a given strategy profile of speculators  $\sigma$ , let  $p(x; \sigma)$  be the proportion of speculators that attack when their signal is  $x$ . Taking into

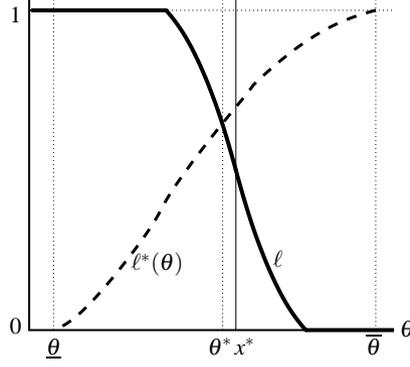


Figure 3: The threshold equilibrium of the model

account the aggregate distribution of the signals, for each  $\theta$  the aggregate fraction of attackers may be calculated as

$$\ell(\theta; \sigma) := \int_S p(\theta + v \cdot \eta; \sigma) \cdot f(\eta) d\eta. \quad (8)$$

Thus, since the policy maker observes  $\theta$  perfectly, her unique optimal strategy is to defend the peg when  $\ell(\theta; \sigma) \leq \ell^*(\theta)$  by setting  $r = r^*(\ell(\theta; \sigma), \theta)$ , and to devalue when  $\ell(\theta; \sigma) > \ell^*(\theta)$ .

The optimal strategy for speculators was derived in section 3.2. A speculator attacks if and only if the expected profit (6) from the attack, conditional on her signal  $x_i$ , is positive. In equilibrium, she also takes into account the behaviour of the policy maker, implying  $c(\ell, \theta) = r^*(\ell(\theta; \sigma), \theta)$ . Substituting this into equation (6) yields the speculator's decision when her signal is  $x_i$ .

#### 4.1. Unique Threshold Equilibrium

We state and discuss two key results about the equilibrium of our model. These key results follow more generally from the payoff conditions stated in section 2. In this subsection, we prove both the existence and the uniqueness of a threshold equilibrium. This is an equilibrium characterised by some  $x^*$ , such that each speculator  $i$  attacks if and only if her signal  $x_i < x^*$ . We will denote this strategy profile for speculators by  $\sigma^{x^*}$ . Since signals are noisy, not all speculators start attacking at the same value of  $\theta$ . The peg collapses if  $\theta$  deteriorates beyond a unique threshold  $\theta^*$ , at which precisely a fraction  $\ell^*(\theta^*)$  of speculators attack, as is depicted in figure 3.

In a threshold equilibrium, speculators' behaviour is monotonic in the information they have about the fundamental. It is the most natural behaviour to consider, and it is also the behaviour that is familiar from the global games literature more generally. For these two reasons, we

regard this first result as our most fundamental. It holds for any noise distribution satisfying the regularity conditions in the previous section. Our second result, in the next subsection, adds that the threshold equilibrium is the unique equilibrium of the model among all possible types of equilibria, provided the noise distribution  $f$  is uniform.

Although threshold equilibria are familiar from global games, recall that our model generalises the usual global game assumptions. It is less straightforward to show that there is a unique threshold equilibrium. In fact, we crucially rely on conditions (3) and (4) to prove the following.

**Proposition 1.** *For sufficiently small  $v$ , the model has a unique threshold equilibrium.*<sup>9</sup>

We will sketch the intuition behind our proposition 1 in two steps and indicate where the conditions are used. For the first step, suppose speculators use the threshold strategy profile  $\sigma^x$ . The expected payoff of the critical agent, that is, the speculator who receives *exactly* the threshold signal is then given by  $\pi^e(x; \sigma^x)$ . The next lemma states two crucial properties of  $\pi^e(x; \sigma^x)$ .

**Lemma 2.** *If the policy maker uses her unique optimal strategy, then  $\pi^e(x; \sigma^x)$  is a continuous function of  $x$ , and, for sufficiently small  $v > 0$ , is strictly decreasing.*

This lemma implies that the expected payoff for the critical agent decreases as her signal indicates stronger fundamentals. Like in [Morris and Shin \(1998\)](#), the intuition is that under stronger fundamentals the policy maker is more willing to resist the attack. In our model,  $\bar{r}_\theta > 0$ , so she is willing to set higher a interest rate, which would indeed lower the payoff.

However, since the interest rate  $r$  is endogenous in our model, it is not immediate that an increase in  $\bar{r}$  actually leads to an increase in  $r$ . In particular, when fundamentals are better, for any given amount of speculative pressure a less vigorous interest rate defence is needed. But there is a counteracting effect. Under stronger fundamentals, the policy maker is also prepared to withstand greater speculative pressure. This means that she may still set a higher interest rate.

In the proof of the lemma, we show that under conditions (3) and (4) the possibility of a high interest rate dominates the expected payoff of the critical agent. Therefore, for the critical agent the expected costs of attacking increase. Because expected benefits fall, the expected payoff

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<sup>9</sup>This result is *not* a limit result:  $v$  does not necessarily need to approach 0 arbitrarily close for the proposition to hold. For instance, if signals are uniformly distributed, any  $v$  will do (as we show below). For other distributions, however,  $v$  may need to be small.

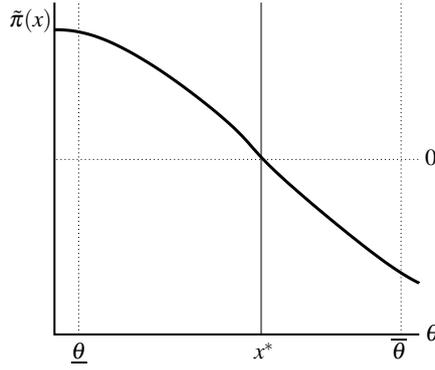


Figure 4: The expected payoff for the critical agent,  $\pi^e(x, \sigma^x)$ .

unambiguously falls, as the lemma states.

The continuity of  $\pi^e(x; \sigma^x)$  implies that, in equilibrium, the expected payoff of the critical agent must be zero—otherwise, the threshold could be adjusted up or down. Since  $\pi^e(x; \sigma^x)$  is strictly decreasing, there can be *at most* one candidate threshold equilibrium, in which speculators use the unique threshold  $x^*$  at which  $\pi^e(x; \sigma^x)$  vanishes—as is shown in figure 4.

It is now a small step to complete the proof of the proposition. Since the noise distribution satisfies the monotone likelihood property, we can apply a result of [Karlin and Rubin \(1956b\)](#) to show that the strategy profile  $\sigma^{x^*}$  is indeed an equilibrium. The argument is broadly similar to lemma 2.3 in [Morris and Shin \(2003\)](#), but it is short and we give it in the appendix for completeness.

**Lemma 3.** *If the policy maker uses her optimal strategy,  $\pi^e(x^*; \sigma^{x^*}) = 0$  and speculators follow the threshold strategy  $\sigma^{x^*}$ , then no speculator has an incentive to deviate since*

$$\pi^e(x_i; \sigma^{x^*}) \begin{cases} > 0 & \text{if } x_i < x^*, \\ < 0 & \text{if } x_i > x^*. \end{cases}$$

Combined, the two lemmas prove proposition 1.

