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Rationalizable Expectations and Sunspot Equilibria in an Overlapping-generations Economy

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There are two theories for the treatment of market uncertainty: rationalizable expectations and sunspot equilibria. This paper shows how the game-theoretic solution concept of rationalizable expectations can be applied to an overlapping-generations exchange economy. Some general properties of these equilibria are discussed. It is shown that rationalizable-expectations equilibria are the predictions yielded by considering sunspot equilibria in which probability beliefs may differ across individuals. This result allows for a new interpretation of sunspot equilibria and helps to understand their relevance.

Keywords: market uncertainty, overlapping generations, rational expectations, rationalizable expectations, sunspots.

JEL classification: D51, D84, D90, E21.

1 Introduction

The economy is a social system composed of individual economic actors who are uncertain about each other's behavior. Uncertainty of this sort is called market uncertainty (Shell, 1987, p. 549). In the 1980s two theories for the treatment of market uncertainty have been developed. One is the concept of sunspot equilibria, introduced by Cass and Shell (1983). The other is the concept of rationalizable expectations, developed by Bernheim (1984) and Pearce (1984) as a solution concept for noncooperative games.

McAllister (1988) and Guesnerie (1992) have shown that rationalizable expectations can also be applied to market economies that cannot be written in strategic form. This paper analyzes the rationalizable-expectations equilibria of an overlapping-generations exchange economy. It is shown that rationalizable-expectations equilibria are the price and consumption paths predicted by the theory of sunspot equilibria if one allows probability beliefs about sunspot activity to differ across individuals. This result allows for a new interpretation of sunspot equilibria and helps to understand their relevance.

Market uncertainty arises whenever the actions of an economic agent depend on his expectations about variables that are or will be influenced by other agents. The simplest paradigm for such a situation is an overlapping-generations economy. Demand in one period depends on the individuals' expectations about future prices. These prices are endogenous variables, depending on the demand of future generations. However, the demand of future generations depends on their expectations about prices in their relative future. Thus, price expectations can be traced back to expectations on expectations. In order to close the model, a solution concept, endogenizing expectations, is needed.

The most popular method of endogenizing expectations is the rational-expectations hypothesis (REH). It is based on the idea that individual expectations should be the same as the predictions of the theory (Muth, 1961). A rational-expectations equilibrium (REE) is a function, assigning determinate values to all endogenous variables for any possible state of the world. The individuals are assumed to behave as though they know this function. In a model with a single REE, the equilibrium is the prediction of the theory and individual expectations are the same.

Intertemporal-allocation models typically have multiple equilibria. In such cases the prediction of the theory is a *set* of equilibria. Beliefs that assign positive probabilities to different equilibria would accord with the theory. But beliefs of this kind are excluded by the REH. In these cases, the concept of rational expectations does not give a complete description of all economic events consistent with rational behavior. Individuals with rational expectations behave *as if* they are able to predict the behavior of other agents for each state of the world correctly. Thus, instead of dealing with market uncertainty, the REH eliminates it.

Morgenstern (1935) recognized the dilemma inflicted by the hypothesis that individual expectations are the same as the predictions of the theory: the agents are assumed to behave as though they know the predictions of a theory which tries to explain their behavior. A prediction of the theory requires well-defined individual expectations and vice versa.

In order for all equilibria to be consistent with rationality, Morgenstern (1935) proposed an iterative procedure which successively eliminates all expectations that contradict the predictions of the theory: the first prediction of the theory consists of the events that are consistent with unrestricted expectations. Eliminating expectations that contradict this first prediction leads to a second prediction which is more accurate. In the third round only expectations according with the second prediction are admitted, and so on. The limit of this mental process describes a set of equilibria that are predicted by the theory whatever positive probabilities the agents assign to these equilibria.

Luce and Raiffa (1957) applied this method as iterative elimination of strictly dominated strategies to noncooperative games. Bernheim (1984) and Pearce (1984) strengthened it by considering the independence of individual strategies. They called the strategies surviving the elimination procedure "rationalizable." The term "rationalizable expectations" has been used accordingly for the probability measures supported by rationalizable strategies.

Bernheim (1986), Brandenburger and Dekel (1987), and Tan and Werlang (1988) have analyzed the decision-theoretic foundation of rationalizable expectations. They demonstrated that this concept is characterized by assuming that the players of a game are choosing strategies independently of each other in order to maximize their expected utilities while the solution concept is common knowledge. Another assumption, characterizing rationalizable expectations, is that the players have the same information as an outside observer (Heinemann, 1995a).

McAllister (1988) adapted rationalizable expectations to a market economy with asymmetric information.[1] Guesnerie (1992) introduced this concept to a macroeconomic model in the spirit of Muth (1961). A general method for the application of decision-theoretic solution concepts to market economies that cannot be written in strategic form has been developed by Heinemann (1995a, b). In this paper rationalizable expectations are applied to an overlapping-generations exchange economy. To make the arguments as clear as possible, we consider an economy with one commodity per period and one unproductive and useless asset that may be interpreted as fiat money or land.

Sunspot equilibria are another theory for the analysis of market uncertainty: Aumann (1974) observed that noncooperative games may have equilibria in which the chosen strategies depend on extrinsic random variables which have no influence on the payoff matrix. He named them "a posteriori equilibria." Cass and Shell (1983) applied this idea to nonstrategic market economies. They designed a nonstochastic economy and introduced a random variable, called "number of sunspots," that is unrelated to the fundamentals of the economy. They showed that there are rational-expectations equilibria in which the endogenous variables depend on the number of sunspots. The additional equilibria, obtained by considering extrinsic uncertainty, can be interpreted as being consistent with rational behavior but excluded by the REH. The prediction of this theory is the set of price paths that can arise in an equilibrium for some sequence of sunspots.

