Reduction method in contact mechanics

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Outline

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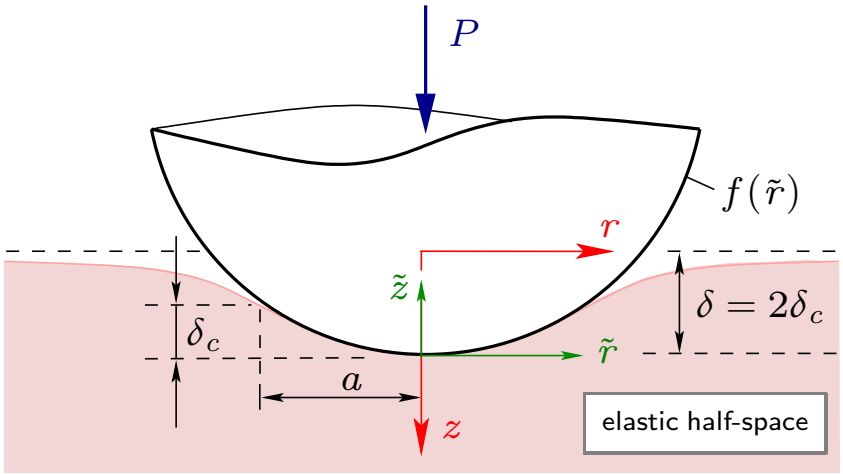
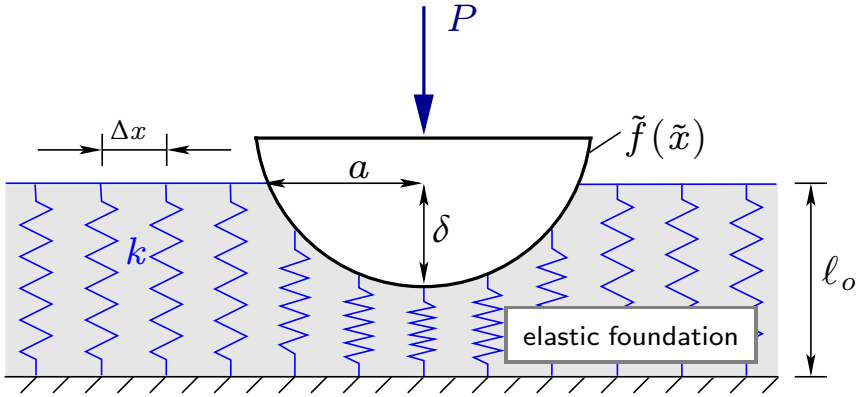
HERTZian contact – global relations

relationship between penetration, load and area of contact





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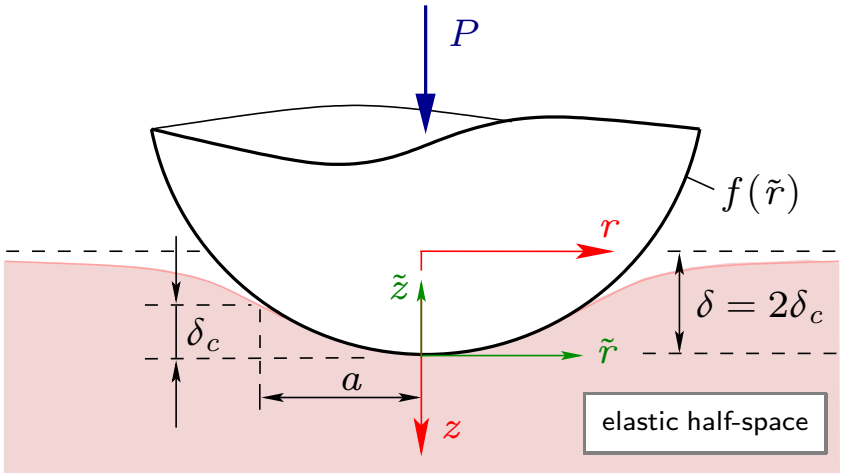