#### 4.2. Uniform Noise Distributions

Under the standard global game assumption of strategic complementarities, the existence and uniqueness of a threshold equilibrium also rules out the existence of other, more complex equilibria. Unfortunately, in our setting this is not automatically the case. But it is true if the noise distribution is uniform (as in e.g. [Morris and Shin, 1998](#)). Our second main result shows

that in this case indeed there can be no other equilibria.<sup>10</sup>

**Proposition 2.** *If the noise distribution  $f$  is a uniform distribution, the threshold equilibrium is the unique equilibrium of the model; there are no other equilibria.*

In case of uniform noise we also do not need the provision that  $v$  is sufficiently small, provided  $D$  is strictly concave in  $\theta$  (as it is in our model).<sup>11</sup> Even though we cannot directly apply the theorem of [Goldstein and Pauzner \(2005\)](#) in our setting, proposition 2 yields a similar conclusion. Our proof differs in the sense that it again relies on conditions (3) and (4), and appears to be somewhat more elementary. It chiefly involves algebraic manipulation.

#### 4.3. The Locus of the Equilibrium Threshold

Let us confine ourselves again to threshold equilibria. What can be said about the locus of the speculators' threshold  $x^*$  in the interval  $[\underline{\theta}, \bar{\theta}]$  in which the weak currency is ripe for attack?

Consider again the critical speculator  $i$  who receives exactly the threshold signal. In equilibrium, we know that her expected payoff  $\pi^e(x^*; \sigma^{x^*}) = 0$ . This (implicitly) determines  $x^*$ , so let us consider  $i$ 's expected payoff in more detail. Since  $i$  does not know whether she has received a relatively high or a relatively low signal about  $\theta$ , and since the other speculators' signals about  $\theta$  are conditionally independent of  $i$ 's signal, in fact  $i$  does not know how many speculators have received signals smaller than  $x^*$ . [Morris and Shin \(2003\)](#) showed that  $i$ 's beliefs about  $\ell$  are given by the uniform distribution, regardless of the noise distribution  $f$  or the standard deviation  $v$ :  $i$  considers all values of  $\ell$  equally likely.

In addition to  $i$ 's uncertainty about  $\ell$ , her expected payoff also depends on the distribution of  $\theta$  conditional on  $x^*$ . But if we consider small  $v$ , so that  $\theta$  is close to  $x^*$ , this uncertainty vanishes, and we may approximate  $i$ 's payoff by

$$\pi^e(x^*; \sigma^{x^*}) \approx \int_0^1 \pi(\ell, x^*) d\ell = \int_0^{\ell^*(x^*)} -r^*(\ell, x^*) d\ell + \int_{\ell^*(x^*)}^1 b(\ell, x^*) d\ell. \quad (9)$$

The locus of  $x^*$  is thus mainly determined by  $i$ 's uncertainty about how many will join her in the attack, which is called the *strategic uncertainty* that she faces, and never vanishes. Its relevance for the threshold  $x^*$  is familiar from the theory of global games.

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<sup>10</sup>We have no counterexamples to proposition 2 for other noise distributions, and it is difficult to imagine what non-threshold equilibria would look like. But we have been unable to extend the proof to non-uniform distributions.

<sup>11</sup>Technically, we use that in this case the inequality in the consequent of (4) is strict. See the proof for details.

The strategic uncertainty captures that during the onset of a speculative attack speculators can only imperfectly coordinate their actions, and have to form expectations about how many other speculators will join the attack. Note that because  $\ell$  is uncertain, so is the likelihood that the attack will elicit a dogged defence which succeeds in warding off the attack. The essential uncertainty about both factors is captured by the inclusion of both  $r$  and  $\ell$  in the critical agent's expected payoff function (9). The possibility of a response of the interest rate to speculative pressure adds an additional source of risk for the critical agent.

The size of the additional risk due to the interest rate response depends on  $\theta$ . For low values of  $\theta$ , only a small group of speculators is necessary to break the defence, so that the odds of success are high, and the interest rate response is small. But for high values of  $\theta$ , a large fraction of speculators is needed to bring down the peg. If the attack on the currency peg is unsuccessful, a large amount of agents attacking the peg leads to a harsh defence by the policy maker, so that the costs of speculation are potentially very high. This makes attacking very risky. In sum, the higher  $\theta$ , the larger the expected interest rate response, and the riskier speculation.

#### 4.4. Implications of an Endogenous Interest Rate

Allowing for an endogenous interest rate defence enriches the standard model with some insights, with theoretical, practical, and policy implications. A theoretical observation is about a weak point of standard global game currency crisis models, pointed out e.g. by Chamley (2003). In standard models, the costs of speculation are often assumed to be exogenous and fixed. If they are interpreted as a small transaction cost, as in Morris and Shin (1998), speculators attack *en masse* when fundamentals are still relatively good and the resulting devaluation from the attack is still modest. This seems at odds with reality.<sup>12</sup> If, on the other hand, the costs of attacking are interpreted as a fixed interest rate differential (as in e.g. Morris and Shin, 2003), this ignores the fact that policy makers usually tighten the conditions on currency and credit markets in response to speculative attacks.

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<sup>12</sup>An important paper by Guimaraes and Morris (2007) also addresses this point. They argue that risk aversion of speculators gives a more plausible locus of the equilibrium threshold compared to standard models, so that in equilibrium speculators attack only when fundamentals are sufficiently deteriorated. The precise impact of risk aversion on the threshold depends on the costs of attacking relative to benefits (see their figure 2) and may reverse when costs are large relative to benefits. So, also in case of risk aversion, it may still be important to study how the costs and benefits vary endogenously.

Our model allows the interest rate differential to become large, explaining large devaluations. Also in reality, the costs associated with speculation may be substantial, as speculators roll over large positions while the policy maker tightens the conditions on markets for the duration of the attack.<sup>13</sup> Our model shows that even when this channel is taken into account, a unique threshold equilibrium still exists. Thus, global games survive both of the above strands of critique.

A more practical implication of our model is about how the interest rate pattern relates to the fundamental  $\theta$ . Recall that the critical speculator expects the policy maker to increase the interest rate to ward off speculative pressure. Yet for realisations of the fundamental sufficiently above  $x^*$  but below  $\bar{\theta}$ , the combination of strategic uncertainty and the threat of an interest rate defence is sufficient to keep speculators out, even though the currency is “ripe for attack”. Hence the model does not predict a substantially increased interest rate until the onset of a crisis. This is, for instance, consistent with observations made about the crisis in the European Monetary System in 1992–1993, that credibility of the exchange rate regime, as measured by realignment expectations, deteriorated only appreciably during the onset of the fall of the regime.<sup>14</sup>

Finally, the impact of the interest rate on the costs of speculation has a policy implication for the effectiveness of other peg defences compared to the interest rate defence. A defence of the peg that is not based on the interest rate instrument, but consists, say, of sterilised foreign exchange interventions in the spot market may be less effective, as it does not raise the costs of speculation for speculators during the attack, and therefore does not directly affect the risk associated with speculation. This implication distinguishes our model from a literature that primarily emphasises the signalling function of an interest rate defence (e.g. [Drazen, 2000](#); and global game papers such as [Angeletos et al., 2006, 2007](#)).