1 McAllister (1988) calls these expectations "weakly admissible."

The meaning of "sunspots" is still open to interpretation. They may be viewed as labels for psychological factors[2] or as selecting devices which affect the economy directly, but are excluded from economic analysis.[3] This paper suggests that the "number of sunspots" should be interpreted as a label for a price system, like a catalogue number. In a sunspot equilibrium the individuals assign probabilities to numbers which stand for different price systems. This is to avoid modelling subjective probabilities for prices directly. The number of sunspots is not a true exogenous variable; it can only be identified by observing the prices and using the rule of numeration. This interpretation follows naturally from the main result of this paper: the price and consumption paths predicted by the rational-expectations equilibria of an overlapping-generations exchange economy with extrinsic uncertainty are the rational*izable*-expectations equilibria of the *non*stochastic economy.

A related result has been provided by Brandenburger and Dekel (1987). In the context of noncooperative game theory they have shown the equivalence of iteratively undominated strategies and Aumann's a posteriori equilibria. Forges and Peck (1995) have demonstrated that common-belief sunspot equilibria of an overlapping-generations economy can be viewed as correlated equilibria of a market game à la Shapley and Shubik (1977). For a wider class of models, Guesnerie (1993a, b) has shown that a sunspot equilibrium is unique if, and only if, there is a unique rationalizable-expectations equilibrium.[4]

The paper is organized as follows: Section 2 introduces the overlapping-generations economy. In Sect. 3 the rational-expectations equilibria of this economy are discussed. Section 4 applies the concept of rationalizable expectations. Section 5 states some properties of rationalizable-expectations equilibria. Section 6 defines sunspot equilibria for the economy and analyzes their relationship to rationalizable-expectations equilibria. Some concluding remarks are given in Sect. 7.

2 Overlapping-generations Economy

Consider an overlapping-generations exchange economy with a pure store of value. Periods are indexed by $t \in \mathbb{N}$. A generation consists of a set of individuals, $I_t \subseteq \mathbb{N}$, who participate in the markets of periods t

2 See Azariadis and Guesnerie (1986, p. 725).

3 "... one may think of political, cultural, and institutional features" (Benhabib and Rustichini, 1994, p. 2).

4 Without formal proof Guesnerie gives convincing arguments for this result in the framework of an abstract one-step forward-looking system.

and $t+1$. An individual $i \in I_t$ possesses an endowment e^i of the one and only consumption good in period t. In the first period there is an exogenously given supply of assets, denoted by a_0. These assets earn no dividends and exist forever. They serve as a pure store of value and are usually interpreted as fiat money. We set the price of the consumption good in each period equal to one and denote by q_t the asset price in period t.

The asset price in each period is assumed to be a Walrasian equilibrium price. Here, we assume that the activities within a period are perfectly coordinated. The price in period t depends on generation t's expectations about the price in period $t+1$ which may (at least in principle) be subject to market uncertainty.

By c_t^i we denote the consumption of individual i in period t. The utility function of individual $i \in I_t$ is $U^i \colon \mathbb{R}_+^2 \to \mathbb{R}$ and assigns the utility level $U^i(c_t^i, c_{t+1}^i)$ to i's consumption plan (c_t^i, c_{t+1}^i). We assume that the underlying preferences satisfy nonsatiation. The individuals maximize expected utilities subject to their budget constraints. This behavior is reflected in the asset–demand correspondences.

An allocation in period $t \in \mathbb{N}$ is a tupel $C_t := (c_t^i)_{i \in I_t \cup I_{t-1}}$, where $I_0 = \emptyset$. It is feasible if

$$\sum_{i \in I_t} (c_t^i - e^i) + \sum_{i \in I_{t-1}} c_t^i \leq 0 \,. \tag{1}$$

An allocation of the economy is a sequence $C = (C_t)_{t \in \mathbb{N}}$. It is feasible if (1) holds for all $t \in \mathbb{N}$.

3 Rational-expectations Equilibria

Let us first assume that every young individual has a point expectation about the asset price next period, and denote by q_{t+1}^i the price individual i expects for period $t+1$. The decision problem of agent $i \in I_t$ in period t is

$$\max_{c_t, c_{t+1}, a} U^i(c_t, c_{t+1}) \quad \text{s.t. } c_t + q_t\, a = e^i \text{ and } c_{t+1} = q_{t+1}^i\, a \,. \tag{2}$$

Replacing the consumption plan by the budget constraints yields the asset–demand correspondence

$$A^i(q_t, q_{t+1}^i) := \arg\max_a U^i(e^i - q_t\, a,\; q_{t+1}^i\, a) \,. \tag{3}$$

A price system $(q_t)_{t \in \mathbb{N}}$ is said to clear the markets for given price expectations $(q^i_{t+1})_{i\in I_t, t\in\mathbb{N}}$ if for each $t \in \mathbb{N}$ there exists a vector $(a^i)_{i\in I_t}$, such that

$$\sum_{i\in I_t} a^i = a_0 \quad \text{and} \quad a^i \in A^i(q_t, q^i_{t+1}), \quad \forall i \in I_t . \tag{4}$$

The set of price systems, which clear the markets for some system of given price expectations, may be quite large. In order to reduce the number of price systems compatible with the model, some condition endogenizing expectations is needed. The most popular condition of this kind is the rational-expectations hypothesis. In a model without exogenous uncertainty, rational expectations are identical with perfect foresight. Thus, in our framework a rational-expectations equilibrium is a price system that clears all markets when it is expected by all individuals plus an appropriate allocation.

