CHARACTERISING EQUILIBRIUM SELECTION IN GLOBAL GAMES WITH STRATEGIC COMPLEMENTARITIES

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Abstract. Global games are widely used to predict behaviour in games with strategic complementarities and multiple equilibria. We establish two results on the global game selection. First, we show that, for any supermodular complete information game, the global game selection is independent of the payoff functions chosen for the game's global game embedding. Second, we give a simple sufficient criterion to derive the selection and establish noise independence in many-action games by decomposing them into games with smaller action sets, to which we may often apply simple criteria. We also report in which small games noise independence may be established by counting the number of players or actions.

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1. Introduction

Games of strategic complementarities, for instance models of financial crises or network externalities, often have multiple equilibria. An important issue, from a theoretical as well as a policy perspective, is how to predict the equilibrium that will be played. One widely used approach to predict behaviour in such games is to turn them into "global games". A global game extends a complete information game to an incomplete information game with a one-dimensional state space, such that the original game can be viewed as one particular realisation of the random state. This state is usually interpreted as an "economic fundamental". Each player receives a noisy private signal about the true state. Then, under certain supermodularity and monotonicity conditions,[1] Frankel, Morris and Pauzner [10] ("FMP") prove *limit uniqueness*: as the noise in private signals vanishes, for almost all realisations of the state parameter players coordinate on a Nash equilibrium of the complete information game given by the payoffs at the true state. This *global game selection* ("GGS") may be used as a prediction and to derive comparative statics results. Applications include models of speculative attacks (Morris and Shin [17], Cukierman,

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[1]FMP assume that there exists an ordering on actions such that players' incentive to switch to a higher action is increasing in both the state and the actions of others; and that at sufficiently low (high) states, each player's lowest (highest) action is strictly dominant. See Section 2 for precise definitions.

Goldstein and Spiegel [9], Guimarães and Morris [12], Corsetti, Dasgupta, Morris and Shin [7], Corsetti, Guimarães and Roubini [8]), banking crises (Goldstein [11], Rochet and Vives [22]), and investment problems (Steiner and Sákovics [23]). Experiments by Heinemann, Nagel and Ockenfels [13, 14] show that the GGS is useful for predicting subjects' behaviour.

Unfortunately, there are many ways to extend a complete information game to a global game, and which equilibrium is selected may depend on the details of the chosen extension. If multiple equilibria are replaced by multiple global game selections, this reduces the value of the GGS as a selection criterion. For instance, it is well-known that the GGS may depend on the distribution of the noisy private signals. To circumvent this problem, FMP provide some conditions under which the GGS is *noise independent*, that is, independent of this distribution.

Furthermore, most applications are limited to binary-action games. In symmetric binary-action games, the GGS can be easily determined as a player's best reply to the belief that the fraction of other players choosing either action is uniformly distributed (see FMP or Morris and Shin [18]; Steiner and Sákovics [23] give a generalisation for a class of asymmetric binary-action games).

In this paper, we extend results on global games in two directions. First, we show that for any supermodular complete information game, the GGS is independent of the extended payoff function, as long as it satisfies the usual supermodularity assumptions. Hence, the GGS does not depend, for example, on the choice of economic fundamental used as a state parameter in the global game. Our result implies that the distribution of private signals is the only source of multiplicity for the GGS.

Second, we provide a new and simple method to determine the GGS and check its noise independence in games with many actions. The GGS is noise independent if the game can be suitably decomposed into smaller noise independent games. For example, we may split up a n-action game into many binary-action games and apply simple known criteria for deriving the GGS. If the smaller games are noise independent and their selections point in direction of the same action profile, then this action profile is the noise independent selection of the larger game.

Our second result gives a new heuristic for the global game equilibrium prediction that is useful in economic applications. For instance, introducing a third action in a binary-action game may change the selection between the two original equilibria. We give an example of a refinancing model where introducing collateralised loans as a third action besides withdrawing and extending unsecured loans changes the GGS from the inefficient withdrawal equilibrium to that of unsecured refinancing. Other models of interest may have many actions and potentially a large number of equilibria. As an example, we analyse a generalised version of Bryant's [3] minimum effort game, to which none of the known easy heuristics may be applied. By decomposing it into binary-action games, we may derive the GGS and establish its noise independence straightforwardly.

The decomposition of large games is particularly useful, because noise independence is typically easier to establish for smaller games, often simply by counting the number of players and actions. In their seminal paper introducing global games, Carlsson and Van Damme [6] proved that all two-player two-action games with multiple equilibria are noise independent. In another application of our methods, we prove the same of all supermodular two-player games in which one player's action set is binary. Together with known results, this completes the characterisation of supermodular games for which noise independence can be established by counting.

Our paper proceeds as follows. Section 2 contains preliminaries. The rest of the paper is organised around a characterisation of the GGS given in Section 3. Instead of analysing a sequence of global games with vanishing noise, we show that we may determine the GGS from a single incomplete information game with simple payoffs. This game is independent of the state-dependent payoff function that is chosen as an extension of the original game. We use the characterisation to prove generic uniqueness of the GGS and extend it to discontinuous global games. In Section 4 we discuss the decomposition method and establish our results on noise independence. Section 5 contains applications. Section 6 concludes. Proofs are in the appendix.

2. Setting and Definitions

Throughout this paper we consider games played by a finite set of players I, who have finite action sets $A_i = \{0, 1, \ldots, m_i\}$, $i \in I$, which we endow with the natural ordering. We define the joint action set A as $\prod_{i \in I} A_i$ and write A_{-i} for $\prod_{j \neq i} A_j$. For action profiles $a = (a_i)_{i \in I}$ and $a' = (a'_i)_{i \in I}$ in A, we write $a \leq a'$ if and only if $a_i \leq a'_i$ for all $i \in I$. The lowest and highest action profiles in A are denoted by 0 and m. A complete information game Γ is specified by its real-valued payoff functions $g_i(a_i, a_{-i})$, $i \in I$, where a_i denotes i's action and $a_{-i} \in A_{-i}$ denotes the opposing action profile. A game Γ is *supermodular* if for all i, $a_i \leq a'_i$, and $a_{-i} \leq a'_{-i}$,

$$(1) \qquad g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i}) \geq g_i(a_i, a'_{-i}) - g_i(a'_i, a'_{-i}).$$

This implies a player's best reply is non-decreasing in the actions of her opponents (see Topkis [24]; note that property (1) is what FMP refer to as *strategic complementarities*).

We define a global game $G(v)$ as in FMP [10]: consider payoff functions $u_i(a_i, a_{-i}, \theta)$ that depend on an additional state parameter $\theta \in \mathbb{R}$. For each θ, let the game given by the $u_i(\cdot, \theta)$ be a supermodular game (Assumption **A1**). In addition, assume there are *dominance regions*: there exist thresholds $\underline{\theta} < \overline{\theta}$ such that the lowest and highest actions in A_i are strictly dominant when payoffs are given by, respectively, $u_i(\cdot, \underline{\theta})$ and $u_i(\cdot, \overline{\theta})$ (Assumption **A2**). Furthermore, each u_i satisfies *state monotonicity* in the sense that higher states make higher actions more appealing (Assumption **A3**): there exists $K > 0$ such that for all $a_i \leq a'_i$ and $\underline{\theta} \leq \theta \leq \theta' \leq \overline{\theta}$ we have

$$0 \leq K(a'_i - a_i)(\theta' - \theta) \leq (u_i(a'_i, a_{-i}, \theta') - u_i(a_i, a_{-i}, \theta')) - (u_i(a'_i, a_{-i}, \theta) - u_i(a_i, a_{-i}, \theta)).$$

Finally, each u_i is continuous in θ (Assumption **A4**). In Section 3.3, we will relax **A3** and **A4**.