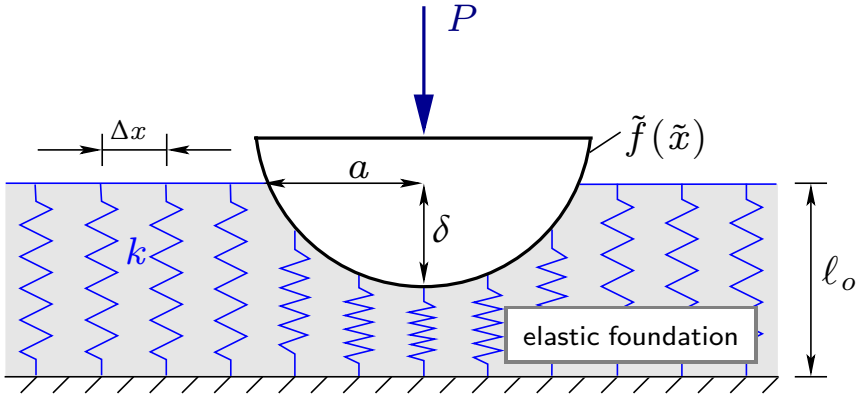

3D-HERTZ relations (HERTZ, 1882):

$$\delta = \frac{a^2}{R} \qquad P = \frac{4}{3}\tilde{E}R^{\frac{1}{2}}\delta^{\frac{3}{2}}$$

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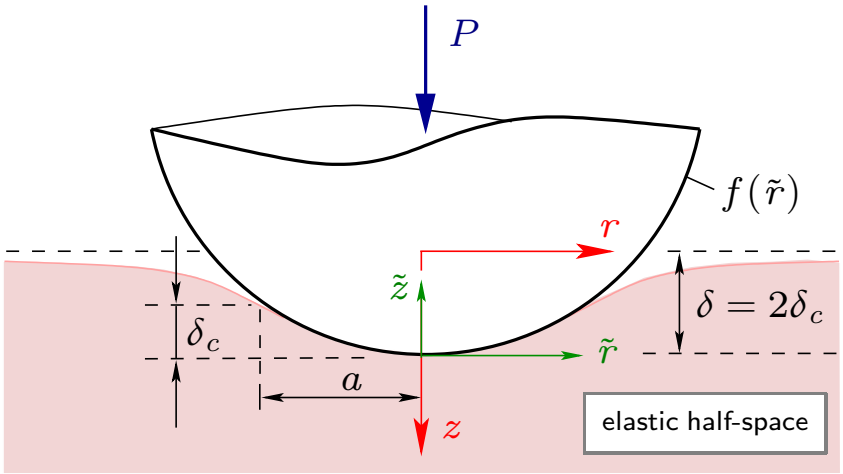

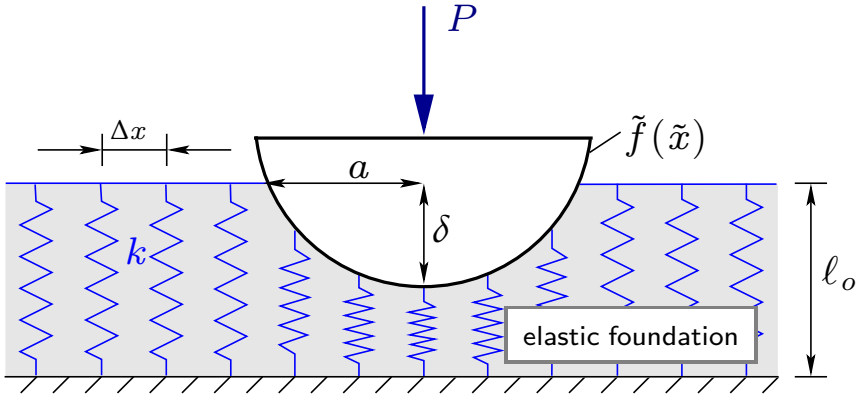

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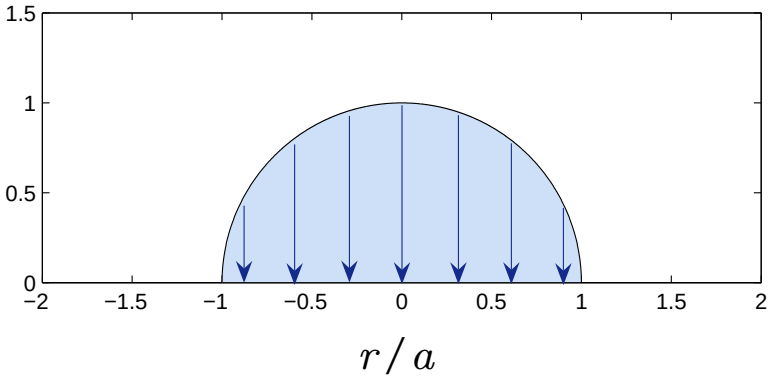
Requirements of mapping: $R_{1D} = \frac{1}{2}R$ und $k = \tilde{E}\Delta x$