Definition 1: A price system $(q_t)_{t\in\mathbb{N}}$ and a feasible allocation C are a rational-expectations equilibrium (REE) if for each $t \in \mathbb{N}$

$$\sum_{i\in I_t}(e^i - c^i_t) = q_t a_0 \quad \text{and} \quad \frac{e^i - c^i_t}{q_t} \in A^i(q_t, q_{t+1}), \quad \forall i \in I_t . \tag{5}$$

The properties of the set of REE have been widely analyzed.[5] Since the nonmonetary steady state ($q_t = 0\ \forall t$) is always an REE, there are multiple REE in all interesting cases. For a wide class of preference profiles the set of REE is a continuum. Although the multiplicity of REE shows that the theory is unable to predict a single equilibrium price path, in an REE individuals behave as though they know future prices.

The formulation of rational expectations eliminates market uncertainty. Two assumptions are responsible for this deficiency: Individuals are assumed to have point expectations and all individuals are assumed to expect the same equilibrium. While point expectations lead to a behavior that is indistinguishable from the behavior of individuals who are sure about next period's price, the second assumption eliminates the coordination problem associated with market uncertainty. The rationality assumption itself does not contradict market uncertainty. Therefore, in the next section we will dispense with point expectations and allow for different individuals to have different expectations, but we will keep the

5 See Geanakoplos and Polemarchakis (1991) for an overview.

rationality assumption in the sense that the agents' expectations accord with the predictions of the theory.

4 Rationalizable Expectations

In the sequel the concept of rationalizable expectations is applied to the overlapping-generations economy. We follow directly Morgenstern's (1935) proposal to eliminate successively all expectations that violate the predictions of the theory. We assume that each individual assigns positive probabilities to denumerably many prices, so that the expectation of an individual i can be described by a function $p^i \in \Delta(\mathbb{R}_+)$, where

$$\Delta(Q) := \left\{ p\colon \mathbb{R}_+ \to [0, 1] \;\middle|\; \sum_{q \in Q} p(q) = 1 \right\}, \quad \forall Q \subseteq \mathbb{R}_+ . \tag{6}$$

$p^i(q)$ is the subjective probability individual $i \in I_t$ assigns to the event $q_{t+1} = q$.

The expected utility of a young individual $i \in I_t$, who decides to buy a units of the asset at price q_t, is

$$\sum_{q \in \mathbb{R}_+} U^i(e^i - q_t\, a,\, q\, a) p^i(q) . \tag{7}$$

Since each individual maximizes expected utility, her asset–demand correspondence is now defined by

$$\tilde{A}^i(q_t, p^i) := \arg\max_a \sum_{q \in \mathbb{R}_+} U^i(e^i - q_t\, a,\, q\, a) p^i(q) . \tag{8}$$

A price q_t is clearing the market in period t for given expectations $(p^i)_{i \in I_t}$ if there exists a vector $(a^i)_{i \in I_t}$, such that

$$\sum_{i \in I_t} a^i = a_0 \quad \text{and} \quad a^i \in \tilde{A}^i(q_t, p^i), \quad \forall i \in I_t . \tag{9}$$

Let $\Phi_t((p^i)_{i \in I_t})$ be the set of prices which are clearing the market in period t for given expectations $(p^i)_{i \in I_t}$. If we allow the individuals to have any expectation $p^i \in \Delta(\mathbb{R}_+)$, then only prices in

$$Q_t^1 := \{ q \in \mathbb{R}_+ \mid \exists (p^i)_{i\in I_t},\ p^i \in \Delta(\mathbb{R}_+) \text{ s.t. } q \in \Phi_t((p^i)_{i\in I_t}) \} \tag{10}$$

are compatible with the model in period t. Thus, the first prediction of the theory is

$$q_t \in Q_t^1 \quad \forall t\ . \tag{11}$$

Let us now assume that the agents in the economy possess the same information as an outside observer. Then the agents know, like we do, that only prices $q_t \in Q_t^1$ can occur in period t. Since an individual $i \in I_t$ will assign probability zero to all prices which she knows to be impossible, her expectation will be

$$p^i \in \Delta(Q_{t+1}^1)\ . \tag{12}$$

Price q_t will then be contained in the subset

$$Q_t^2 := \{ q \in \mathbb{R}_+ \mid \exists (p^i)_{i\in I_t},\ p^i \in \Delta(Q_{t+1}^1) \text{ s.t. } q \in \Phi_t((p^i)_{i\in I_t}) \}\ . \tag{13}$$

This is the second prediction of the theory. By our assumption, the agents know the predictions of the theory. Hence, the expectation of $i \in I_t$ will be

$$p^i \in \Delta(Q_{t+1}^2)\ . \tag{14}$$

Therefore, prices must be $q_t \in Q_t^3$ $\forall t$ and so on. This mental process successively eliminates all prices which contradict the assumption that agents possess the same information as an outside observer. Individual expectations coincide with the predictions of the theory if we take the elimination procedure to its limit. To do this, define $Q_t^0 := \mathbb{R}_+$ for all $t \in \mathbb{N}$ and

$$Q_t^k := \{ q \in \mathbb{R}_+ \mid \exists (p^i)_{i\in I_t},\ p^i \in \Delta(Q_{t+1}^{k-1}) \text{ s.t. } q \in \Phi_t((p^i)_{i\in I_t}) \}\ , \quad \forall k, t \in \mathbb{N}\ . \tag{15}$$

Since $Q_t^k \subseteq Q_t^{k-1}$ for all k and t, there exists a limit set

$$Q_t^\infty = \bigcap_{k=1}^{\infty} Q_t^k \tag{16}$$

for each t. Q_t^∞ is the set of prices that can occur in period t if the members of generation t have rationalizable expectations. Therefore, a price $q_t \in Q_t^\infty$ is called rationalizable.