In the global game $G(\nu)$, the state θ is realised according to a continuous density with connected support that includes the thresholds $\underline{\theta}$ and $\overline{\theta}$ in its interior. Players observe θ with some noise, and then act simultaneously. Formally, let $f = (f_i)_{i\in I}$ denote a tuple of probability densities, whose supports are subsets of $[-\frac{1}{2}, \frac{1}{2}]$. Each player i receives a private signal $x_i = \theta + \nu\eta_i$, where each η_i is drawn independently according to the density f_i, and $\nu \in (0, 1]$ is a scale factor. Thus, a global game $G(\nu)$ is specified by the payoff functions u_i, a prior distribution of states, a noise distribution f, and a scale factor ν, each of which is common knowledge among players.

A strategy s_i is a function that maps a player's signal onto an action.[2] Joint strategy profiles are denoted by $s = (s_i)_{i\in I}$ and $s_{-i} = (s_j)_{j\in I-\{i\}}$. Slightly abusing notation, denote by $s(x)$ the joint action profile obtained when each player receives the same signal $x_i = x$. We write $s \leq s'$ if and only if $s(x) \leq s'(x)$ for all x. A strategy profile s is *increasing*, if each s_i is weakly increasing in x_i; it is a (Bayes-Nash) equilibrium if each $s_i(x_i)$ is a best reply against s_{-i}, given u_i and using Bayes's rule to derive the conditional densities of θ and x_{-i}, given x_i.

Following FMP, we also define the *simplified* global game $G^*(\nu)$, differing from the global game $G(\nu)$ in that its prior is uniform and payoffs are given by $u_i(a_i, a_{-i}, x_i)$ and thus depend directly on the signals. FMP prove that the equilibrium strategy profile in $G^*(\nu)$ is unique up to its points of discontinuity (Lemma A1 in FMP) while $G(\nu)$ may have multiple equilibrium strategy profiles. However, the key result on global games says that as the scale factor ν goes to zero, the equilibrium strategy profiles of the games $G(\nu)$ and $G^*(\nu)$ all converge to the same limiting strategy profile, which is increasing and unique up to its finitely many discontinuities (Theorem 1 in FMP).

Theorem 0 (Limit Uniqueness). *The global games $G(\nu)$ and $G^*(\nu)$ have an essentially unique, common limit equilibrium strategy profile as the scale factor ν goes to zero. More precisely, there exists an increasing pure limit strategy profile s such that, for each $\nu > 0$, if s_ν is an equilibrium strategy profile of $G(\nu)$, and s^*_ν is the unique equilibrium strategy profile of $G^*(\nu)$, then* $\lim_{\nu\to 0} s_\nu(x) = \lim_{\nu\to 0} s^*_\nu(x) = s(x)$ *for all x except possibly at the finitely many discontinuities of s.*

Since a global game's limit strategy profile s is well-defined up to its points of continuity, we use $\overline{s}$ and $\underline{s}$ to denote its right and left continuous versions.

Finally, we will use the following fact about supermodular incomplete information games. For a given game and strategy profile s, let $\beta(s)$ denote the joint upper best reply to s, i.e. the profile in which each player uses her highest best reply to s_{-i}. Then, β is monotonically increasing in s. Moreover, if $s \leq \beta(s)$, the upper best reply iteration $s, \beta(s), \beta(\beta(s)), \dots$ converges monotonically

[2]Focusing on pure strategies is without loss of generality and simplifies notation. Supermodularity implies that highest and lowest equilibrium strategy profiles exist in pure strategies and are bounds on any mixed strategy profile Since these two equilibria converge almost everywhere to a common limit as the scale factor ν goes to zero (Theorem 0), surviving strategies may only mix on a null set of signals, which has no effect on other players' incentives.

to an equilibrium strategy profile. This applies in particular to the global games $G(v)$ and $G^*(v)$.

3. Equilibrium Selection via Global Games

Let Γ be a given supermodular complete information game. Often, such a game has multiple Nash equilibria. In that case, a global game gives a method to resolve equilibrium indeterminacy. Consider a global game, in which payoffs depend on a state parameter, and suppose that for some fixed state θ^*, these payoffs coincide with those of Γ. We say that the global game *embeds* Γ at state θ^*. If—as the noise in private signals vanishes—the global game's limit equilibrium strategy profile is continuous at θ^*, its value at that state determines a particular Nash equilibrium of the complete information game.[3] More generally, consider the left continuous version $\underline{s}$ and right continuous version $\overline{s}$ of the limit equilibrium strategy profile. These determine two (perhaps distinct) Nash equilibria: $\underline{s}(\theta^*)$ and $\overline{s}(\theta^*)$.[4] We refer to them as the lowest and, respectively, highest *global game selection* (GGS, following Heinemann et al. [14]). If they coincide, the GGS is *unique*.

In this section, we give a simple characterisation of the GGS, which shows that it depends only on the payoff functions of the complete information game Γ and the noise distribution of private signals f in the global game. The additional modelling choices of the global game—its prior and its payoff functions at states other than θ^*—do not affect the GGS. We also show that (for a given noise distribution) the lowest GGS is identical to the highest GGS for almost all supermodular games. Thus, the GGS is generically unique.

Let us first note that this approach applies to any supermodular complete information game.

Lemma 1. *For any supermodular game Γ, there is a global game that embeds it.*

To see this, extend the payoff functions g_i of the complete information game with a state parameter θ by setting $u_i(a_i, a_{-i}, \theta) = g_i(a_i, a_{-i}) + \theta a_i$. The payoff functions u_i satisfy the global game assumptions (**A1**)–(**A4**). Choose an appropriate prior with sufficiently wide support and some noise distribution f for private signals, and our claim is satisfied at $\theta = 0$.

§3.1. A Characterisation of the Global Game Selection

We now show that the GGS induced by the global game may be determined without analysing the full global game under vanishing noise. Instead, we introduce a much simpler game that allows us to establish the GGS more directly. Following [10], consider a new incomplete information game E, in which each agent i has a payoff function $\tilde{u}_i(a_i, a_{-i}, x_i)$ that depends directly on her signal x_i. The payoff functions $\tilde{u}_i$ are equal to the payoff functions g_i of the complete information game Γ for high signals, but for low signals they make the lowest action dominant (Figure 1):

[3]Theorem 2 in FMP implies that the limit strategy profile selects a Nash equilibrium of Γ if it is continuous at θ.

[4]Continuity of the payoff functions u_i ensures that $\underline{s}(\theta^*)$ and $\overline{s}(\theta^*)$ are also Nash equilibria of Γ.

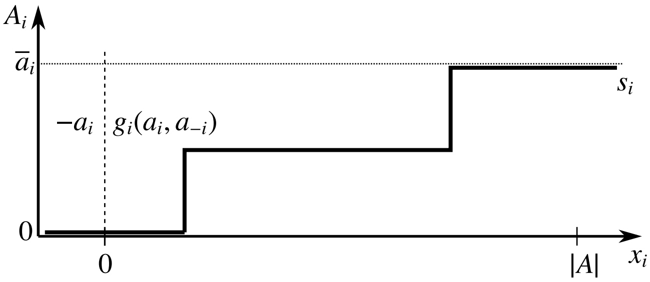

FIGURE 1. The lower-f-elaboration E and the attained action $\overline{a}_i$

$$\tilde{u}_i(a_i, a_{-i}, x_i) := \begin{cases} -a_i & \text{if } x_i < 0, \\ g_i(a_i, a_{-i}) & \text{if } x_i \geq 0. \end{cases} \tag{2}$$

Players' signals are given by $x_i = \theta + \eta_i$, where the common part θ is drawn uniformly from a large interval[5] and, for each player i, the idiosyncratic part η_i is drawn according to a noise distribution f_i, just like in a global game. We refer to E as the (*lower-f-*)*elaboration* of Γ.

Standard results on supermodular games ensure that the elaboration E has a highest equilibrium strategy profile s and ensure it is increasing. Therefore, s must consist of monotonic step functions, where at each step at least one player switches to a higher action. Since the joint action set is finite, s must reach a highest action profile (see again Figure 1). We denote this profile by $\overline{a}$ and refer to it as the action profile *attained* in E. Furthermore, s must be constant for signals above $|A|$. This is because the number of steps must be less than the size of the action space; and a player with signal x_i knows that all her opponents receive signals within $[x_i - 1, x_i + 1]$. Therefore, the distance between the steps must be less than 1—otherwise the distance can be shortened without affecting expected payoffs at the steps, contradicting the maximality of s. Thus, the attained action profile $\overline{a}$ is equal to $s(|A|)$.