HERTZian contact – pressure distribution

$$\frac{p_{\text{3D}}(r)}{p_o} = \sqrt{1 - \left(\frac{r}{a}\right)^2}$$



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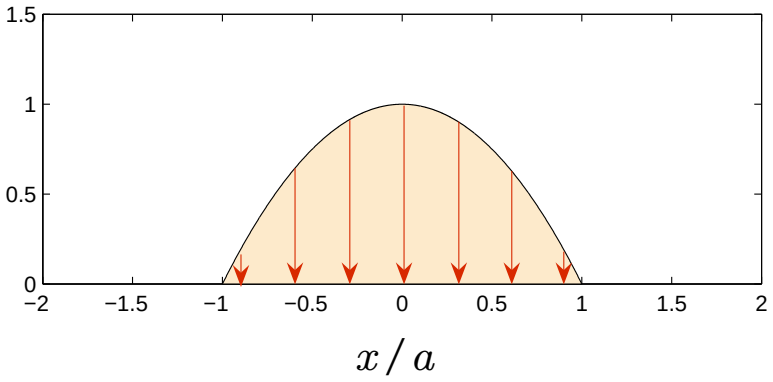

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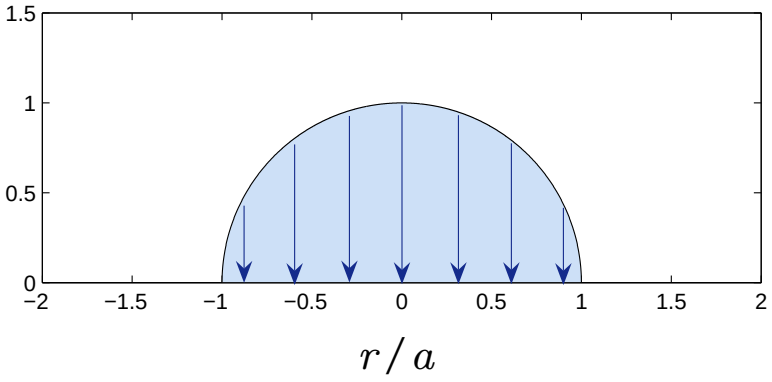
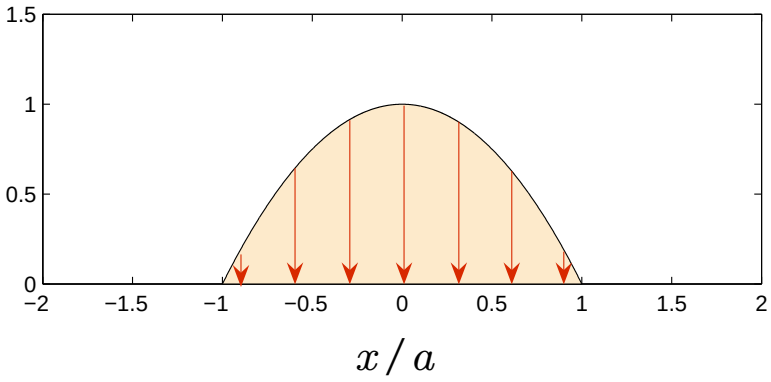


$$\frac{p_F(x)}{p_{o_F}} = 1 - \left(\frac{x}{a}\right)^2$$



New definition of „normal stresses"

$$p_{3D}(r) = \frac{p_F(r)}{c\sqrt{u_F(r)\,R_{1D}}}$$

(POPOV / GEIKE, 2007)