Definition 2: A price system $(q_t)_{t\in\mathbb{N}}$ and a feasible allocation C are a rationalizable-expectations equilibrium (RBEE) if for each $t \in \mathbb{N}$ there exists a system of expectations $p^i \in \Delta(Q_{t+1}^\infty)$, $i \in I_t$, such that

$$q_t \in Q_t^\infty, \quad \sum_{i\in I_t}(e^i - c_t^i) = q_t a_0, \quad \text{and} \quad \frac{e^i - c_t^i}{q_t} \in \tilde{A}(q_t, p^i) \ . \quad (17)$$

5 Properties of Rationalizable-expectations Equilibria

Since point expectations can be represented by special functions $p^i \in \Delta(\mathbb{R}_+)$, any REE is also rationalizable. Therefore, the existence of REE guarantees

$$Q_t^\infty \neq \emptyset \quad \forall t \in \mathbb{N} \ . \quad (18)$$

Note that there may be RBEE even if there is no REE. Furthermore, if there is a unique RBEE then it is the unique REE.[6]

An RBEE allows for apparently erratic fluctuations of the asset price. These fluctuations are purely driven by expectations. There is no need for exogenous random shocks to explain these fluctuations, but they can neither be predicted nor can economic theory assign any probabilities to the different equilibria. The prediction of the theory is a set of price paths. The set of all rationalizable price paths is

$$Q^\infty := \prod_{t\in\mathbb{N}} Q_t^\infty \ . \quad (19)$$

In the context of game theory, Pearce (1984) and Basu and Weibull (1991) have defined some properties for set-valued solution concepts that can easily be adapted to the overlapping-generations model:

Let Q be a set of price sequences, for which there exists a sequence of subsets of prices $(Q_t)_{t\in\mathbb{N}}$, such that

$$Q = \prod_{t\in\mathbb{N}} Q_t \subseteq \prod_{t\in\mathbb{N}} \mathbb{R}_+ \ . \quad (20)$$

6 Guesnerie (1992, p. 1258) calls a unique RBEE a "strongly rational-expectations equilibrium."

Q has the best-response property if for any price sequence $(q_t)_{t\in\mathbb{N}} \in Q$ there exist expectations $p^i \in \Delta(Q_{t+1})$ for each i and for all t, such that these prices are market clearing. Q is closed under rational behavior (curb) if for any system of expectations $p^i \in \Delta(Q_{t+1})$, $i \in I_t$, $t \in \mathbb{N}$, there exists a sequence of market-clearing prices $(q_t)_{t\in\mathbb{N}} \in Q$. Formally, Q has the best-response property [is curb] if

$$Q_t \subseteq [\supseteq] \{ q \in \mathbb{R}_+ \mid \exists (p^i)_{i\in I_t},\ p^i \in \Delta(Q_{t+1}) \text{ s.t. } q \in \Phi_t((p^i)_{i\in I_t}) \} ,\quad \forall t \in \mathbb{N} . \tag{21}$$

A set Q, which is curb *and* has the best-response property, is called "tight curb" by Basu and Weibull (1991). A tight curb set consists of all price sequences which can arise when all individuals expect that only price systems out of this set will occur. A tight curb set is a set-valued self-fulfilling prophecy. The two most important properties of rationalizable-expectations equilibria are that Q^∞ is tight curb and that any set Q with the best-reponse property is a subset of Q^∞.

Theorem 1: 1. The set of rationalizable price paths Q^∞ is tight curb.
2. If a subset $\prod_{t\in\mathbb{N}} Q_t \subseteq \prod_{t\in\mathbb{N}} \mathbb{R}_+$ has the best-response property, then $Q_t \subseteq Q_t^\infty$ for all t.

Proof: See appendix.

An immediate corollary is that Q^∞ is the union of all tight curb sets and, therefore, the largest self-fulfilling prophecy. These properties allow the application of Tarski's fixed-point theorem (Balkenborg, 1992), and they are a useful tool for calculations of the set of rationalizable-expectations equilibria. Theorem 1 is also very useful in proofing further results related to the concept of rationalizable expectations. The latter is demonstrated in the next section.

6 Sunspot Equilibria

A random variable is called extrinsic if it lacks any direct influence on the fundamentals of the economy, like preferences, endowments, money supply, etc. Cass and Shell (1983) proved the existence of rational-expectations equilibria in which extrinsic random variables matter. They designed a nonstochastic economy, introduced an extrinsic random variable, which they called "number of sunspots," and showed that prices and allocation may depend on the number of sunspots if the market participants believe in this dependence. If the agents have different probability assessments at equilibrium, then sunspots matter trivially. But, even if agents are restricted to share a common probability belief, there exist equilibria in which sunspots matter. This has been shown for models with incomplete markets, incomplete market participation, monopolistic competition, externalities, increasing returns, and for all models with multiple nonstochastic equilibria.[7] In particular, sunspot equilibria exist in overlapping-generations economies.

For many economists the mere existence of equilibria in which sunspots matter is no convincing argument for considering these equilibria in their analysis. For them, sunspot equilibria remain a theoretical curiosity. Although the belief in the matter of sunspots is self-fulfilling, some of these economists are troubled by the question of why people should believe that economic data depend on sunspots in the first place. Another barrier to the use of this concept are the technical difficulties in calculating the set of all sunspot equilibria. In the sequel we compare the sunspot equilibria of the overlapping-generations economy with its rationalizable-expectations equilibria. We find that the rationalizable-expectations equilibria are the price and consumption paths that can occur in sunspot equilibria when the agents are allowed to hold diverse beliefs about the probabilities of sunspots. In this sense the predictions of both theories are identical. This result helps to understand the relevance of sunspot equilibria, and it shows that the iterative definition of rationalizable expectations can serve as an algorithm to calculate the predictions of sunspot equilibria.