Dually, define the *upper-f-elaboration* E^∂ of the complete information game Γ, similar to the lower-f-elaboration E, except that payoffs are given by g_i if the signal $x_i \leq 0$ and by a_i if $x_i > 0$. We consider the lowest equilibrium strategy profile in E^∂, and denote by $\underline{a}$ the lowest action profile that is attained in it. It is found by evaluating the lowest equilibrium strategy profile at $x = -|A|$.

The following result gives (for the noise distribution f) the relation between the GGS, the action profile $\overline{a}$ attained in the elaboration E, and the action profile $\underline{a}$ attained in the elaboration E^∂.

Theorem 1. *Let $G(v)$ be any global game with noise distribution f that embeds the complete information game Γ at some state θ^*. Let $\overline{s}$ and $\underline{s}$ be the left and right continuous versions of the essentially unique[6] limit equilibrium strategy profile in $G(v)$. Then $\overline{s}(\theta^*) = \overline{a}$ and $\underline{s}(\theta^*) = \underline{a}$.*

Note the following implication. In any global game, at any state, as the noise in private signals

[5] We assume this interval is a superset of $[-|A| - 1, |A| + 1]$, where $|A|$ denotes the cardinality of the joint action set.
[6] Recall that the limit strategy profile is unique up to its (finitely many) points of discontinuity.

vanishes, the value taken by its essentially unique limit strategy profile depends only on the complete information game given by the payoffs at this state and the noise distribution f. This holds because our choice of the embedded complete information game Γ at the start of the section was entirely arbitrary, and Γ and f are the only ingredients of the global game appearing in E and E^{∂}.

The irrelevance of the prior distribution in the global game for establishing its limit equilibrium strategy profile was already shown by FMP. It may be surprising that the extended payoff function is also irrelevant. If one thinks of the GGS as determined by an infection process starting from the dominance regions, one might imagine that if the complete information game Γ is embedded at some state θ^* close to the lower dominance region, this may influence the GGS in such a way that it selects a lower equilibrium compared to a global game in which Γ is embedded near the upper dominance region. However, Theorem 1 tells us this is not the case.

A practical way to think about Theorem 1 is the following. In economic applications, the state θ is typically interpreted as an economic fundamental affecting the decision problem of players. But several economic variables may be candidates for the state θ. Theorem 1 says that the choice of the fundamental used to perturb the decision problem is irrelevant: the GGS will be the same.

We sketch the proof of Theorem 1, concentrating on the highest GGS. To this end, we connect the elaboration E and its attained action profile $\overline{a}$ with the simplified global game $G^*(\nu)$ that determines the GGS. We do this by introducing two additional parameters in E: the scale factor ν and an explicit threshold signal y at which its payoffs change (normalised to 0 in Equation (2)). Formally, let $E_y(\nu)$ denote a version of E in which agents receive scaled signals $x_i = \theta + \nu\eta_i$, and in which payoffs are given by g_i for $x_i \geq y$ and by $-a_i$ for $x_i < y$. Note that due to the simple structure of E, any equilibrium strategy profile of E has a scaled and shifted counterpart in $E_y(\nu)$. In particular, the highest equilibrium strategy profile s of E has a scaled counterpart in $E_y(\nu)$ where the distance between steps is scaled down by a factor ν and all steps are shifted to the right by y. Thus, it prescribes that player i plays action $\overline{a}_i$ for all signals higher than $x_i = y + \nu|A|$.

Now, we compare the simplified global game $G^*(\nu)$ and the elaboration $E_{\theta^*}(\nu)$ (i.e., the threshold y is set to the state θ^*). For any signal x_i and any opposing strategy profile, the upper-best reply in $G^*(\nu)$ is weakly higher than in $E_{\theta^*}(\nu)$: for signals below θ^* the lowest action is strictly dominant in $E_{\theta^*}(\nu)$; for signals above θ^* a player's incentive to increase her action increases in x_i in $G^*(\nu)$ but is constant in $E_{\theta^*}(\nu)$. In particular, this means that any equilibrium in $E_{\theta^*}(\nu)$ must be a lower bound on the unique equilibrium of $G^*(\nu)$. Letting the scale factor ν go to zero, the fact that action $\overline{a}_i$ is played at $x_i = \theta^* + \nu|A|$ in an equilibrium of $E_{\theta^*}(\nu)$, together with the right-continuity of the limit profile $\overline{s}$ of $G^*(\nu)$, establishes that the attained action profile $\overline{a}$ is a *lower bound* on the highest GGS at θ^*. This argument is also used in the proof of Theorem 4 in FMP.

The key observation behind Theorem 1 is that a converse result also holds: the attained action profile $\overline{a}$ also determines an *upper bound* for the highest GGS. To see this, compare the simplified global game $G^*(v)$ with the elaboration $E_{\underline{\theta}}(v)$, so that the scale factor v and dominance regions of both games coincide (the threshold y is set to the lower dominance threshold $\underline{\theta}$ of $G^*(v)$). For signals $x_i \in [\underline{\theta}, \theta^*]$, payoffs in $E_{\underline{\theta}}(v)$ are given by $g_i(\cdot) = u_i(\cdot, \theta^*)$, while payoffs in $G^*(v)$ are given by $u_i(\cdot, x_i)$. State monotonicity of payoff functions u_i (assumption **A3**) implies that for all signals up to θ^*, best replies in $E_{\underline{\theta}}(v)$ will be weakly higher than in $G^*(v)$. This suggest that, at least for small v, $\overline{a}$ can serve as an upper bound on the action profile played at θ^* in the unique equilibrium of $G^*(v)$. In the appendix, we prove this intuition to be correct.

§3.2. The Global Game Selection is Generically Unique

FMP define a global game as a family of supermodular games ordered along a one-dimensional state space, and find that the GGS is unique at almost all states. Theorem 1 says that for any given supermodular game, the lowest and highest GGS depend only on the choice of the noise distribution f and can be determined without reference to any particular global game embedding. We will complement FMP's observation by showing that (given f), if one picks an individual supermodular game at random, the lowest GGS and highest GGS typically coincide.

More precisely, consider the set of games with player set I and joint action set A, which may be identified with the Euclidean space $\mathbb{R}^{|I \times A|}$. E.g., for two-player two-action games, $|I \times A| = 8$, corresponding to the number of entries that characterise the payoff matrix. Let $S \subseteq \mathbb{R}^{|I \times A|}$ be the subset of supermodular games. For any fixed noise distribution f, denote by S^f the subset of supermodular games in which the GGS is unique; S^{-f} denotes its complement in S, the set of games in which the GGS is not unique. Then the set S^{-f} is small relative to S^f, both in measure and in a topological sense.

Theorem 2. *For any noise distribution f, the set S^f of supermodular games with a unique GGS is open and dense in S, while its complement S^{-f} is closed and nowhere dense in S. Moreover, S^f is of infinite Lebesgue measure, while S^{-f} is of zero Lebesgue measure.*

Recall that S^f is dense in S if each supermodular game $\Gamma \in S^{-f}$ can be approximated by games in S^f. As jumps in the limit strategy profile of a global game are isolated, we can approximate any $\Gamma \in S^{-f}$ if we embed it in a global game $G(v)$ with noise distribution f and choose a sequence of games in S^f along the one-dimensional state space of $G(v)$. In the proof, we also establish that S^f is open in S, and thus S^{-f} is closed and nowhere dense. We then show that S^{-f} has Lebesgue measure zero, by applying a result that connects its topological properties to its measure.

§3.3. Global Game Selection in Discontinuous Global Games

Since Theorem 1 says that the GGS may be determined without reference to any particular global game embedding, this suggests that the monotonicity (**A3**) and continuity (**A4**) assumptions imposed on the embedding, can be weakened. To see why this may be important in an applied context, consider the following n-player speculative attack game:

		$\alpha > \xi$	$\alpha \leq \xi$
player i	("don't attack") 0	0	0
	("attack") 1	-1	1

where α is the fraction of players who choose 0.