#### *4.5. A Comparison to Global Games with Rational Expectations Equilibria*

By including an uncertain  $r$  in equation (9), our model also departs from a number of general equilibrium global game models in which speculators can fully observe or control the costs of speculation before acting, such as [Angeletos and Werning \(2006\)](#), [Hellwig et al. \(2006\)](#) and [Tarashev \(2007\)](#). In these models, speculators do not simply choose to speculate or not, but simultaneously condition their decision on the market rate that results from their behaviour,

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<sup>13</sup>[Lall \(1997\)](#) analyses how a policy maker may “squeeze” speculators by tightening market conditions.

<sup>14</sup>This conclusion emerges from the well-known study by [Rose and Svensson \(1994\)](#), who write (p. 1185) that “[t]here were few indications of poor ERM credibility before late August 1992”.

treating it parametrically in their speculation decision.

To bring out some essential differences with our own model, we will compare it in more detail to the model of [Hellwig et al. \(2006\)](#), which is exemplary for the above approach. In the latter model, speculators submit a full bid function for the weak currency conditional on the behaviour of the interest rate during the attack. Subsequently, the model is solved for a rational expectations equilibrium, in which the market clears at an interest rate that is consistent with the bidding functions of speculators. The policy maker abandons the peg if this rate is too high. Market clearing interest rates are based on a stochastic demand function that plays a role similar to the (deterministic)  $D(\cdot)$  function of our model, but is augmented with a shock  $s$ . To solve the model, [Hellwig et al.](#) impose the following functional form:

$$D^{\text{HMT}}(s, r) = \Phi(s - \gamma\Phi^{-1}(r)), \quad (10)$$

where  $\Phi$  is the normal distribution's c.d.f.,  $\gamma$  an elasticity parameter, and the shock  $s$  is normally distributed. In contrast to our model,  $D^{\text{HMT}}$  does not depend on the fundamental  $\theta$ . Introducing that dependency would lead to some of the additional complications that we describe in section 2.

The key difference between solving for a rational expectations equilibrium and our approach is the ability to *condition* on the market clearing interest rate, and thereby implicitly the outcome of the model.<sup>15</sup> Since the market clearing  $r$  reflects  $\ell$ , and thus whether the speculators reach critical mass, speculators are effectively able to coordinate their actions through their bid functions, and therefore may condition their decisions both on the fraction of attackers and the devaluation decision. In effect, they face virtually no strategic uncertainty. Our model, by contrast, emphasises exactly this kind of uncertainty.

For a detailed comparison, two parameterisations of equation (10) seem especially relevant, as they lead to relatively uninformative interest rates, which seems *prima facie* most comparable with our approach. The first arises if the variance of the shock  $s$  becomes large. In this case, the interest rate signal becomes uninformative about the bond demand of other traders. However, it is still a public signal on which the speculators may coordinate, reducing strategic uncertainty. That this leads to a crucial difference with our model is witnessed, among other things, by the

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<sup>15</sup>[Hellwig et al. \(2006\)](#) write (p. 1782): “the fact that  $r$  must be consistent with market-clearing allows traders to infer [...] the devaluation outcome.”

fact that our model has a unique equilibrium when signals are precise, whereas under the same conditions the rational expectations approach typically has multiple. This is because, while there is a *unique* optimal bid function for speculators, it generally admits *multiple* market clearing interest rates—as is often the case in rational expectations models—at each of which speculators conditionally demand the amount of weak currency consistent with the actual outcome.

A second, degenerate, case arises when the interest rate is uninformative because it remains unchanged due to a perfectly elastic demand function when  $\gamma \rightarrow \infty$ . In this case, the setup of [Hellwig et al.](#) reduces to the model of Morris and Shin with fixed transaction costs. It can also be reproduced in our model, by assuming that the cost function  $c$  is constant. However, more generally, in our model the costs of attacking can change endogenously even if the interest rate signal remains uninformative. The value added is that our model provides a tractable analysis of a global game model where  $r$  is endogenous but where this does not eliminate strategic uncertainty.

In sum, the rational expectations approach and our own are different ways to capture the interplay between interest rates and speculators' decisions, which in reality is a complicated dynamic decision process of rolling over speculative positions (as our example in the introduction shows). The two approaches represent two opposite angles to model this interplay: the rational expectations approach emphasises the informational role of the interest rate, by allowing agents to condition on it, while our approach emphasises the impact of interest rate risk and strategic uncertainty, by making the opposite assumption.

We offer two reasons why we feel that our approach captures important aspects of the decision problem of speculators in reality. First, interpreted as a model of the onset of a speculative attack, it captures the uncertainty surrounding the decision whether to speculate or not: whether the attack will succeed, and if so, how long speculative positions will have to be maintained, and against what costs, which are ultimately endogenous and reflected in the value of their positions. They depend crucially on how many other speculators will eventually join the attack and what will be the resulting policy response. From this perspective, an approach that focuses on uncertainty about  $\ell$  and  $r$  is realistic.

Second, for the purpose of modelling speculative attacks, the rational expectations approach may overstate the informational role of interest rates by treating it as a public signal. This is the argument of [Morris and Shin \(2006\)](#), who compare the rational expectations equilibrium

and standard approach in more detail. They argue that, even though interest rates are essentially public in reality, as a practical matter they do not seem to substantially reduce the uncertainty about the decisions of other market participants.<sup>16</sup> The explanation is that pure public signals, in the theoretical sense, do not exist in reality. They also show that a slight modification of the rational expectations approach yields a model in which the interest rate is no longer a pure public signal. Unfortunately, they indicate this modification becomes intractable.

In any case, to study the implications of uncertainty about  $r$ , one can use our model. In the rational expectations approach, this is more difficult, due to the typical presence of multiple equilibria. To see this, consider a situation where a low interest rate equilibrium, in which no-one attacks, coexists with a high interest rate, attack equilibrium. The high interest rate materialises purely for exogenous—“sun-spot”—reasons. Speculators, conditioning on the interest rate, attack in a coordinated way if and only if the interest rate is high, in this case facing essentially no risk that the attack might fail. In this setup, the policy maker’s willingness to raise interest rates cannot prevent the attack from happening. Thus, it is difficult to interpret the threat of an interest rate defence in a rational expectations model.

## 5. Conclusion

During currency crises, the policy maker undertakes defensive actions aimed at increasing the financing costs of speculators, such as raising the interest rate. Any approach that abstracts from these actions, and solely focuses on the policy maker’s decision whether or not to devalue under speculative pressure, gives a partial picture of the strategic interaction. In particular, it only focuses on the strategic complementarities in the speculators’ decisions, whereas the policy maker’s defence can also easily generate strategic substitutes.

We have incorporated the threat of an interest rate defence into the well-known global game speculative attack model of [Morris and Shin \(1998\)](#). In our model, the policy maker’s interest rate defence is endogenous and the payoff function varies non-monotonically both with respect to the fraction of attackers and the fundamentals. We have proved that the model still has a

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<sup>16</sup>Consider the remark of [Tabellini \(1994\)](#), p. 1222) on the EMS crisis, that “[t]he expectation of an imminent realignment needed more than just a misaligned [...] exchange rate. It also needed the perception that enough investors were ready to participate in the speculative attack.” It emphasises the significance of strategic uncertainty during the onset of a speculative attack.

unique threshold equilibrium, just like the usual global game models. Thus global games are robust to this more realistic specification of the payoff function. While we have focused on currency crises, our results rely only on the payoff structure, so that they also apply in other settings.