Let $s_t \in \mathbb{N}_0$ be the number of sunspots in period t and denote by $\hat{p}^i(s)$ the probability that individual $i \in I_t$ assigns to the event $s_{t+1} = s$. The expectation of $i \in I_t$ concerning the number of sunspots in period $t+1$ is

$$\hat{p}^i \in \Delta(\mathbb{N}_0) \ . \tag{22}$$

The description of an extrinsic state in one period may be so complex that it contains all past sunspot numbers.[8] Thus, we do not lose generality if we restrict ourselves to assume that in equilibrium the asset price and the allocation in some period t are a function of the number of sunspots in that period. We denote these functions by $\rho_t\colon \mathbb{N}_0 \to \mathbb{R}_+$ and $\zeta_t\colon \mathbb{N}_0 \to \mathbb{R}_+^{|I_t \cup I_{t-1}|}$, with

$$q_t = \rho_t(s_t) \quad \text{and} \quad C_t = \zeta_t(s_t) \ . \tag{23}$$

7 For surveys, see Shell (1987) and Benhabib and Rustichini (1994).

8 This is because the product set $\prod_{t \in \mathbb{N}} \mathbb{N}_0$ is isomorph to $\mathbb{N}_0$.

The expected utility of a young individual $i \in I_t$, who decides to buy a units of the asset at price q_t, is

$$\sum_{s \in \mathbb{N}_0} U^i(e^i - q_t a, \ \rho_{t+1}(s) a) \hat{p}^i(s) \ . \tag{24}$$

Her asset demand is

$$\hat{A}^i(q_t, \rho_{t+1}, \hat{p}^i) = \arg\max_a \sum_{s \in \mathbb{N}_0} U^i(e^i - q_t a, \ \rho_{t+1}(s) a) \hat{p}^i(s) \ . \tag{25}$$

There are two definitions of REE: one, from Radner (1979) and used especially in microeconomic models, assumes that all individuals know the functions ρ_t but need not agree on the probability distribution of exogenous random variables. The other, given by Lucas and Prescott (1971) and common in macroeconomics, requires furthermore that all agents agree on these probabilities. Most of the literature on sunspot equilibria considers only those equilibria in which agents have a common probability assessment. In this paper we use the weaker definition that allows for diverse beliefs.

Definition 3: A sequence of functions $(\rho_t, \zeta_t)_{t \in \mathbb{N}}$, ρ_t: $\mathbb{N}_0 \to \mathbb{R}_+$, ζ_t: $\mathbb{N}_0 \to \mathbb{R}_+^{|I_t \cup I_{t-1}|}$, is a rational-expectations equilibrium of the sunspot economy (sunspot equilibrium)[9] if for each $t \in \mathbb{N}$ there exists a system of expectations $\hat{p}^i \in \Delta(\mathbb{N}_0)$, $i \in I_t$, such that for all $s \in \mathbb{N}_0$, $\zeta_t(s)$ is a feasible allocation in period t,

$$\sum_{i \in I_t} (e^i - \zeta_t^i(s)) = \rho_t(s) a_0, \quad \text{and} \quad \frac{e^i - \zeta_t^i(s)}{\rho_t(s)} \in \hat{A}^i(\rho_t(s), \rho_{t+1}, \hat{p}^i) \ , \quad \forall i \in I_t \ . \tag{26}$$

Definition 4: A sunspot equilibrium $(\rho_t, \zeta_t)_{t \in \mathbb{N}}$ is called stationary if $\rho_t = \rho$ and $\zeta_t = \zeta \ \forall t$.

The literature has paid special attention to stationary sunspot equilibria in order to analyze the dynamic behavior of a stationary and in-

9 Cass and Shell (1983) use the term "sunspot equilibrium" only for such equilibria in which at least one of these functions is not constant. If $\rho_t(s) = q_t$ and $\zeta_t(s) = C_t$ for all s and t, then $(q_t)_{t \in \mathbb{N}}$ and C are a rational-expectations equilibrium according to Definition 1.

trinsically nonstochastic economy. The stationarity of the fundamentals of our overlapping-generations economy is expressed by Assumption 1.

Assumption 1: All generations are identical in size, preference profile, and endowments, i.e.,

$$|I_t| = |I_1|, \quad (U^i)_{i\in I_t} = (U^i)_{i\in I_1}, \quad \text{and} \quad (e^i)_{i\in I_t} = (e^i)_{i\in I_1}, \quad \forall t \, . \tag{27}$$

Theorem 2: 1. If $(\rho_t, \zeta_t)_{t\in\mathbb{N}}$ is a sunspot equilibrium, then for each sequence of numbers $s_t \in \mathbb{N}_0$, $t \in \mathbb{N}$, the price system and the allocation given by

$$q_t = \rho_t(s_t) \quad \text{and} \quad C_t = \zeta_t(s_t), \quad t \in \mathbb{N} \, , \tag{28}$$

are a rationalizable-expectations equilibrium.

2. If $(q_t)_{t\in\mathbb{N}}$ and C are a rationalizable-expectations equilibrium and $(s_t)_{t\in\mathbb{N}}$ is a sequence of numbers $s_t \in \mathbb{N}_0$ $\forall t$, then there exists a sunspot equilibrium $(\rho_t, \zeta_t)_{t\in\mathbb{N}}$, such that

$$q_t = \rho_t(s_t) \quad \text{and} \quad C_t = \zeta_t(s_t), \quad \forall t \in \mathbb{N} \, . \tag{29}$$

3. If Assumption 1 holds and if $(q_t)_{t\in\mathbb{N}}$ and C are a rationalizable-expectations equilibrium, then there exist a sequence of numbers $(s_t)_{t\in\mathbb{N}}$, $s_t \in \mathbb{N}_0$ $\forall t$, and a stationary sunspot equilibrium $(\rho, \zeta)_{t\in\mathbb{N}}$, such that

$$q_t = \rho(s_t) \quad \forall t \in \mathbb{N} \, . \tag{30}$$

Proof: See appendix.