It is an exemplary regime change game, where players' payoffs depend on whether they reach a critical mass ξ. Intuitively, there are at least two ways to embed the game in a global game: one can perturb the payoffs as in Lemma 1, or one can perturb the critical mass by setting $\xi = \theta$. In the latter case, which is often considered in the applied literature (e.g. [7, 12, 17, 23]), payoffs remain unchanged for most changes in the state, jumping at the point where a change in the state leads to a regime change. Thus the global game payoff functions violate state monotonicity (**A3**) and continuity (**A4**). But even in this case, they satisfy the following assumption:

(**A3***) *Weak state monotonicity*. Higher states make higher actions weakly more appealing:

(3) for all i, a_{-i} and $a_i < a_i'$, $u_i(a_i', a_{-i}, \theta) - u_i(a_i, a_{-i}, \theta)$ is weakly increasing in θ.

We can show that, even if a global game embedding satisfies only these weakened conditions, the GGS may be determined analogous to Theorem 1. As before, let Γ be a given supermodular complete information game. Suppose it is embedded at state θ^* in a *generalised global game*, with payoffs differing from an ordinary global game in that they satisfy (**A3***) but not necessarily (**A3**) or (**A4**). By standard results on supermodular games, for each scale factor $v > 0$, the highest and lowest equilibrium strategy profiles $\hat{s}_v$ and $\check{s}_v$ exist nonetheless. We define their pointwise limits $\hat{s} = \limsup_{v \to 0} \hat{s}_v$ and $\check{s} = \liminf_{v \to 0} \check{s}_v$. Also, for each state θ, consider the complete information game with payoffs $u_i(\cdot, \theta)$. We may determine its attained action profiles $\overline{a}_\theta$ and $\underline{a}_\theta$ from the associated elaborations E_θ and E_θ^∂; these attained action profiles may be regarded as functions of θ.[7] Using these definitions, we obtain the following result, parallel to Theorem 1:

Theorem 3. *Let $\tilde{G}(v)$ be a generalised global game with noise distribution f that embeds the complete information game Γ at θ^*. Let $\hat{s}$ and $\check{s}$ be its highest and lowest limit equilibrium strategy profiles. If (i) the attained equilibria $\overline{a}$ and $\underline{a}$ of the game Γ coincide and (ii) the functions $\overline{a}_\theta$ and $\underline{a}_\theta$ are continuous at θ^*, then $\hat{s}(\theta^*) = \overline{a} = \underline{a} = \check{s}(\theta^*)$.*

To prove this result, we "sandwich" the payoff function of the global game $\tilde{G}(v)$ between that of two "ordinary" global games that approximate it. Then, we use Theorem 1 to show that their limit strategy profiles coincide at θ^*. This pins down the limit strategy profile of $\tilde{G}(v)$ at θ^* as well.

The regulatory conditions (i) and (ii) are needed because of the weakening of (**A3**) and (**A4**). Note that Theorem 2 guarantees that (i) is satisfied for almost all supermodular games. As for (ii),

[7] We formally define these concepts by substituting $u_i(\cdot, \theta)$ for $g_i(\cdot)$ in the definition of E, E^∂, $\overline{a}$, and $\underline{a}$.

Symmetric Games			
actions:	2 each	3 each	4 each
2 players	✓ [a]	✓ [c]	× [b]
3 players	✓ [b]	× [d]	
n players	✓ [b]		

Asymmetric Games			
actions:	2 each	2 by n	3 each
2 players	✓ [a]	✔ [g]	× [c]
3 players	× [e]	n/a	
n players	× [f]	n/a	

✓ Always noise independent. × Counterexample to noise independence exists. [a]Carlsson and Van Damme [6]. [b]Frankel, Morris and Pauzner [10]. [c]Basteck and Daniëls [1]. [d]Basteck et al. [2]. [e]Carlsson [4]. [f]Corsetti et al. [7]. [g]This paper, see Section 5: Two-player games with 2-by-n-actions. For empty cells noise dependence follows from an example in smaller games.

TABLE 1. Noise (In)dependence in Supermodular Games

(**A3***) implies that the attained profiles $\overline{a}_\theta$ and $\underline{a}_\theta$ are increasing in θ and thus have only finitely many discontinuities, as the joint action set is finite. But unlike in an ordinary global game, (i) does not imply (ii), as $\overline{a}_\theta$ and $\underline{a}_\theta$ may jump because of a discontinuity in the payoff difference (3).

4. A Decomposition Approach to Noise Independence

A supermodular complete information game Γ, embedded in a global game at a state θ^*, is called *noise independent* if, as the noise in private signals vanishes, the global game's limit strategy profile takes on the same value at θ^* regardless of the choice of the noise distribution f. Theorem 1 implies that noise independence is a well-defined property of the game Γ: it is noise independent in one global game embedding if and only if it is noise independent in every other.

Many small games, with few players or few actions, are noise independent. Typically, for such games there are also easy heuristics to find the GGS. For instance, a well-known elementary condition to judge whether a game is noise independent is the "p-dominance" criterion.

Definition. Given a tuple $p = (p_i)_{i\in I}$, an action profile a^* is said to be p-dominant if each player i who expects her opponents to play a^*_{-i} with probability p_i would choose a^*_i as a best reply.

If a supermodular game Γ has a p-dominant action profile a^* with $\sum_{i\in I} p_i < 1$, then a^* is an equilibrium robust to incomplete information in the sense of Kajii and Morris [15]. Moreover, a^* is robust in all games with payoffs close to those of Γ. This implies that a^* is the unique GGS in Γ, regardless of the noise distribution.[8] For some games, an even simpler way to determine noise independence is to count the number of players and actions. Table 1 summarises when this is possible; for supermodular games indicated with a "✓", noise independence always holds.

Table 1 also shows that noise independence may fail quickly as we enlarge the action sets of players. In this section, we will show that a game may nevertheless be noise independent if we can suitably decompose it into smaller games. We start by making a more basic observation: in certain games with large action sets, the GGS may be determined by solving smaller games.

[8]See Oury and Tercieux [19] or our working paper version [2]; Morris and Shin [18] provide a heuristic argument.

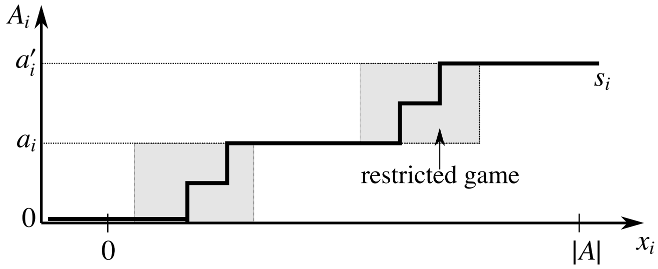

FIGURE 2. Exploiting the GGS of restricted games

Definition. Consider a supermodular complete information game Γ with joint action set A. For action profiles $a \leq a'$, we define $[a, a'] := \{\tilde{a} \in A \mid a \leq \tilde{a} \leq a'\}$. The *restricted* game $\Gamma|[a, a']$ and elaboration $E|[a, a']$ are given by restricting the joint action set of Γ and its elaboration E to $[a, a']$.

Figure 2 now illustrates the idea. If certain action profiles a and a' are played in equilibrium strategy profiles of the restricted elaborations $E|[0, a]$ and $E|[a, a']$, we can "patch" these profiles together to obtain a strategy profile s in the elaboration E, such that an upper best reply iteration starting from s is weakly increasing. Hence the attained action profile $\overline{a}$ in E must be weakly higher than a'. By Theorem 1, a' provides a bound on the GGS. We may also do this iteratively:

Lemma 2. *Fix a supermodular game* Γ *and noise distribution* f. *An action profile* a^n *is the unique global game selection, if there is a sequence* $0 = a^0 \leq a^1 \cdots \leq a^n \leq a^{n+1} \cdots \leq a^m = m$ *such that*
(i) a^j *is the unique global game selection in* $\Gamma|[a^{j-1}, a^j]$ *for all* $j \leq n$, *and*
(ii) a^{j-1} *is the unique global game selection in* $\Gamma|[a^{j-1}, a^j]$ *for all* $j > n$.