Our focus on currency crises leads to a number of observations that are broadly consistent with how crises develop in reality. A relatively large deterioration of fundamentals is necessary to trigger a speculative attack, which may be accompanied by a spike in the interest rate, while much less stress on markets (in terms of an elevated interest rate) will be observed for smaller misalignments of fundamentals. A successful speculative attack will lead to a substantial jump of the exchange rate following the attack. As a policy implication, we find that a defence of the peg not based on the interest rate instrument, but consisting, say, of interventions in the foreign exchange spot market is less effective insofar as the costs of speculation are not increased.

## Appendix: Proofs

*Proof of Lemma 1.* Together with the spot market equilibrium condition (7),  $c(\ell, \theta) = r^*(\ell, \theta)$  implies that, for fixed  $\theta$ ,  $c(\ell, \theta)$  is the inverse of  $D(r, \theta)$ . Thus the derivative  $c_\ell$  is given by  $1/D_r(c(\ell, \theta), \theta)$ . Since  $c$  is increasing in  $\ell$ , the (weak) concavity of  $D(r, \theta)$  in  $r$  implies (weak) convexity of its inverse, proving condition (3). Moreover, if  $c(\ell, \theta) = c(\ell', \theta')$  then  $\theta < \theta'$  implies:

$$c_\ell(\ell, \theta) = \frac{1}{D_r(c(\ell, \theta), \theta)} = \frac{1}{D_r(c(\ell', \theta'), \theta)} \geq \frac{1}{D_r(c(\ell', \theta'), \theta')} = c_\ell(\ell', \theta'),$$

where the inequality follows because  $D_r$  is weakly increasing in  $\theta$ . Hence, condition (4) is satisfied. In addition, in the model  $c(\ell^*(\theta), \theta)$  equals  $\bar{r}(\theta)$ , which is a weakly increasing function of  $\theta$  by assumption, so that condition (2) is also satisfied.  $\square$

The following lemma is used in the proofs of lemma 2 and of proposition 2.

**Lemma A 1.** *Let  $f_a$  and  $f_b$  be twice continuously differentiable, real valued functions with domain  $[0, h]$ , satisfying (i)  $f_a(h) \leq f_b(h)$ , (ii)  $f_a(x) = f_b(y) \implies f'_a(x) \geq f'_b(y)$  for all  $x, y \in [0, h]$ , and (iii)  $f'_b \geq 0$ . Then  $f_a(x) \leq f_b(x)$  for all  $x \in [0, h]$ . (If the inequality in the consequent of (ii) holds strictly, condition (iii) may be dropped and the conclusion still holds.)*

*Proof.* Define  $g : [0, h] \rightarrow \mathbb{R}$  by  $g(x) = f_b(x) - f_a(x)$  and note that  $g$  is continuous. Let  $G = \{x \in [0, h] \mid g(x) < 0\}$ . If  $G = \emptyset$ , the lemma is proved. We will show that the converse

leads to a contradiction. Assume  $s_0 \in G \neq \emptyset$  and let  $s_1 = \inf\{x \in [s_0, h] | g(x) \geq 0\}$ . Observe  $s_0 < s_1$ . By (i)  $g(h) \geq 0$ , and the continuity of  $g$  implies  $g(s_1) = 0$ . Now, for  $f_b(s_0) \leq k \leq f_b(s_1)$ , let  $f_b^{-1}(k) = \sup\{y \in [s_0, h] | f_b(y) = k\}$ . Since  $f_b$  is continuous, this function is well defined,  $f_b(f_b^{-1}(f_a(x))) = f_a(x)$  for all  $x$ . Moreover, for all  $x$ ,  $g(x) < 0$  implies  $f_b(x) < f_a(x)$  which in turn implies  $f_b^{-1}(f_a(x)) > x$ . Using the fundamental theorem of calculus we deduce:

$$\begin{aligned} g(s_0) &= f_b(s_0) - f_a(s_0) = \int_{s_0}^{s_1} f'_a(x) - f'_b(x) dx \\ &\geq \int_{s_0}^{s_1} f'_a(x) - f'_b(f_b^{-1}(f_a(x))) dx \\ &\geq \int_{s_0}^{s_1} f'_a(x) - f'_a(x) dx = 0. \end{aligned}$$

The first inequality follows from the convexity of  $f_b$  implied by (iii). The second follows from (ii), since  $f_b(f_b^{-1}(f_a(x))) = f_a(x)$ , and thus  $f'_a(x) \geq f'_b(f_b^{-1}(f_a(x)))$ . The final equality gives  $g(s_0) \geq 0$ , contradicting the choice of  $s_0$ . (If  $f_b$  is not convex but the inequality in the consequent of (ii) is strict, we have  $f'_a(x) > f'_b(x)$  in a neighbourhood  $U$  of  $s_1$ . In that case we may choose  $s_2 \in U$  such that  $s_0 < s_2 < s_1$ . The argument based on the fundamental theorem is valid with  $s_2$  replacing  $s_0$ , without the step that appeals to the convexity of  $f_b$ .)  $\square$

*Proof of Lemma 2.* Let  $\ell(\theta, x)$  be the fraction of speculators who attack when the true state is  $\theta$  under the joint threshold strategy around  $x$ ,

$$\ell(\theta, x) = F\left(\frac{1}{v}(x - \theta)\right), \quad (11)$$

where  $F : \mathbb{R} \rightarrow [0, 1]$  is the c.d.f. of the noise distribution  $f$ . Now let  $H(\theta|x)$  denote the probability that the true state is less than or equal to  $\theta$  given the signal  $x$ . For a critical agent receiving the signal  $x_i = x$ , this probability is equal to the probability that her error  $\eta_i$  exceeds  $\frac{1}{v}(x - \theta)$ . Hence  $H(\theta|x) = 1 - F\left(\frac{1}{v}(x - \theta)\right) = 1 - \ell(\theta, x)$ , and the expectation of  $\pi$  conditional on the signal  $x$  is

$$\pi^e(x; \sigma^x) = \int_{-\infty}^{\infty} \pi(\ell(\theta, x), \theta) dH(\theta | x) = \int_{-\infty}^{\infty} \pi(\ell(\theta, x), \theta)(-\ell_\theta(\theta, x)) d\theta. \quad (12)$$

Since  $f > 0$ ,  $F$  is strictly increasing on  $S$ . Thus (11) is invertible and its inverse is:

$$\theta(\ell, x) = x - vF^{-1}(\ell). \quad (13)$$

By change of variables, using the transformation  $\theta \mapsto 1 - \ell(\theta, x)$ , we rewrite (12) as

$$\pi^e(x; \sigma^x) = \int_0^1 \pi(\ell, \theta(\ell, x)) d\ell. \quad (14)$$

This is the familiar observation (see [Morris and Shin, 2003](#)) that the density of  $\ell$  is uniform on the interval  $[0, 1]$  from the perspective of the critical agent.