Theorem 2 states that the set of rationalizable-expectations equilibria coincides with the price and consumption paths predicted by the theory of sunspot equilibria. In this sense the two theories are equivalent. This allows for a new interpretation of sunspot equilibria: the variable "number of sunspots" serves as an enumeration of price systems. Each realization of the variable is associated with a price consistent with rationalizable expectations. This number acts only as a label for the associated price. Instead of modelling expectations about the endogenous variable "price of next period" directly, in a sunspot economy these prices are numbered, and expectations are modelled as probabilities for these numbers. In equilibrium, individuals behave as though they know which numbers are associated with each price. But this only means that they know the rule of numeration. A subjective probability distribution of the number of sunspots effects demand in the same way as does an appropriate distribution of prices.

In rationalizable-expectations equilibria, market uncertainty is modelled directly by defining beliefs about endogenous variables (prices). In sunspot equilibria, exogenous variables (sunspots) are inserted. Beliefs are probabilities for sunspots and the number of sunspots determines the endogenous variables. This detour may be justified because we are more familiar with modelling expectations about exogenous random terms. But the number of sunspots is no independent and directly observable random variable. It is just a product of mind, a catalogue number for a price system. This number cannot be observed; it can only be identified by observing prices and using the rule of numeration.

Theorem 2 part 2 states that for any sequence of sunspot numbers $(s_t)_{t \in \mathbb{N}}$ there exists a sunspot equilibrium $(\rho_t, \zeta_t)_{t \in \mathbb{N}}$, such that the sequence $(\rho_t(s_t), \zeta_t(s_t))_{t \in \mathbb{N}}$ equals a given rationalizable-expectations equilibrium. In other words, any sequence of sunspot numbers is compatible with any sequence of rationalizable prices. So, even if we had a real and observable extrinsic random variable with a known probability distribution, we would not gain any information about the probabilities of prices. Hence, the two theories are not only predicting the same set of price paths, but they are also both incapable of assigning probabilities to these prices.

Theorem 2 part 3 says that any price sequence of an RBEE can occur in a stationary sunspot equilibrium on the premises that the number of sunspots follows an appropriate path. Thus, the restriction to stationary sunspot equilibria plus the assumption of a Markovian process on the number of sunspots sheds some light on the probabilities of prices.

The concept of sunspot equilibria assumes that individuals know the rule of numeration, i.e., the equilibrium. However, the prediction of this theory is a set of equilibria. One might then ask whether there are additional price paths that can arise only when individuals assign positive probabilities to different sunspot equilibria. The answer is no, and the reasoning is based on the two theorems stated above. Suppose that individuals assign positive probabilities to different sunspot equilibria. By Theorem 2 part 1 this has the same effect on demand as assigning positive probabilities to different rationalizable prices. By Theorem 1 part 1 the market-clearing prices for such beliefs are rationalizable. But then, by Theorem 2 part 2, these prices can arise in a sunspot equilibrium.

While the existence of equilibria in which sunspots matter has been shown for a variety of models, there has been no algorithm capable to calculate the set of all sunspot equilibria. In consequence of Theorem 2, the iterative procedure described in Sect. 4 can serve this purpose. Sometimes this algorithm is not finite, but in "good" cases the rationalizable-expectations equilibria can also be calculated by using the properties stated in Sect. 5.

7 Concluding Remarks

We have seen how the idea of rationalizable expectations can be applied to an overlapping-generations economy. The definition of rationalizable-expectations equilibria (RBEE) is straightforward and can easily be extended to more elaborate overlapping-generations models. Compared to REE, the set of RBEE is quite large. This may be seen as an argument in favor of rational instead of rationalizable expectations. But note that rationality alone does not exclude these additional equilibria. The relatively small size of the set of REE follows from assuming highly coordinated point expectations that do not capture the idea of market uncertainty.

The equivalence of the concepts of rationalizable expectations and sunspot equilibria seems to hold for far more general models than the one considered in this paper. However, one must be careful in the definitions of these equilibria. We allowed only expectations that assign a positive probability to denumerably many prices because the variable "number of sunspots" should be denumerable. Note that we allowed for diverse beliefs about the probabilities of sunspots, while most of the literature concentrates on sunspot equilibria in which all individuals share a common belief about the sequence of sunspot numbers. The set of common-belief sunspot equilibria is much smaller, and there are RBEE that cannot occur in any sunspot equilibrium with common beliefs.

Appendix

Proof of Theorem 1 Part 1

We have to show that

$$Q_t^\infty = \{\, q \in \mathbb{R}_+ \mid \exists (p^i)_{i\in I_t},\ p^i \in \Delta(Q_{t+1}^\infty) \text{ s.t. } q \in \Phi_t((p^i)_{i\in I_t}) \,\}\ , \quad \forall t \in \mathbb{N}\ . \tag{31}$$

By construction of the sequences $(Q_t^k)_{k\in\mathbb{N}_0}$ and by definition of Q_t^∞, the statements (32) to (35) are equivalent.

$$q_t \in Q_t^\infty\ , \tag{32}$$

$$q_t \in Q_t^k \quad \forall k, \tag{33}$$

$$\exists (p^i)_{i \in I_t} \text{ s.t. } p^i \in \Delta(Q_{t+1}^{k-1}) \ \forall k \in \mathbb{N}, \forall i \in I_t$$
$$\text{and } q_t \in \Phi_t((p^i)_{i \in I_t}) \ , \tag{34}$$

$$\exists (p^i)_{i \in I_t} \text{ s.t. } p^i \in \Delta(Q_{t+1}^{\infty}) \ \forall i \in I_t \text{ and } q_t \in \Phi_t((p^i)_{i \in I_t}) \ . \tag{35}$$