A noise independence result follows as an immediate corollary. If the restricted games in the hypothesis of Lemma 2 are noise independent, we can use the same decomposition of the game Γ, regardless of the noise distribution f, and Γ must be noise independent.

Theorem 4. *Fix a supermodular game* Γ. *An action profile* a^n *is the unique noise independent global game selection, if there is a sequence* $0 = a^0 \leq a^1 \cdots \leq a^n \leq a^{n+1} \cdots \leq a^m = m$ *such that*
(i) a^j *is the unique noise independent global game selection in* $\Gamma|[a^{j-1}, a^j]$ *for all* $j \leq n$, *and*
(ii) a^{j-1} *is the unique noise independent global game selection in* $\Gamma|[a^{j-1}, a^j]$ *for all* $j > n$.

In Section 5 we apply these results to derive the noise independent GGS in two examples.

The concept of p-dominance reveals another connection between noise independence and robustness to incomplete information. Together, Proposition 2.7 and 3.8 in Oyama and Tercieux [21] imply that if a supermodular game Γ can be decomposed—as above—into restricted games, all of which have a strict p-dominant equilibrium with sufficiently small p, uniform across these games, then a^n is also the unique robust equilibrium of Γ (and thus the unique GGS).

However, Theorem 4 establishes a more direct result about equilibrium selection in global games. First, its conclusion holds independently of whether the action profile a^n is robust to incomplete information, which is a sufficient, but not a necessary condition for noise independence (see the combined results of Basteck and Daniëls [1] and Oyama and Takahashi [20]). Second, it allows application of a range of known criteria for noise independence besides p-dominance; e.g., the fact that symmetric three-player three-action games, symmetric n-player binary-action games, and (as we show below) two-player 2-by-n-action games are noise independent, none of which are equivalent to the p-dominance criterion. Indeed, we can apply a different criterion to each restricted game. Thus, our theorem applies under strictly more general conditions.

5. Applications

It remains to show that our results may be applied in economically interesting settings. We report two examples. In addition, we use Theorem 1 to establish a new noise independence result.

Refinancing Game. Consider a complete information game Γ in which 3 lenders decide whether or not to refinance a firm invested in a long term project. Each lender can lend one unit of cash. The firm promises to repay R_h at maturity. However, if the firm cannot fully refinance the project, it may not be able to repay in full. It may repay $R_l < R_h$, or default outright and not repay at all.

The firm has the option to issue collateralised debt that gives lenders an additional payoff of c in both the partial and the complete default scenarios, but yields a lower repayment $R_m < R_h$ when the project succeeds, as the provision of collateral reduces balance sheet flexibility and thus is costly for the firm. The following matrix summarises the possible outcomes of the game, identifying the different lending decisions with actions 0, 1, and 2 as indicated.

		lenders $I - \{i\}$		
		$(0, 0)$	$(0, 1), (0, 2)$ or $(1, 1)$	$(1, 2)$ or $(2, 2)$
	("don't lend") 0	1	1	1
lender i	("secured loan") 1	c	$R_l + c$	R_m
	("unsecured loan") 2	0	R_l	R_h

For $R_l = \frac{2}{3}$ and $R_h = 2$, and in the absence of collateralised debt, the GGS implies that lenders will not finance the project—recall that in symmetric binary-action games, the GGS is the best reply to the belief assigning equal probability to opponents' profiles $(0, 0)$, $(0, 2)$ and $(2, 2)$.

Introducing collateralised loans may change this result. For example, if $c = \frac{2}{3}$ and $R_m = \frac{3}{2}$, we find that in the restricted game $\Gamma|[0, 1]$ (where lenders choose 0 or 1), the GGS is 1,[9] as it is the best reply to the belief assigning equal probability to $(0, 0)$, $(0, 1)$ and $(1, 1)$. Similarly, the GGS in $\Gamma|[1, 2]$ is 2. Thus, by Lemma 2, the GGS in Γ is 2: lenders are willing to provide *unsecured* loans if we include option 1. Intuitively, this change occurs because the possibility of opponents extending secured loans makes action 2 less risky. Note that all of this holds regardless of the noise distribution, as noise independence is inherited from the two symmetric binary-action games by Theorem 4, and despite the fact that 3-player 3-action games are not in general noise independent.

[9] In this section, for brevity we use $n \in \mathbb{N}$ to denote the action profile where all players use action n.

Asymmetric Minimum Effort Game. Each player $i \in I$ produces an intermediate good that is a necessary input for a final good. Players choose their production level $a_i \in \{0, \dots, m\}$ and production of the final good good is $a_{\min} := \min\{a_i | i \in I\}$. Players' individual payoff functions are given by $g_i(a_i, a_{-i}) = b_i(a_{\min}) - c_i(a_i)$, where b_i and c_i are increasing benefit and cost functions. This game typically has many equilibria. It generalises a model of Bryant [3] studied by Carlsson and Ganslandt [5] and Van Huyck et al. [25].[10]

To solve the game for the GGS, we decompose it into m restricted binary-action games with joint action sets $\{k-1, k\}^{|I|}$, $0 < k \leq m$. We will determine the GGS in these restricted games. By Theorem 1, the GGS equals k if and only if in the associated elaboration $E|[k-1, k]$ we can construct an increasing strategy profile such that players choose action k for sufficiently high signals, and from which a best reply iteration leads upwards.

Consider a strategy profile for the elaboration $E|[k-1, k]$, given by thresholds $z = (z_i)_{i \in I}$ such that each player i switches from action $k-1$ to k at $z_i \in [0, |A|]$. Let $P_i(z)$ be the probability that player i attaches to all her opponents playing action k, given their thresholds, when she gets signal z_i. The highest GGS equals k if and only if we can adjust the $(z_i)_{i \in I}$ such that each individual player prefers to play k at her threshold; it is unique if they can be made to strictly prefer this, i.e.

$$p_i := \frac{c_i(k) - c_i(k-1)}{b_i(k) - b_i(k-1)} < P_i(z), \tag{4}$$

To solve the restricted game, we use the following fact (proved in the appendix): the beliefs at the thresholds z_i always satisfy the constraint $\sum_{i \in I} P_i(z) = 1$. So, summing (4) over all players gives

$$\sum_{i \in I} \frac{c_i(k) - c_i(k-1)}{b_i(k) - b_i(k-1)} < 1. \tag{5}$$

Therefore, a necessary condition for k to be the unique GGS is that players' *aggregate* marginal cost-benefit ratios, when everyone switches from $k-1$ to k, are strictly smaller than 1. This is also sufficient, since k is p-dominant with $\sum p_i < 1$ when (5) holds, implying noise independent selection. Conversely, if (5) holds with the inequality reversed, $k-1$ must (necessarily) be the unique GGS—though in this case the restricted game is generally not p-dominant solvable.

Provided the left hand side of (5) crosses 1 at most once and from below for increasing $k = 1, 2, \dots, m$, this decomposition yields a generically unique, noise independent, GGS. (A sufficient, but not necessary, condition is that the individual c_i are convex and b_i are concave.)

Two-player games with 2-by-n-actions. To conclude, we will use Theorem 1 to prove a new noise independence result for supermodular 2-by-n games. This completes the characterisation of supermodular games for which noise independence follows from the size of individual action sets (Table 1). Let Γ be a supermodular game with player set $I = \{1, 2\}$, and action sets $A_1 = \{0, 1\}$

[10] Carlsson and Ganslandt [5] consider a form of trembling hand perfection in a symmetric version of this model; Van Huyck et al. [25] experimentally test a variant with linear and symmetric payoffs.

and $A_2 = \{0, 1, \ldots, m_2\}$. E.g., player 1 is a government contemplating an infrastructure project and player 2 is a firm choosing a plant capacity, and their investments are complements.