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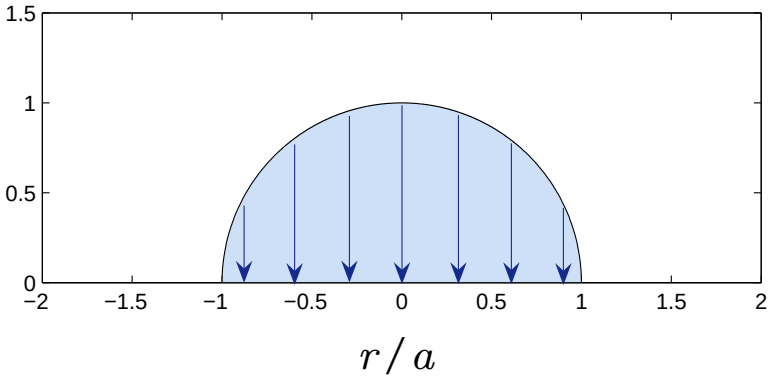

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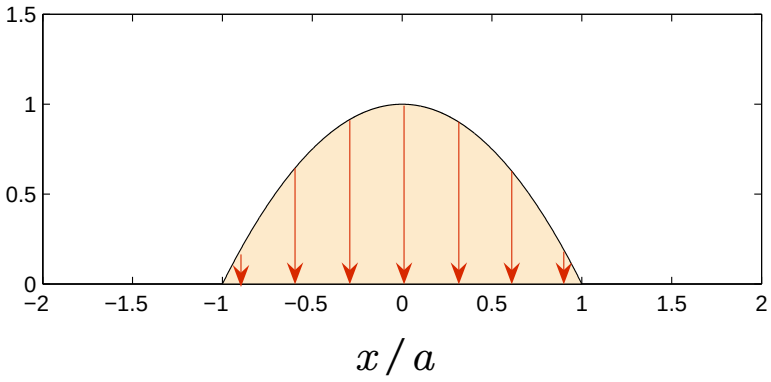

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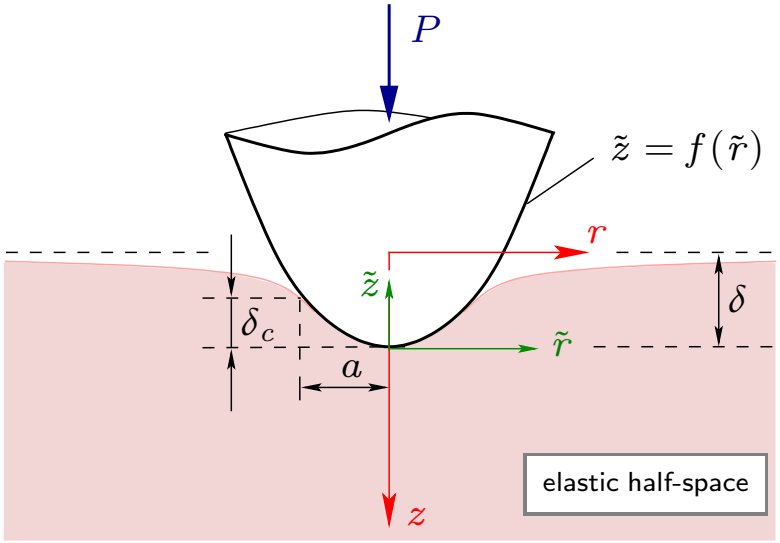
Guiding concept of reduction method

„As simply as possible, but nevertheless exactly".

Outline

Single axisymmetric contact of arbitrary shape



Boundary conditions:

$$
\begin{aligned}
u_z(r,0) &= \delta - f(r)\,,\ 0 \le r \le a \\
\sigma_{zz}(r,0) &= 0\,,\ r > a \\
\tau_{rz}(r,0) &= 0\,,\ r \ge 0
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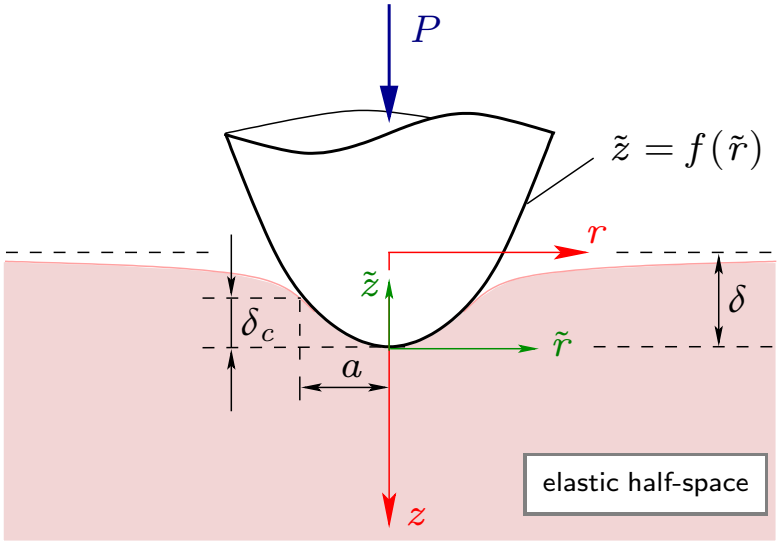