□

Proof of Theorem 1 Part 2

Let $\prod_{t \in \mathbb{N}} Q_t \subseteq \prod_{t \in \mathbb{N}} \mathbb{R}_+$ be a set with the best-response property, i.e.,

$$Q_t \subseteq \{ q \in \mathbb{R}_+ \mid \exists (p^i)_{i \in I_t}, \ p^i \in \Delta(Q_{t+1}) \text{ s.t. } q \in \Phi_t((p^i)_{i \in I_t}) \} \ ,$$
$$\forall t \in \mathbb{N} \ . \tag{36}$$

If there is a $k \in \mathbb{N}_0$ such that $Q_t \subseteq Q_t^k$ for all t, then for each t and for any $q_t \in Q_t$ there exist expectations $p^i \in \Delta(Q_{t+1}) \subseteq \Delta(Q_{t+1}^k)$, $i \in I_t$, such that $q \in \Phi((p^i)_{i \in I_t})$. Hence, $q_t \in Q_t^{k+1}$. Therefore, $Q_t \subseteq Q_t^k \ \forall t$ implies $Q_t \subseteq Q_t^{k+1} \ \forall t$. Since $Q_t \subseteq Q_t^0 \ \forall t$, we have $Q_t \subseteq Q_t^k$ for all k and t and, therefore, $Q_t \subseteq Q_t^{\infty} \ \forall t \in \mathbb{N}$. □

Proof of Theorem 2 Part 1

Let $(\rho_t, \zeta_t)_{t \in \mathbb{N}}$ be a sunspot equilibrium. Then for each $t \in \mathbb{N}$ there exists a system of expectations $\hat{p}^i \in \Delta(\mathbb{N}_0)$, $i \in I_t$, such that $\zeta_t(s)$ is feasible and (26) holds for all $s \in \mathbb{N}_0$. For each $t \in \mathbb{N}$ and for each $i \in I_t$ define $p^i \in \Delta(\mathbb{R}_+)$ by

$$p^i(q) := \sum_{s \mid q = \rho_{t+1}(s)} \hat{p}^i(s) \ . \tag{37}$$

Then $p^i \in \Delta(\rho_{t+1}(\mathbb{N}_0))$, with

$$\rho_{t+1}(\mathbb{N}_0) := \{ q \in \mathbb{R}_+ \mid \exists s \in \mathbb{N}_0 \text{ s.t. } q = \rho_{t+1}(s) \} \ . \tag{38}$$

Then, for all $t \in \mathbb{N}$, for each $i \in I_t$ and for every $s_t \in \mathbb{N}_0$:

$$\hat{A}^i(\rho_t(s_t), \rho_{t+1}, \hat{p}^i) = \tilde{A}^i(\rho_t(s_t), p^i) \ . \tag{39}$$

Hence, $\rho_t(s_t) \in \Phi_t((p^i)_{i\in I_t})$ for all $s_t \in \mathbb{N}_0$ and for all $t \in \mathbb{N}$. Therefore,

$$\rho_t(\mathbb{N}_0) \subseteq \{ q \in \mathbb{R}_+ \mid \exists (p^i)_{i\in I_t},\ p^i \in \Delta(\rho_{t+1}(\mathbb{N}_0)) \text{ s.t. } q \in \Phi_t((p^i)_{i\in I_t}) \} . \tag{40}$$

Theorem 1 part 2 implies $\rho_t(\mathbb{N}_0) \subseteq Q_t^\infty$ for all t. □

Proof of Theorem 2 Part 2

Let $(s_t)_{t\in\mathbb{N}}$ be an arbitrary sequence of numbers $s_t \in \mathbb{N}_0$ $\forall t \in \mathbb{N}$, and let $(q_t)_{t\in\mathbb{N}}$ and C be a rationalizable-expectations equilibrium. Then $q_t \in Q_t^\infty$ $\forall t \in \mathbb{N}$. Because of Theorem 1 part 1, for any $q \in Q_t^\infty$ there exist expectations $(p^i[q])_{i\in I_t}$, with $p^i[q] \in \Delta(Q_{t+1}^\infty)$ and $q \in \Phi_t((p^i[q])_{i\in I_t})$. Let us choose $p^i[q_t]$, such that (17) holds for all i and t. For each t define $B_t^0 := \{q_t\}$ and for $k \in \{1, 2, \ldots, t\}$

$$B_{t+1}^k := \{ q \in \mathbb{R}_+ \mid \exists i \in I_t,\ \exists q' \in B_t^{k-1} \text{ s.t. } p^i[q'](q) > 0 \} . \tag{41}$$

Further define $B_t := \bigcup_{k=0}^{t-1} B_t^k$. Then

$$p^i[q] \in \Delta(B_{t+1}) \quad \forall q \in B_t,\ \forall i \in I_t,\ \forall t \in \mathbb{N} . \tag{42}$$

Note that $B_t \subseteq Q_t^\infty$. Since B_t is denumerable, there exists a function $\rho_t\colon \mathbb{N}_0 \to B_t$, such that $\rho_t(\mathbb{N}_0) = B_t$ and $\rho_t(s_t) = q_t$.