For an arbitrary noise distribution f, consider the associated elaboration E and its highest equilibrium strategy profile s, in which players jointly play $\overline{a} = (\overline{a}_1, \overline{a}_2)$ at sufficiently high signals. If the attained action $\overline{a}_i = 0$ for some player i, she plays 0 for all signals in the elaboration E. Thus the attained action $\overline{a}_j$, $j \neq i$, must be j's highest best reply to 0. Then, the joint strategy profile given by $s_i(x_i) = 0$ for all x_i, and $s_j(x_j) = 0$ if $x_j < 0$ and $s_j(x_j) = \overline{a}_j$ if $x_j \geq 0$ is the highest equilibrium in a lower-f'-elaboration under any other noise distribution f'.

Alternatively, suppose attained action $\overline{a}_1 = 1$ and $\overline{a}_2 = k > 0$. Then the highest strategy profile s may be identified with a threshold z_1^1 and a tuple of thresholds $z_2^1, z_2^2, \ldots, z_2^k$, at which players 1 and 2 switch to higher actions. The opposing action distribution faced by player 1 at signal $x_1 = z_1^1$ is given by the probabilities $Q_j = \Pr(x_2 < z_2^j | x_1 = z_1^1)$ that player 2 chooses an action below $j = 1, ..., k$ given that $x_1 = z_1^1$. But the opposing action distribution that player 2 faces at each of her thresholds z_2^j is also described by these probabilities, since for all $j = 1, ..., k$ we find

$$\begin{aligned}\Pr(x_1 > z_1^1 | x_2 = z_2^j) &= \Pr(x_1 - x_2 > z_1^1 - z_2^j | x_2 = z_2^j) \\ &= \Pr(x_1 - x_2 > z_1^1 - z_2^j | x_1 = z_1^1) = \Pr(x_2 < z_2^j | x_1 = z_1^1) = Q_j,\end{aligned}$$

where the second equality follows from the uniform prior distribution of θ. Now, if we consider any other noise distribution f' and associated lower-f'-elaboration E', we can always construct an increasing strategy profile s' in which players jointly play $(1, a_2)$ for sufficiently high signals, by putting $z_1^1 = 1$ and arranging the remaining k thresholds $\{z_2^1, z_2^2, \ldots, z_2^k\}$ such that the k independent equations that determine the Q_j, $j = 1, ..., k$, hold under the new noise distribution.

In this way, the action distributions that players face at their thresholds remain unchanged. Since they are willing to switch to higher actions given these beliefs, a best reply iteration in elaboration E' starting at the profile s' is monotonic and leads to an equilibrium strategy profile $s^* \geq s'$ of E'. By Theorem 1, the highest GGS under the noise distribution f', $\overline{a}'$, is weakly higher than $\overline{a}$, the highest GGS under f. Since f and f' were arbitrary, we find that $\overline{a}' = \overline{a}$. A dual argument establishes that $\underline{a}' = \underline{a}$, proving the noise independence of the game Γ.

6. Conclusion

We have shown how, for any supermodular complete information game, we may deduce its global game selection directly using solely the complete information game's payoffs and the global game's noise distribution (Theorem 1). For almost all games this gives a unique global game selection (Theorem 2). Our results may be used to establish selection under weakened assumptions on the global game (Theorem 3), which is useful from an applied perspective.

From a practical point of view, our most powerful result is Theorem 4. It implies that the global game selection may be derived by decomposing a many-action game into smaller games, for which existing heuristics and noise independence results can be applied. As we showed in Section 5, simplified conditions for the global game selection and a manageable heuristic to derive it in many-action games make it easier to apply the theory of global games. That should facilitate new research on topics where strategic complementarities are crucial.

Appendix

Proof of Theorem 1. The lower bound argument used in FMP and given in Section 3.1 yields $\overline{s}(\theta^*) \geq \overline{a}$. We now prove the converse: $\overline{s}(\theta^*) \leq \overline{a}$. By duality, $\underline{s}(\theta^*) \leq \underline{a}$ and $\underline{s}(\theta^*) \geq \underline{a}$ also hold.

Consider a simplified global game $G^*(\nu)$ with noise distribution f, and its right continuous, increasing equilibrium strategy profile $\overline{s}^*_\nu$. As $\nu \to 0$, $\overline{s}^*_\nu$ converges to the right continuous version $\overline{s}$ of the limit strategy profile of $G^*(\nu)$ at all points of continuity (Theorem 0). For now, assume $\overline{s}$ is continuous at θ^*. Then there exist $\overline{\nu}$ and $\delta > 0$ such that for all $\nu < \overline{\nu}$ and $x \in [\theta^* - \delta, \theta^* + \delta]$ we have that $\overline{s}^*_\nu(x)$ equals the (highest) GGS $\overline{s}(\theta^*)$; an equilibrium a^* of the game Γ. Now fix some $\nu < \min\{\delta, \overline{\nu}\}$ and consider $E_{\underline{\theta}}(\nu)$, the scaled and shifted version of elaboration E as defined at the end of Section 3.1. The lower dominance regions, scale factors, and noise distributions of $E_{\underline{\theta}}(\nu)$ and $G^*(\nu)$ coincide. Assume that in the game $E_{\underline{\theta}}(\nu)$ players use the strategy profile s' given by:

$$s'(x) = \begin{cases} \overline{s}^*_\nu(x) & \text{if } x \leq \theta^*, \\ \overline{s}^*_\nu(\theta^*) & \text{if } x > \theta^*. \end{cases}$$

For any player i, and any signal $x_i < 0$, action 0 is dominant both in $E_{\underline{\theta}}(\nu)$ and in $G^*(\nu)$, so for $x < 0$, the upper-best reply in $E_{\underline{\theta}}(\nu)$ is $\beta(s')(x) = 0 = \overline{s}^*_\nu(x) = s'(x)$. For $x_i \in [\underline{\theta}, \theta^*]$, i's opponents receive signals smaller than $\theta^* + \delta$ and follow $\overline{s}^*_{\nu,-i}$ in $E_{\underline{\theta}}(\nu)$. Since the distributions of players' signals are identical in $G^*(\nu)$ and $E_{\underline{\theta}}(\nu)$, but i's payoff is given by $g_i(\cdot) = u_i(\cdot, \theta^*)$ in $E_{\underline{\theta}}(\nu)$ and by $u_i(\cdot, x_i)$ in $G^*(\nu)$, supermodularity implies that the upper-best reply in $E_{\underline{\theta}}(\nu)$ is $\beta(s')(x) = \beta(\overline{s}^*_\nu)(x) \geq \overline{s}^*_\nu(x) = s'(x)$, for $\underline{\theta} \leq x \leq \theta^*$. For $x_i > \theta^*$, player i's opponents receive signals higher than $\theta^* - \delta$ and, following s'_{-i}, play a^*_{-i}. As a^* is a Nash equilibrium of Γ, $\beta(s')(x) \geq a^* = s'(x)$.

In sum, in the elaboration $E_{\underline{\theta}}(\nu)$, $\beta(s') \geq s'$. Hence, an upper-best reply iteration starting at s' yields a monotonically increasing sequence of strategy profiles that converges to an equilibrium profile $s^* \geq s'$. It follows that $s^*(\theta^*) \geq s'(\theta^*) = \overline{s}^*_\nu(\theta^*) = \overline{s}(\theta^*)$. As the attained action profile $\overline{a}$ is defined as the highest action profile attained in any equilibrium strategy profile in E, and each equilibrium profile in $E_{\underline{\theta}}(\nu)$ has a scaled and shifted counterpart in E, we have $\overline{a} \geq s^*(\theta^*) \geq \overline{s}(\theta^*)$.

If the limit strategy profile $\overline{s}$ is not continuous at θ^*, we may choose a decreasing sequence $\theta^0, \theta^1, \theta^2, \ldots$ converging to θ^* such that $\overline{s}$ is continuous at each θ^n. For each game Γ^n embedded at θ^n, consider the elaboration E^n, identical to E except that payoffs are given by $u_i(\cdot, \theta_n)$ if $x_i \geq 0$.