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Integral equations

of penetration and load:

$$
\delta = \int_0^1 \frac{f'(x)}{\sqrt{1-x^2}}\,dx + \frac{\pi}{2}\chi(1)
$$

$$
P = 2\tilde{E}a \int_0^1 \left[\delta - t \int_0^t \frac{f'(x)}{\sqrt{t^2-x^2}}\,dx \right] dt
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(Sneddon, 1965)

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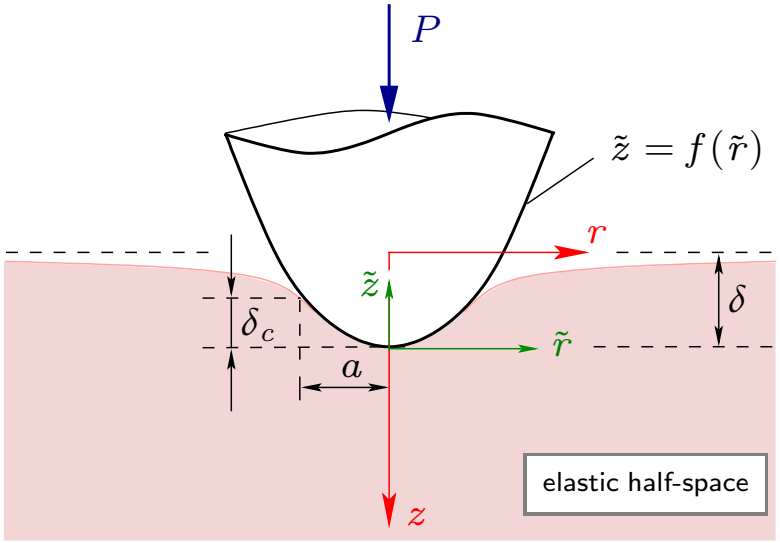

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Power-series as shape function:

$$
f(r) := \sum_{n=1}^{\infty} f_n(r) = \sum_{n=1}^{\infty} c_n r^n
$$

(Segedin, 1957)

$$P = 2\tilde{E} \sum_{n=1}^{\infty} \int_0^a \frac{\sqrt{\pi}}{2} \frac{n\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)} \left[\underbrace{\boldsymbol{f_n\left(a\right)}}_{\equiv \delta_{cn}} - f_n\left(r\right) \right] dr + 2\tilde{E}\, a \frac{\pi}{2} \chi\left(1\right)$$

After some simple manipulations ... integrand has equal structure as in the 1D-model!

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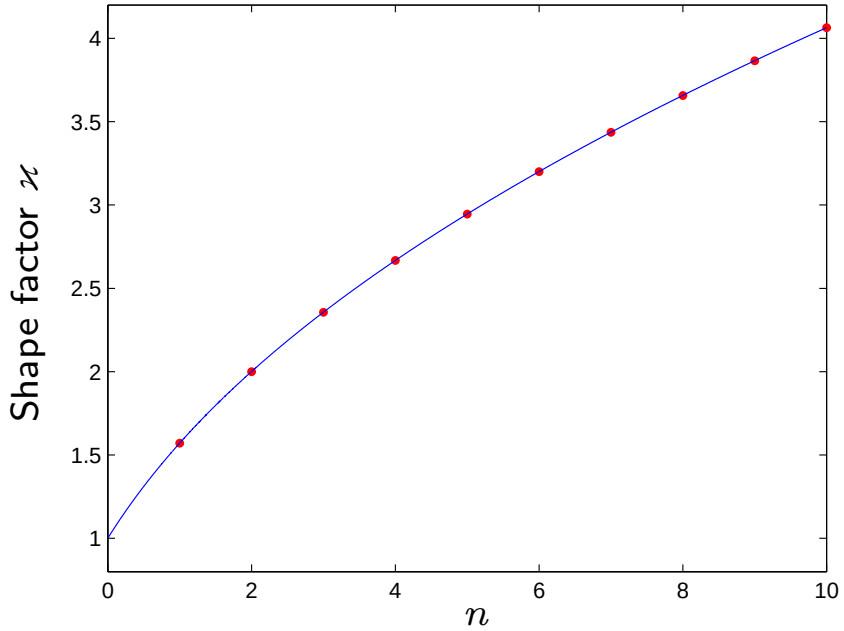
Necessary change of shape