In the remainder of the proof it will be shown that there are functions ζ_t, such that $\zeta_t(s_t) = C_t$ for all t and $(\rho_t, \zeta_t)_{t\in\mathbb{N}}$ is a sunspot equilibrium. For each $t \in \mathbb{N}$, for each $i \in I_t$, and for each $p^i \in \Delta(B_t)$ define $\hat{p}^i[p^i] \in \Delta(\mathbb{N}_0)$ by

$$\hat{p}^i[p^i](s) = \frac{p^i(\rho_{t+1}(s))}{|\{s' \in \mathbb{N}_0 \text{ with } \rho_{t+1}(s') = \rho_{t+1}(s)\}|} \quad \forall s \in \mathbb{N}_0 . \tag{43}$$

Then the following equations hold for all $t \in \mathbb{N}$, $q_t \in \mathbb{R}_+$, $i \in I_t$, and for every $p^i \in \Delta(B_t)$

$$\hat{A}^i(q_t, \rho_{t+1}, \hat{p}^i[p^i]) = \arg\max_a \sum_{s=0}^{\infty} U^i(e^i - q_t\, a,\ \rho_{t+1}(s)\, a)\hat{p}^i[p^i](s) \tag{44}$$

$$= \arg\max_a \sum_{s=0}^{\infty} U^i(e^i - q_t\, a,\ \rho_{t+1}(s)\, a) \times$$
$$\times \frac{p^i(\rho_{t+1}(s))}{|\{s' \in \mathbb{N}_0 \text{ with } \rho_{t+1}(s') = \rho_{t+1}(s)\}|} \tag{45}$$

$$= \arg\max_a \sum_{q=0}^{\infty} \sum_{s|\rho_{t+1}(s)=q} U^i(e^i - q_t\, a,\ q\, a) \times$$
$$\times \frac{p^i(q)}{|\{s' \in \mathbb{N}_0 \text{ with } \rho_{t+1}(s') = q\}|} \tag{46}$$

$$= \arg\max_a \sum_{q=0}^{\infty} U^i(e^i - q_t\, a,\ q\, a) p^i(q) = \tilde{A}^i(q_t, p^i)\ . \tag{47}$$

Since $\rho_t(s) \in B_t \subseteq Q_t^{\infty}$ for all s and t,

$$p^i[\rho_{t+1}(s)] \in \Delta(B_{t+1}) \quad \forall s \in \mathbb{N}_0,\ \forall i \in I_t,\ \forall t \in \mathbb{N} \tag{48}$$

and

$$\rho_t(s) \in \Phi_t((p^i[\rho_t(s)])_{i \in I_t}) \quad \forall s \in \mathbb{N}_0,\ \forall t \in \mathbb{N}\ . \tag{49}$$

The latter implies that for each $t \in \mathbb{N}$ and for each $s \in \mathbb{N}_0$ there exists a feasible allocation $(\zeta_t^i(s))_{i \in I_t \cup I_{t-1}}$ with $\sum_{i \in I_t}(e^i - \zeta_t^i(s)) = \rho_t(s) a_0$ and

$$\frac{e^i - \zeta_t^i(s)}{\rho_t(s)} \in \tilde{A}^i(\rho_t(s), p^i[\rho_t(s)])$$
$$= \hat{A}^i(\rho_t(s), \rho_{t+1}, \hat{p}^i[p^i[\rho_t(s)]]) \quad \forall i \in I_t\ , \tag{50}$$

where $\zeta_t^i(s_t)$ can be chosen to be c_t^i for all i and t. This shows that $(\rho_t, \zeta_t)_{t \in \mathbb{N}}$ is a sunspot equilibrium. □

Proof of Theorem 2 Part 3

Let $(q_t)_{t \in \mathbb{N}}$ and C be an RBEE. In consequence of Assumption 1, $\Phi_t = \Phi_1$ and $Q_t^{\infty} = Q_1^{\infty}$ for all t. Define

$$B^0 := \{\, q \in \mathbb{R}_+ \mid \exists t \text{ s.t. } q = q_t \,\}\ , \tag{51}$$

for $k \in \mathbb{N}$

$$B^k := \{q \in \mathbb{R}_+ \mid \exists i \in I, \exists q' \in B^{k-1} \text{ s.t. } p^i[q'](q) > 0\} , \tag{52}$$

where $I := \bigcup_{t\in\mathbb{N}} I_t$, and $B := \bigcup_{k=0}^{\infty} B^k$. Then $p^i[q] \in \Delta(B)$ for all $q \in B$ and $i \in I$, $B \subseteq Q_1^{\infty}$, and, since B is denumerable, there exists a function ρ: $\mathbb{N}_0 \to B$, such that $\rho(\mathbb{N}_0) = B$.

For each $i \in I$ and for every $p^i \in \Delta(B)$ define $\hat{p}^i[p^i] \in \Delta(\mathbb{N}_0)$ by

$$\hat{p}^i[p^i](s) = \frac{p^i(\rho(s))}{|\{s' \in \mathbb{N}_0 \text{ with } \rho(s') = \rho(s)\}|} \quad \forall s \in \mathbb{N}_0 . \tag{53}$$

Then for all $q \in \mathbb{R}_+$, $i \in I$, and for every $p^i \in \Delta(B)$

$$\hat{A}^i(q, \rho, \hat{p}^i[p^i]) = \tilde{A}^i(q, p^i) . \tag{54}$$

Note that $\rho(s) \in \Phi_t((p^i[\rho(s)])_{i\in I_t})$ for all s and t. This implies that for each s and t there exists a feasible allocation $\zeta_t(s) = \zeta(s)$ with $\sum_{i\in I_t}(e^i - \zeta^i(s)) = \rho(s)a_0$ and

$$\begin{aligned}\frac{e^i - \zeta^i(s)}{\rho(s)} &\in \tilde{A}^i(\rho(s), p^i[\rho(s)]) \\ &= \hat{A}^i(\rho(s), \rho, \hat{p}^i[p^i[\rho(s)]]) \quad \forall i \in I .\end{aligned} \tag{55}$$

Hence, $(\rho, \zeta)_{t\in\mathbb{N}}$ is a sunspot equilibrium. Since $q_t \in B = \rho(\mathbb{N}_0)$, there exists an $s_t \in \mathbb{N}_0$, such that $q_t = \rho(s_t)$ for each $t \in \mathbb{N}$. □

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