The function  $\ell(\theta, x)$  is weakly decreasing in  $\theta$  for fixed  $x$ , while  $\ell^*(\theta)$  is strictly increasing. Thus these functions intersect at a unique  $\theta$ , which we denote  $\phi(x)$ . By the implicit function theorem  $\phi$  is a continuous function near the point  $x$ , and  $\phi_x(x) > 0$ . Define  $\kappa^*(x) := \ell^*(\phi(x))$ , which is a strictly increasing function of  $x$ . Indeed, letting

$$\underline{f} := \min\{f(s) | s \in S\} > 0, \quad \underline{\ell}_\theta^* := \min\{\ell_\theta^*(\theta) | \theta \in [\underline{\theta}, \bar{\theta}]\} > 0, \quad \bar{\ell}_\theta^* := \max\{\ell_\theta^*(\theta) | \theta \in [\underline{\theta}, \bar{\theta}]\} > 0,$$

for each  $v > 0$  we can bound the derivative  $\kappa_x^*$  both from above, and away from 0:<sup>17</sup>

$$\bar{\ell}_\theta^* \geq \kappa_x^*(x) \geq \frac{1}{(1/\underline{\ell}_\theta^*) + (v/\underline{f})} =: \underline{\kappa}_x^*(v) > 0.$$

Now we decompose (14) as follows:

$$\pi^e(x; \sigma^x) = \int_0^{\kappa^*(x)} -c(\ell, \theta(\ell, x)) d\ell + \int_{\kappa^*(x)}^1 b(\ell, \theta(\ell, x)) d\ell. \quad (15)$$

This function is continuous and differentiable in  $x$ . Differentiate the second summand with respect to  $x$  to get:

$$b(\kappa^*(x), \phi(x)) \cdot -\kappa_x^*(x) + \int_{\kappa^*(x)}^1 b_\theta(\ell, \theta(\ell, x)) \cdot \theta_x(\ell, x) d\ell. \quad (16)$$

This summand is decreasing in  $x$ , since  $\kappa^*(x)$  is a strictly increasing function,  $b_\theta$  is non-positive, and, from equation (13),  $\theta_x(\ell, x) = 1$ . Denoting  $\underline{b} = \min\{b(\theta, \ell) | \theta \in [\underline{\theta}, \bar{\theta}], \ell \in [0, 1]\}$ , expression (16) is smaller than  $-\underline{\kappa}_x^*(v) \cdot \underline{b}$ .

It remains to be shown that the changes in the first summand in (14) do not counteract the negative effect. We will show this in two steps. First, we take the derivative of the first summand:

$$-c(\kappa^*(x), \phi(x)) \cdot \kappa_x^*(x) - \int_0^{\kappa^*(x)} c_\theta(\ell, \theta(\ell, x)) \cdot \theta_x(\ell, x) d\ell. \quad (17)$$

Since  $c$  is twice continuously differentiable on the compact set  $[\underline{\theta}, \bar{\theta}]$ ,  $c_\theta$  is Lipschitz continuous

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<sup>17</sup>Observe that  $0 < \phi_x(x) = \frac{f\left(\frac{1}{v}(x - \phi(x))\right)}{f\left(\frac{1}{v}(x - \phi(x))\right) + v \cdot \ell_\theta^*(\phi(x))} < 1$ , implying  $\kappa_x^*(x) = \ell_\theta^*(\phi(x)) \cdot \phi_x(x) \leq \bar{\ell}_\theta^*$  and:

$$\kappa_x^*(x) = \frac{\ell_\theta^*(\phi(x))}{1 + \frac{v}{f\left(\frac{1}{v}(x - \phi(x))\right)} \ell_\theta^*(\phi(x))} \geq \frac{\ell_\theta^*(\phi(x))}{1 + (v/\underline{f}) \ell_\theta^*(\phi(x))} = \frac{1}{(1/\ell_\theta^*(\phi(x))) + (v/\underline{f})} \geq \frac{1}{(1/\underline{\ell}_\theta^*) + (v/\underline{f})}.$$

on  $[\underline{\theta}, \bar{\theta}]$ . Let  $k$  be a Lipschitz constant—for each  $\theta, \theta'$ ,  $c_\ell(\ell, \theta)$  is within  $k \cdot (\theta - \theta')$  of  $c_\ell(\ell, \theta')$ . Let  $\bar{s} = \max S - \min S$ . Since  $\kappa^*(x) \leq 1$ , and since from the perspective of the critical agent the distance between  $\phi(x)$  and the true fundamental  $\theta$  must be smaller than  $v \cdot \bar{s}$ , we find that

$$-vk\bar{s} \leq \left[ \int_0^{\kappa^*(x)} c_\theta(\ell, \phi(x)) d\ell - \int_0^{\kappa^*(x)} c_\theta(\ell, \theta(\ell, x)) d\ell \right] \leq vk\bar{s}. \quad (18)$$

Now consider the derivative of the function  $\int_0^{\kappa^*(x)} -c(\ell, \phi(x)) d\ell$ , which is given by

$$-c(\kappa^*(x), x) \cdot \kappa_x^*(x) - \int_0^{\kappa^*(x)} c_\theta(\ell, \theta(\ell, x)) \cdot \phi_x(x) d\ell. \quad (19)$$

Since  $\phi_x \rightarrow 1$  uniformly as  $v \rightarrow 0$ , expression (19) is within  $2vk\bar{s}$  of the true derivative (17) for sufficiently small  $v$ . In our second step, we will show that the derivative (19) must be negative, by showing that  $\int_0^{\kappa^*(x)} -c(\ell, \phi(x)) d\ell$  is a decreasing function. Since for sufficiently small  $v$ , we have  $2vk\bar{s} < \underline{\kappa_x^*}(v)\underline{b}$ , this in turn implies that the total effect of an increase in  $x$  on  $\pi^e(x, \sigma^x)$  must be strictly negative, which completes the proof of the lemma.

So for our second step consider  $x_a$  and  $x_b$  such that  $x_a^* < x_b^*$ . Then  $\kappa^*(x_a) < \kappa^*(x_b)$ ; let  $\delta = \kappa^*(x_b) - \kappa^*(x_a)$ . We will prove that

$$\int_\delta^{\kappa^*(x_b)} -c(\ell, \phi(x_b)) d\ell \leq \int_0^{\kappa^*(x_a)} -c(\ell, \phi(x_a)) d\ell. \quad (20)$$

This suffices, since  $\int_0^\delta -c(\ell, \phi(x_b)) d\ell$  is clearly negative. Now rewriting (20) as

$$\int_0^{\kappa^*(x_a)} c(\ell, \phi(x_a)) d\ell \leq \int_0^{\kappa^*(x_a)} c(\ell + \delta, \phi(x_b)) d\ell, \quad (21)$$

we can verify that the integrand functions  $c(\ell, \phi(x_a))$  and  $c(\ell + \delta, \phi(x_b))$  satisfy the conditions of lemma A 1 on the domain  $[0, \kappa^*(x_a)]$ . Therefore  $c(\ell, \phi(x_a)) \leq c(\ell + \delta, \phi(x_b))$  for all  $\ell \in [0, \kappa^*(x_a)]$ , so that the inequality in (21) indeed holds.  $\square$