$$\tilde{r} \longmapsto |\tilde{x}|\,, \quad f_n\left(\tilde{r}\right) := c_n \tilde{r}^n \longmapsto \tilde{f}_n\left(\tilde{x}\right) := \tilde{c}_n |\tilde{x}|^n \quad \text{with } \tilde{c}_n := \varkappa_n c_n\,,\ n \in \mathbb{R}^+$$

$$\text{Shape factor:} \quad \varkappa_n := \frac{\delta_n}{\delta_{cn}}$$



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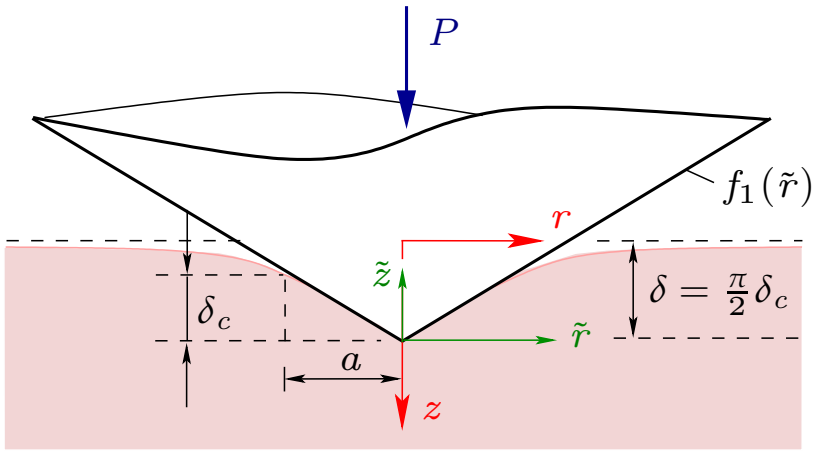
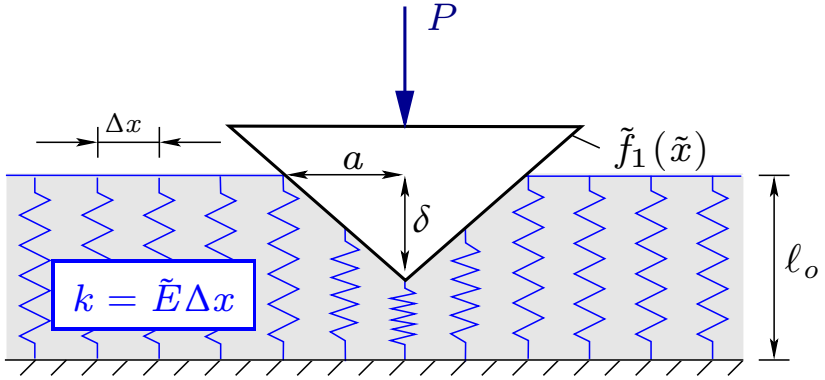
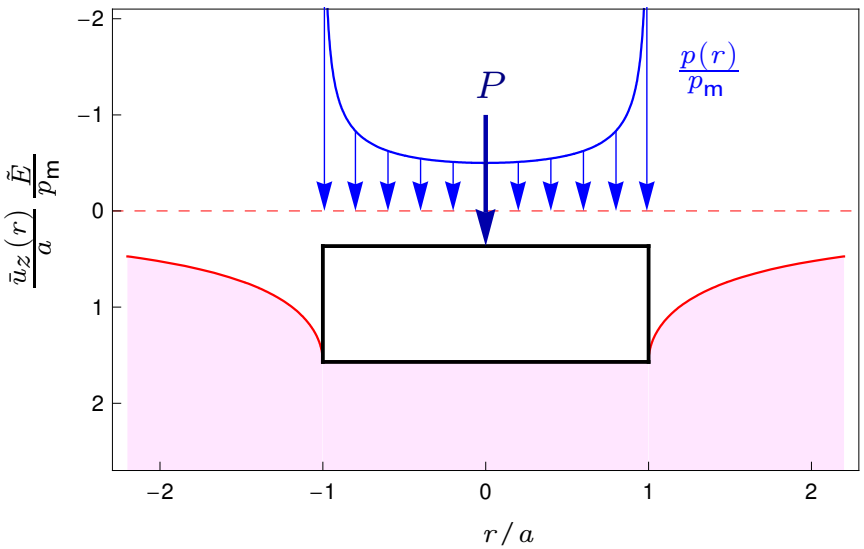
HERTZ: $\varkappa_2 = 2$