Let s^n be the highest equilibrium profile of E^n and recall $s^n(|A|)$ is the highest action profile played in s^n. By the first part of the proof, $s^n(|A|) \geq \overline{s}(\theta^n)$. As s^n is an equilibrium of E^n,

$$\forall i \in I, \forall a_i \in A_i, \forall x_i \geq 0, \qquad \int_{\mathbb{R}^{|I|-1}} \left(u_i(s_i^n(x_i), s_{-i}^n(x_{-i}), \theta^n) - u_i(a_i, s_{-i}^n(x_{-i}), \theta^n)\right) \pi_i(x_{-i}|x_i) \;\; \mathrm{d}x_{-i} \geq 0,$$

where $\pi_i(\cdot|x_i)$ is the conditional density over opponents' signals. By state monotonicity (**A3**), the sequence of equilibria s^n converges to $s^* = \inf\{s^n | n \in \mathbb{N}\}$. As payoffs u_i are bounded on the compact set $A \times [\theta^*, \theta^0]$, the dominated convergence theorem ensures expected payoffs also converge:

$$\begin{aligned}\forall i \in I, a_i \in A_i, x_i \geq 0, \quad & \int_{\mathbb{R}^{|I|-1}} \left(u_i(s_i^*(x_i), s_{-i}^*(x_{-i}), \theta^*) - u_i(a_i, s_{-i}^*(x_{-i}), \theta^*)\right) \pi_i(x_{-i}|x_i) \;\; \mathrm{d}x_{-i} \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^{|I|-1}} \left(u_i(s_i^n(x_i), s_{-i}^n(x_{-i}), \theta^n) - u_i(a_i, s_{-i}^n(x_{-i}), \theta^n)\right) \pi_i(x_{-i}|x_i) \;\; \mathrm{d}x_{-i} \geq 0\end{aligned}$$

So s^* is an equilibrium strategy profile in the elaboration E of the game Γ. Finally, since $s^*(|A|) = \inf\{s^n(|A|) | n \in \mathbb{N}\} \geq \inf\{\overline{s}(\theta^n) | n \in \mathbb{N}\} = \overline{s}(\theta^*)$, we conclude that the attained profile $\overline{a} \geq \overline{s}(\theta^*)$. ■

Proof of Theorem 2. Fix f. In this proof, for any game $\Gamma \in S$, denote its associated upper- and lower-f-elaborations by $E(\Gamma)$ and $E^\partial(\Gamma)$ and their respective attained action profiles by $\underline{a}(\Gamma)$ and $\overline{a}(\Gamma)$. Also, for any game $\Gamma \in S$ (with payoffs g_i) define a global game embedding as in Lemma 1 by setting $u_i^\Gamma(a_i, a_{-i}, \theta) = g_i(a_i, a_{-i}) + \theta a_i$. Given this embedding, Γ_θ denotes the complete information game embedded at θ. For $r > 0$, let $B_r(\Gamma)$ be the open ball in $\mathbb{R}^{|I \times A|}$ with radius r around Γ.

To prove that S^f is dense in S we may show that if $\Gamma \in S^{-f}$, there is a game arbitrarily close to Γ in which the GGS is unique. But this is always true, as the limit equilibrium strategy profile of Γ's embedding given by the payoffs u_i^Γ is unique up to its finitely many discontinuities (Theorem 0). To prove that S^f is open in S, note that if $\Gamma \in S^f$, the limit equilibrium strategy profile of its embedding given by u_i^Γ is constant over some interval $(-2\varepsilon, 2\varepsilon)$, as the joint action set A is finite. Then, by the following result, S^f is open in S (and hence S^{-f} is closed and nowhere dense in S):

Claim 1. *If $\Gamma \in S^f$ and, for some $\varepsilon > 0$ and $a^* \in A$, $a^* = \underline{a}(\Gamma_\theta) = \overline{a}(\Gamma_\theta)$ for all $\theta \in (-2\varepsilon, 2\varepsilon)$, then $a^* = \underline{a}(\Gamma') = \overline{a}(\Gamma')$ for all supermodular games Γ' in an ε-neighbourhood of Γ.*

Proof. Let Γ' be a supermodular game in an ε-neighbourhood of Γ. Then for all i, a_{-i}, and $a_i' < a_i$,

$$\begin{aligned} u_i^\Gamma(a_i, a_{-i}, -2\varepsilon) - u_i^\Gamma(a_i', a_{-i}, -2\varepsilon) &= g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) - 2\varepsilon(a_i - a_i') \\ &\leq g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) - 2\varepsilon \leq g_i'(a_i, a_{-i}) - g_i'(a_i', a_{-i}), \end{aligned}$$

where g_i' denotes the payoffs of Γ'. Thus, for any opposing action distribution, the upper-best reply in the elaboration $E(\Gamma')$, is weakly higher than in $E(\Gamma_{-2\varepsilon})$. But then the same is true for their highest equilibrium strategy profiles, so that $\overline{a}(\Gamma') \geq \overline{a}(\Gamma_{-2\varepsilon}) = a^*$. Using a symmetric argument, we establish that $\overline{a}(\Gamma') \leq a^*$. Dually, we may show that $\underline{a}(\Gamma') = a^*$, proving the claim.

Now, we may establish measure theoretic genericity. A subset P of $\mathbb{R}^{|I \times A|}$ is called *porous* if there are $\lambda \in (0,1)$ and $k > 0$ such that for any $\Gamma \in P$ and $\varepsilon \in (0,k)$, there exists $\Gamma' \in \mathbb{R}^{|I\times A|}$ such that $B_{\lambda\varepsilon}(\Gamma') \subseteq B_\varepsilon(\Gamma) - P$. Any porous subset of $\mathbb{R}^{|I\times A|}$ is a Lebesgue null set ([16], p. 220–222). Let:

$$S_k^{-f} := \{\Gamma \in S^{-f} \mid \Gamma_\theta \in S^f, \forall \theta \in (-k,0) \cup (0,k)\}.$$

We will prove that S_k^{-f} is porous. Assume $\Gamma \in S_k^{-f}$ and choose $\varepsilon \in (0,k)$. Setting $\Gamma' := \Gamma_{\frac{\varepsilon}{2}}$, we know that the GGS will be unique and identical to that of Γ' for all games $\{\Gamma'_\theta \in S \mid \theta \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})\}$. By Claim 1, we know that the GGS is unique for *all* supermodular games in an $\frac{\varepsilon}{4}$-neighbourhood of Γ', thus $B_{\frac{\varepsilon}{4}}(\Gamma') \cap S_k^{-f} = \emptyset$. Setting $\lambda = \frac{1}{4}$, we have for $\varepsilon \in (0,k)$ that $B_{\lambda\varepsilon}(\Gamma') \subseteq B_\varepsilon(\Gamma) - S_k^{-f}$, i.e., S_k^{-f} is porous. Thus $S^{-f} = \bigcup_{\{k\in\mathbb{Q} \mid k>0\}} S_k^{-f}$ is a countable union of Lebesgue null sets and hence a null set itself. To see that, by contrast, S is of infinite Lebesgue measure, pick a game such that the inequalities in (1) hold strictly, and note it is contained in an open ball $B \subseteq \mathbb{R}^{|I\times A|}$ of supermodular games. Moreover, for each open ball B in S, we find another ball $B' \subseteq S$ of arbitrarily large measure, if we multiply the payoffs of all games in B with a sufficiently large constant. ■

Proof of Theorem 3. For two different games write $\Gamma^1 \prec \Gamma^2$ if for all i, $a'_i < a_i$ and a_{-i} the corresponding payoffs satisfy $g_i^1(a_i, a_{-i}) - g_i^1(a'_i, a_{-i}) < g_i^2(a_i, a_{-i}) - g_i^2(a'_i, a_{-i})$. Given any opposing action distribution, the lowest best reply in Γ^1 will then be weakly lower than the lowest best reply in Γ^2.