*Proof of Lemma 3.* For  $\phi(x^*)$  in the proof of lemma 2 we have  $\ell(\theta, x^*) > \ell^*(\theta)$  iff  $\theta < \phi(x^*)$ . So:

$$\pi(\ell(\theta, x^*), \theta) = \begin{cases} -c(\ell(\theta, x^*), \theta) < 0 & \text{if } \theta \geq \phi(x^*), \\ b(\ell(\theta, x^*), \theta) > 0 & \text{if } \theta < \phi(x^*). \end{cases}$$

Thus, the function  $\pi(\ell(\theta, x^*), \theta)$  changes sign exactly once in  $\theta$ . Using Bayes rule to find the likelihood of  $\theta$  conditional on  $x_i$ , we calculate the expected payoff conditional on  $x_i$  as

$$\pi^e(x_i; \sigma^{x^*}) = \frac{1}{v} \int_{-\infty}^{\infty} \pi(\ell(\theta, x^*), \theta) \cdot f\left(\frac{1}{v}(x_i - \theta)\right) d\theta.$$

Since  $f$  satisfies the monotone likelihood ratio property, Lemma 1 in [Karlin and Rubin \(1956b\)](#) implies that  $\pi^e(x_i; \sigma^{x^*})$  changes sign at most once in  $x_i$  and, if so, in the same direction as  $\pi(\ell(\theta, x^*), \theta)$ . Using  $\pi^e(x^*; \sigma^{x^*}) = 0$ , we see that  $\pi^e(x_i; \sigma^{x^*})$  in fact changes sign at  $x^*$ .  $\square$

*Proof of Proposition 2.* For the purpose of this proof we will assume  $f$  is the uniform distribution with mean 0. First, we will give a slightly modified argument for the existence of a unique threshold strategy that is valid even when  $v$  is not small. In the proof of lemma 2, we applied lemma A 1 to the integrands in inequality (21). This inequality approximates the following inequality:

$$\int_0^{\kappa^*(x_a)} c(\ell, \theta(\ell, x_a^*)) d\ell \leq \int_0^{\kappa^*(x_a)} c(\ell + \delta, \theta(\ell + \delta, x_b^*)) d\ell. \quad (22)$$

Clearly, the existence of a unique threshold equilibrium is proved regardless of the choice of  $v$  if we can apply lemma A 1 directly to prove inequality (22). To this end, denote the left and right hand side integrands by  $f_a(\ell)$  and  $f_b(\ell)$  respectively. Under uniform noise  $\theta(\ell, x_b^*)$  is a linear function with slope  $-v$ , so the derivative of  $f_a$  is

$$f'_a(\ell) = c_\ell(\ell, \theta(\ell, x_a^*)) - v \cdot c_\theta(\ell, \theta(\ell, x_a^*)) > 0,$$

and the derivative of  $f_b$  may be found analogously. Now suppose that there are  $\ell_a, \ell_b$  in the domain  $[0, \kappa^*(x_a)]$  such that  $f_a(\ell_a) = f_b(\ell_b) = r$  for some  $r$ . Then, using condition (4) and that  $\theta_a := \theta(\ell_a, x_a^*) < \theta(\ell_b + \delta, x_b^*) =: \theta_b$ , we find that  $f'_a(\ell_a) = c_\ell(\ell_a, \theta_a) + v \cdot -c_\theta(\ell_a, \theta_a) > c_\ell(\ell_b + \delta, \theta_b) + v \cdot -c_\theta(\ell_b + \delta, \theta_b) = f'_b(\ell_b)$ —here we use the additional fact that

$$-c_\theta(\ell_a, \theta_a) = \frac{D_\theta(r, \theta_a)}{D_r(r, \theta_a)} > \frac{D_\theta(r, \theta_b)}{D_r(r, \theta_a)} \geq \frac{D_\theta(r, \theta_b)}{D_r(r, \theta_b)} = -c_\theta(\ell_b + \delta, \theta_b).$$

Under these conditions, lemma A 1 also applies, so a unique threshold equilibrium exists.

Next, we will prove every equilibrium must be a threshold equilibrium. This broadly follows the tactic of [Goldstein and Pauzner \(2005\)](#), but in our case this conclusion will follow from yet another application of lemma A 1. Consider any equilibrium of the model, and let  $\sigma$  denote the joint strategy profile of the speculators. Equation (8) gives the fraction  $\ell(\theta; \sigma)$  of attackers when the true state of fundamentals is  $\theta$ . Under uniform noise, the expected payoff from attacking is

$$\pi^e(x_i; \sigma) = \frac{1}{v} \int_{x_i - \frac{v}{2}}^{x_i + \frac{v}{2}} \pi(\ell(\theta; \sigma), \theta) d\theta. \quad (23)$$

Under the equilibrium conditions, a speculator who receives the signal  $x_i$  attacks if (23) is strictly positive, refrains if (23) is negative, and is indifferent when (23) vanishes. Let  $\underline{x}$

be the smallest signal at which speculators do not strictly prefer to refrain from attacking:  $\underline{x} = \inf\{x \in \mathbb{R} \mid \pi^e(x; \sigma) \leq 0\}$ . Since  $\pi^e$  is continuous in  $x_i$ ,  $\pi^e(\underline{x}; \sigma) = 0$ . The derivative of (23) evaluated at  $\underline{x}$  is  $\frac{1}{v} \left[ \pi(\ell(x_i + \frac{v}{2}); \sigma), x_i + \frac{v}{2} \right] - \pi(\ell(x_i - \frac{v}{2}); \sigma), x_i - \frac{v}{2} \right] < 0$ .

Now assume  $\sigma$  is not a threshold equilibrium. Then there are speculators who attack at signals exceeding  $\underline{x}$ , and hence by the continuity of  $\pi^e(x_i; \sigma)$  a smallest signal  $\bar{x}$  satisfying  $\bar{x} > \underline{x}$  exists such that  $\pi^e(\bar{x}; \sigma) = 0$ . Let  $Z := [\underline{x} - \frac{v}{2}, \underline{x} + \frac{v}{2}] \cap [\bar{x} - \frac{v}{2}, \bar{x} + \frac{v}{2}]$ , and let  $\underline{z} := \inf Z$  and  $\bar{z} := \sup Z$  if  $Z$  is non-empty, and  $\underline{z} := \underline{x} + \frac{v}{2}$  and  $\bar{z} := \bar{x} - \frac{v}{2}$  otherwise. Then  $\pi^e(\underline{x}; \sigma) = \pi^e(\bar{x}; \sigma) = 0$  implies

$$V_{\underline{x}} := \int_{\underline{x} - \frac{v}{2}}^{\underline{z}} \pi(\ell(\theta; \sigma), \theta) d\theta = \int_{\bar{z}}^{\bar{x} + \frac{v}{2}} \pi(\ell(\theta; \sigma), \theta) d\theta =: V_{\bar{x}}. \quad (24)$$

We will go on to deduce that  $V_{\bar{x}} < V_{\underline{x}}$  must hold, which contradicts equation (24), thus gives an absurdity. The inescapable conclusion is that  $\sigma$  must be a threshold equilibrium after all.

First note that  $\ell$  weakly decreases on  $[\underline{x} - \frac{v}{2}, \bar{z}]$ . To see this, note that, by equation (8),  $\ell(\underline{x} - \frac{v}{2}; \sigma) = 1$ ; as we move through the interval  $[\underline{x} - \frac{v}{2}, \bar{z}]$  and evaluate equation (8), we replace speculators with signals smaller than  $\underline{x}$  (who always attack), with speculators with signals above  $\underline{x}$  (who may or may not attack). Hence  $\ell$  weakly decreases on this interval.

Now we prove that  $V_{\underline{x}} \geq 0$ . If  $Z = \emptyset$ , then this follows from  $\pi^e(\underline{x}; \sigma) = 0$ . If  $Z \neq \emptyset$  and  $V_{\underline{x}} < 0$ , this implies there is some  $\theta' \in [\underline{x} - \frac{v}{2}, \underline{z}]$  such that  $\pi(\ell(\theta'), \theta') < 0$ . Since  $\ell$  weakly decreases and  $\theta$  increases on  $[\underline{x} - \frac{v}{2}, \bar{z}]$ , the functional form of  $\pi$  then implies  $V_Z := \int_Z \pi(\ell(\theta), \theta) d\theta < 0$ . But then  $V_Z + V_{\underline{x}} < 0$ , contradicting  $\pi^e(\underline{x}; \sigma) = 0$ . So indeed we must have  $V_{\underline{x}} \geq 0$ .

Next, we define  $\vec{\theta} := \underline{x} + \bar{x} - \theta$ . Thus,  $\vec{\theta}$  is the mirror image of  $\theta$  in the midpoint  $\frac{1}{2}(\underline{x} + \bar{x})$  between  $\underline{x}$  and  $\bar{x}$ . We claim that  $\ell(\vec{\theta}; \sigma) \leq \ell(\theta; \sigma)$  for all  $\theta \leq \frac{1}{2}(\underline{x} + \bar{x})$ . To see this, first note that the intervals  $[\theta - \frac{v}{2}, \theta + \frac{v}{2}]$  and  $[\vec{\theta} - \frac{v}{2}, \vec{\theta} + \frac{v}{2}]$  may or may not be disjoint. Taking this into account by adjusting the limits of integration in equation (8), it suffices to prove for the disjoint parts that

$$\int_{\theta - \frac{v}{2}}^{\min\{\theta + \frac{v}{2}, \vec{\theta} - \frac{v}{2}\}} p(x; \sigma) dx = \int_{\theta - \frac{v}{2}}^{\min\{\underline{x}, \vec{\theta} - \frac{v}{2}\}} dx \leq \int_{\max\{\vec{\theta} - \frac{v}{2}, \theta + \frac{v}{2}\}}^{\vec{\theta} + \frac{v}{2}} p(x; \sigma) dx = \int_{\max\{\bar{x}, \theta + \frac{v}{2}\}}^{\vec{\theta} + \frac{v}{2}} p(x; \sigma) dx, \quad (25)$$

where the equalities follow from the fact that no speculator attacks for signals  $x$ ,  $\underline{x} < x < \bar{x}$ . Using the definition of  $\vec{\theta}$  and that  $\vec{\theta} + \theta = \underline{x} + \bar{x}$ , the left hand side of inequality (25) equals

$$\begin{aligned} \min\{\underline{x}, \vec{\theta} - \frac{v}{2}\} - (\theta - \frac{v}{2}) &= \min\{\theta + \vec{\theta} - \bar{x}, \underline{x} + \bar{x} - \theta - \frac{v}{2}\} - (\underline{x} + \bar{x} - \vec{\theta} - \frac{v}{2}) \\ &= \min\{-\bar{x}, -\theta - \frac{v}{2}\} - (-\vec{\theta} - \frac{v}{2}) = \vec{\theta} + \frac{v}{2} - \max\{\bar{x}, \theta + \frac{v}{2}\}, \end{aligned}$$

which is clearly weakly greater than the right hand side of inequality (25), proving our claim.

Finally, to complete the proof, we distinguish two cases.

*Case 1:*  $\pi(\ell(\underline{z}; \boldsymbol{\sigma}), \underline{z}) > 0$ . In this case,  $\pi(\ell(\theta; \boldsymbol{\sigma}), \theta) = b(\ell(\theta; \boldsymbol{\sigma}), \theta) > 0$  for all  $\theta \in [\underline{x} - \frac{v}{2}, \underline{z}]$  since  $\ell$  is weakly decreasing on this interval, and since  $b$  is weakly increasing in  $\ell$  and decreasing in  $\theta$ . Now, for all  $\theta \in [\underline{x} - \frac{v}{2}, \underline{z}]$ ,  $\pi(\ell(\vec{\theta}; \boldsymbol{\sigma}), \vec{\theta}) < b(\ell(\theta; \boldsymbol{\sigma}), \theta)$ . If  $\pi(\ell(\vec{\theta}; \boldsymbol{\sigma}), \vec{\theta}) = -c(\ell(\vec{\theta}; \boldsymbol{\sigma}), \vec{\theta})$  then this is immediate; if  $\pi(\ell(\vec{\theta}; \boldsymbol{\sigma}), \vec{\theta}) = b(\ell(\vec{\theta}; \boldsymbol{\sigma}), \vec{\theta})$ , this follows since  $\ell(\theta; \boldsymbol{\sigma}) \geq \ell(\vec{\theta}; \boldsymbol{\sigma})$  implies  $b(\ell(\theta; \boldsymbol{\sigma}), \theta) \geq b(\ell(\vec{\theta}; \boldsymbol{\sigma}), \theta)$ , and  $\vec{\theta} > \theta$  in turn implies  $b(\ell(\vec{\theta}; \boldsymbol{\sigma}), \theta) > b(\ell(\vec{\theta}; \boldsymbol{\sigma}), \vec{\theta})$ . So  $b(\ell(\theta; \boldsymbol{\sigma}), \theta) > b(\ell(\vec{\theta}; \boldsymbol{\sigma}), \vec{\theta})$ . It follows readily that  $V_{\bar{x}} < V_{\underline{x}}$  and thus the proof is complete.

*Case 2:*  $\pi(\ell(\underline{z}; \boldsymbol{\sigma}), \underline{z}) < 0$ . Suppose that  $V_{\bar{x}} \geq V_{\underline{x}} \geq 0$  holds. Then there must be  $\theta' \in [\bar{z}, \bar{x} + \frac{v}{2}]$  such that  $\pi(\ell(\theta'; \boldsymbol{\sigma}), \theta') > 0$ . This implies  $\ell(\theta'; \boldsymbol{\sigma}) > \ell^*(\theta')$ . By continuity there exists  $\theta^* = \inf\{\theta \in [\bar{z}, \bar{x} + \frac{v}{2}] | \ell(\theta; \boldsymbol{\sigma}) \geq \ell^*(\theta)\}$ , which must satisfy  $\ell(\theta^*; \boldsymbol{\sigma}) = \ell^*(\theta^*)$ .