Now, consider a generalised global game $\tilde{G}(v)$ with noise distribution f and payoffs functions u_i. Write Γ_θ for the complete information game with payoffs $u(\cdot,\theta)$. Assume that (i) $\overline{a}_{\theta^*} = \underline{a}_{\theta^*} = a^*$ and (ii) $\overline{a}_\theta$, $\underline{a}_\theta$ continuous at θ^*. Note that $\hat{s} \geq \check{s}$, so it suffices to show that $\check{s}(\theta^*) \geq a^* \geq \hat{s}(\theta^*)$. We will prove the first inequality; the second follows by duality. To do so, we will compare the payoff functions u_i satisfying (**A1**)–(**A3***) to payoff functions u'_i satisfying (**A1**)–(**A4**).

First, by continuity of $\overline{a}_\theta$ and $\underline{a}_\theta$ at θ^*, there is some nearby, game $\Gamma_{\theta^*-\varepsilon}$ embedded at $\theta^* - \varepsilon$ whose GGS is unique and equal to a^*. Next, consider the game Γ' given by $g'_i(a_i, a_{-i}) = u_i(a_i, a_{-i}, \theta^* - \varepsilon) - ka_i$, $k > 0$. By Claim 1, we can choose k such that the GGS in Γ' is unique and equal to a^*. Furthermore, w.l.o.g., assume that there exist extreme values $\check{\theta} < \underline{\theta}$ and $\overline{\theta} < \hat{\theta}$ such that we have a chain of four games satisfying $\Gamma_{\check{\theta}} \prec \Gamma' \prec \Gamma_{\theta^*} \prec \Gamma_{\hat{\theta}}$. Using this chain, we construct u':

$$u'_i(a_i, a_{-i}, \theta) = \begin{cases} u_i(a_i, a_{-i}, \check{\theta}) & \text{if } \theta < \theta^* - \varepsilon, \\ \frac{\theta^*-\theta}{\varepsilon} u_i(a_i, a_{-i}, \check{\theta}) + \frac{\theta-(\theta^*-\varepsilon)}{\varepsilon} g'_i(a_i, a_{-i}) & \text{if } \theta^* - \varepsilon \leq \theta < \theta^*, \\ \frac{\hat{\theta}-\theta}{\hat{\theta}-\theta^*} g'_i(a_i, a_{-i}) + \frac{\theta-\theta^*}{\hat{\theta}-\theta^*} u_i(a_i, a_{-i}, \theta^*) & \text{if } \theta^* < \theta < \hat{\theta}, \\ (\hat{\theta} + 1 - \theta) u_i(a_i, a_{-i}, \theta^*) + (\theta - \hat{\theta}) u_i(a_i, a_{-i}, \hat{\theta}) & \text{if } \hat{\theta} \leq \theta < \hat{\theta} + 1, \\ u_i(a_i, a_{-i}, \hat{\theta}) & \text{if } \hat{\theta} + 1 \leq \theta, \end{cases}$$

Comparing the payoffs u' with u, we see that under u' the dominance regions have been shifted to the right, the game $\Gamma_{\check{\theta}}$ is now embedded at $\theta^* - \varepsilon$, Γ' at θ^*, Γ_{θ^*} at $\hat{\theta}$, $\Gamma_{\hat{\theta}}$ at $\hat{\theta} + 1$ and the remaining games are linear interpolations. Thus, for any θ, the lowest best reply under u is weakly higher than under u'. Also, since payoffs are linearly interpolated between $\Gamma_{\check{\theta}} \prec \Gamma' \prec \Gamma_{\theta^*} \prec \Gamma_{\hat{\theta}}$, payoff differences are piecewise linear in θ, thus satisfy (**A3**). Clearly, (**A4**) is satisfied as well.

Finally, consider the global game $G'(\nu)$ with the newly constructed payoff function u', and the same noise distribution f and prior as $\tilde{G}(\nu)$. For any $\nu > 0$, $G'(\nu)$ has a lowest equilibrium strategy profile, denoted s'_ν. As best replies are higher under u than under u', for the lowest equilibrium strategy profile in $\tilde{G}(\nu)$ we find $\check{s}_\nu \geq s'_\nu$. Thus, $\check{s}(\theta^*) = \liminf_{\nu\to 0} \check{s}_\nu(\theta^*) \geq \lim_{\nu\to 0} s'_\nu(\theta^*) = a^*$, where the last equality follows by Theorem 1 and the fact that a^* is the unique GGS of Γ'. ∎

Proof of Lemma 2. Let us first prove two claims about restricted games of Γ:

Claim 2. *Consider four action profiles a, b, c, d such that $a \leq b \leq d$ and $a \leq c \leq d$. Then the highest GGS in $\Gamma|[a, b]$ is weakly lower than the highest GGS in $\Gamma|[c, d]$.*

Proof. Consider the highest GGS $\overline{a}$ in $\Gamma|[a, b]$. By Theorem 1 there exists an equilibrium strategy profile s in $E|[a, b]$ prescribing the action profile $\overline{a}$ for high signals. Define a strategy profile s' pointwise as $\max\{c, s(x)\}$ for each signal tuple x. Due to supermodularity, an upper best reply iteration in $E|[c, d]$ starting from s' will be increasing. Thus, there exists an equilibrium strategy profile in $E|[c, d]$ that prescribes actions weakly higher than $\overline{a}$ for high enough signals.

Claim 3. *Consider three action profiles $a \leq b \leq c$. If, for fixed f, a is the unique GGS in $\Gamma|[a, b]$ and b is the unique GGS in $\Gamma|[b, c]$, then a is the unique GGS in $\Gamma|[a, c]$.*

Proof. Consider the highest GGS $\overline{a}$ in $\Gamma|[a, c]$. Since b is the highest GGS in $\Gamma|[b, c]$, Claim 2 implies $\overline{a} \leq b$. In addition, $\overline{a}$ is the highest GGS in $\Gamma|[a, \overline{a}]$, as the highest equilibrium strategy profile in $E|[a, c]$ (which attains $\overline{a}$) is also an equilibrium profile in $E|[a, \overline{a}]$. Since $a \leq \overline{a} \leq b$, Claim 2 applied to the games $\Gamma|[a, \overline{a}]$ and $\Gamma|[a, b]$ yields $\overline{a} \leq a$, proving Claim 3.

Now, applying Claim 3 iteratively to the sequence $a^n \leq a^{n+1} \cdots \leq a^m$, we see that a^n is the unique GGS in $\Gamma|[a^n, m]$. Hence, by Claim 2, the highest GGS in $\Gamma = \Gamma|[0, m]$, $\overline{a}$, is weakly lower than a^n. By a dual argument, a^n is the unique GGS in $\Gamma|[0, a^n]$, and the lowest GGS in $\Gamma = \Gamma|[0, m]$, $\underline{a}$, is weakly higher than a^n. Together, this yields $\underline{a} = \overline{a} = a^n$. ∎

Proof of inequality (5). Fix a noise distribution f; let F_i be the c.d.f. of player i's density f_i. Then

$$\sum_{i\in I} P_i = \sum_{i\in I} \prod_{j\in I-\{i\}} \Pr(x_j \geq z_j | x_i = z_i) = \sum_{i\in I} \int_{-|A|-1}^{|A|+1} f_i(z_i - \theta) \prod_{j\in I-\{i\}} (1 - F_j(z_j - \theta))\, d\theta.$$

Picking some player $t \in I$ and integrating by parts the summand corresponding to $i = t$ gives

$$\left[(1 - F_1(z_1 - \theta)) \prod_{j\in I-\{t\}} (1 - F_j(z_j - \theta))\right]_{-|A|-1}^{|A|+1} - \int_{-|A|-1}^{|A|+1} \sum_{i\in I-\{t\}} \left(f_i(z_i - \theta) \prod_{j\in I-\{i\}} (1 - F_j(z_j - \theta))\right) d\theta$$

$$+ \sum_{i\in I-\{t\}} \int_{-|A|-1}^{|A|+1} f_i(z_i - \theta) \prod_{j\in I-\{i\}} (1 - F_j(z_j - \theta))\, d\theta = \left[\prod_{j\in I} (1 - F_j(z_j - \theta))\right]_{-|A|-1}^{|A|+1} = 1. \quad ∎$$

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