Now note that on the interval  $[\underline{x} - \frac{v}{2}, \underline{z}]$ ,  $\ell$  decreases linearly at the fastest possible rate,  $-\frac{1}{v}$ . This is because for all  $\theta$  satisfying  $\underline{x} - \frac{v}{2} < \theta < \underline{z}$ , we have  $\theta - \frac{v}{2} < \underline{x}$  and  $\underline{x} \leq \theta + \frac{v}{2} \leq \bar{x}$ , so that (by equation (8)) under uniform noise the derivative of  $\ell(\theta, \boldsymbol{\sigma})$  simply equals  $-\frac{1}{v}$ .

Now choose  $\tilde{\ell}(\theta)$  to be the linear function with positive slope  $\frac{1}{v}$  and such that  $\tilde{\ell}(\theta^*) = \ell^*(\theta^*)$ . Let  $\theta_0$  be the solution to  $\tilde{\ell}(\theta) = \ell(\underline{z}; \boldsymbol{\sigma})$ . Since  $\ell(\underline{z}; \boldsymbol{\sigma}) \geq \ell(\bar{z}; \boldsymbol{\sigma})$ , we must have  $\bar{z} \leq \theta_0 \leq \theta^*$ .

Rewrite the right hand side of equation (24) using the following decomposition of  $V_{\bar{x}}$ :

$$\begin{aligned} V_{\bar{x}} &= \int_{\bar{z}}^{\bar{x} + \frac{v}{2}} b(\ell(\theta; \boldsymbol{\sigma}), \theta) \cdot \mathbf{1}[\ell(\theta; \boldsymbol{\sigma}) > \ell^*(\theta)] - c(\ell(\theta; \boldsymbol{\sigma}), \theta) \cdot \mathbf{1}[\ell(\theta; \boldsymbol{\sigma}) \leq \ell^*(\theta)] d\theta \\ &= \int_{\theta^*}^{\bar{x} + \frac{v}{2}} b(\ell(\theta; \boldsymbol{\sigma}), \theta) \cdot \mathbf{1}[\ell(\theta; \boldsymbol{\sigma}) > \ell^*(\theta)] - c(\ell(\theta; \boldsymbol{\sigma}), \theta) \cdot \mathbf{1}[\ell(\theta; \boldsymbol{\sigma}) \leq \ell^*(\theta)] d\theta - \int_{\bar{z}}^{\theta^*} c(\ell(\theta; \boldsymbol{\sigma}), \theta) d\theta \\ &\leq \int_{\theta^*}^{\bar{x} + \frac{v}{2}} b(\tilde{\ell}(\theta), \theta) d\theta - \int_{\bar{z}}^{\theta^*} c(\tilde{\ell}(\theta), \theta) d\theta \leq \int_{\theta^*}^{\bar{x} + \frac{v}{2}} b(\tilde{\ell}(\theta), \theta) d\theta - \int_{\theta_0}^{\theta^*} c(\tilde{\ell}(\theta), \theta) d\theta. \end{aligned}$$

The first inequality follows since the slope  $\frac{1}{v}$  of  $\tilde{\ell}(\theta)$  guarantees that  $\tilde{\ell}(\theta) \geq \ell(\theta; \boldsymbol{\sigma})$  for  $\theta \geq \theta^*$ , and conversely  $\tilde{\ell}(\theta) \leq \ell(\theta; \boldsymbol{\sigma})$  for  $\theta \leq \theta^*$ . The second inequality follows since  $\bar{z} \leq \theta_0$ .

Now let  $\theta_*$  be the (unique) solution to  $\ell^*(\theta) = \ell(\theta; \boldsymbol{\sigma})$  on the interval  $[\underline{x} - \frac{v}{2}, \underline{z}]$ . Let  $q := (\theta_0 - \bar{z}) \geq 0$  and  $w := \theta^* - \vec{\theta}_* > q$ . Rewrite the left hand side of equation (24) as

$$\begin{aligned} V_{\underline{x}} &= \int_{\underline{x} - \frac{v}{2}}^{\theta_*} b(\ell(\theta; \boldsymbol{\sigma}), \theta) d\theta - \int_{\theta_*}^{\underline{z}} c(\ell(\theta; \boldsymbol{\sigma}), \theta) d\theta = \int_{\vec{\theta}_*}^{\bar{x} + \frac{v}{2}} b(\ell(\vec{\theta}; \boldsymbol{\sigma}), \vec{\theta}) d\theta - \int_{\bar{z}}^{\vec{\theta}_*} c(\ell(\vec{\theta}; \boldsymbol{\sigma}), \vec{\theta}) d\theta \\ &\geq \int_{\vec{\theta}_*}^{\bar{x} + \frac{v}{2}} b(\ell(\vec{\theta}; \boldsymbol{\sigma}), \theta) d\theta - \int_{\bar{z}}^{\vec{\theta}_*} c(\ell(\vec{\theta}; \boldsymbol{\sigma}), \vec{\theta}) d\theta = \int_{\vec{\theta}_*}^{\bar{x} + \frac{v}{2}} b(\tilde{\ell}(\theta + q), \theta) d\theta - \int_{\bar{z}}^{\vec{\theta}_*} c(\tilde{\ell}(\theta + q), \vec{\theta}) d\theta \\ &> \int_{\theta^*}^{\bar{x} + \frac{v}{2}} b(\tilde{\ell}(\theta + q), \theta) d\theta - \int_{\bar{z}}^{\vec{\theta}_*} c(\tilde{\ell}(\theta + q), \vec{\theta}) d\theta. \end{aligned}$$

The first inequality follows since in the range of integration  $\vec{\theta} < \theta$  and  $b$  is decreasing in  $\theta$ . The

next equality since, by construction,  $\tilde{\ell}(\theta + q) = \ell(\vec{\theta}; \sigma)$  in the range of integration. The second and strict inequality from the fact that  $\vec{\theta}_* < \theta^*$  (because  $\ell^*$  is decreasing).

$$\text{Surely } \int_{\vec{\theta}_*}^{\bar{x} + \frac{v}{2}} b(\tilde{\ell}(\theta + q), \theta) d\theta \geq \int_{\vec{\theta}_*}^{\bar{x} + \frac{v}{2}} b(\tilde{\ell}(\theta), \theta) d\theta. \text{ Thus to establish } V_{\bar{x}} < V_{\underline{x}} \text{ it suffices to show}$$

$$\int_{\bar{z}}^{\vec{\theta}_*} c(\tilde{\ell}(\theta + q), \vec{\theta}) d\theta = \int_0^{\vec{\theta}_* - \bar{z}} c(\tilde{\ell}(\theta + q + \bar{z}), \vec{\theta} + \bar{z}) d\theta \leq \int_0^{\vec{\theta}_* - \bar{z}} c(\tilde{\ell}(\theta + w + \bar{z}), \theta + w + \bar{z}) d\theta \leq \int_{\theta_0}^{\theta^*} c(\tilde{\ell}(\theta), \theta) d\theta.$$

To prove the first inequality, we verify that the integrands under the inner integrals, as functions of  $\theta$ , satisfy the conditions of lemma A 1 on the domain  $[0, \vec{\theta}_* - \bar{z}]$ . The second inequality holds because the length of the support of the integrals differs. This gives a contradiction to our assumption that  $V_{\bar{x}} \geq V_{\underline{x}} \geq 0$ . So we find  $V_{\bar{x}} < 0 \leq V_{\underline{x}}$ , a contradiction to inequality (24).  $\square$

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