Cone: $\varkappa_1 = \frac{\pi}{2}$





Indentation by flat cylindrical punch



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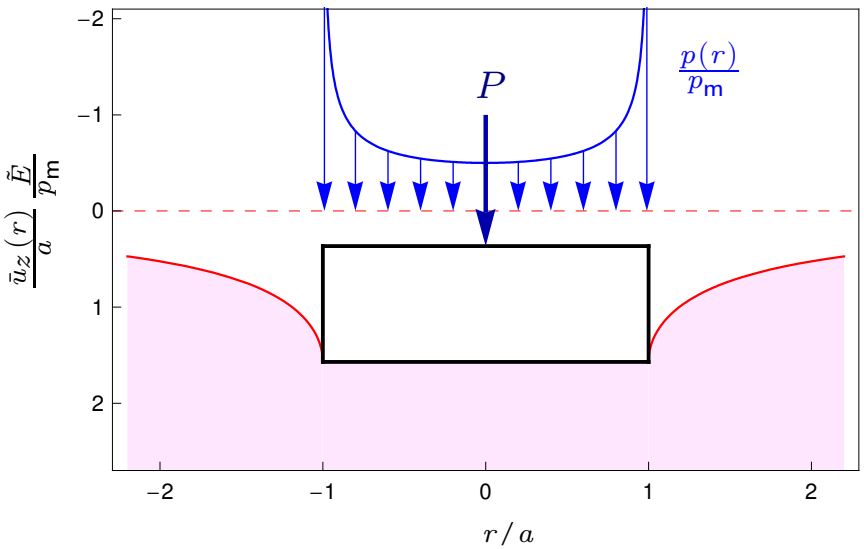

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Universal stiffness of normal contact: $k_{\text{ges}} := \frac{dP}{d\delta} = 2\tilde{E}a$ (Pharr / Oliver, 1992)

$k = \tilde{E}\Delta x$, 1D-model has the same stiffness!

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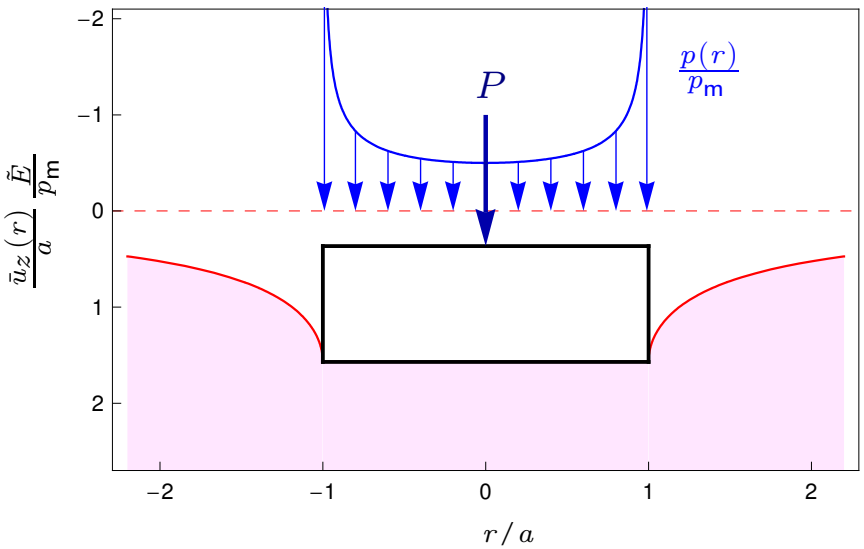

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The indentation with a flat cylindrical punch can also be mapped exactly by an one-dimensional system!

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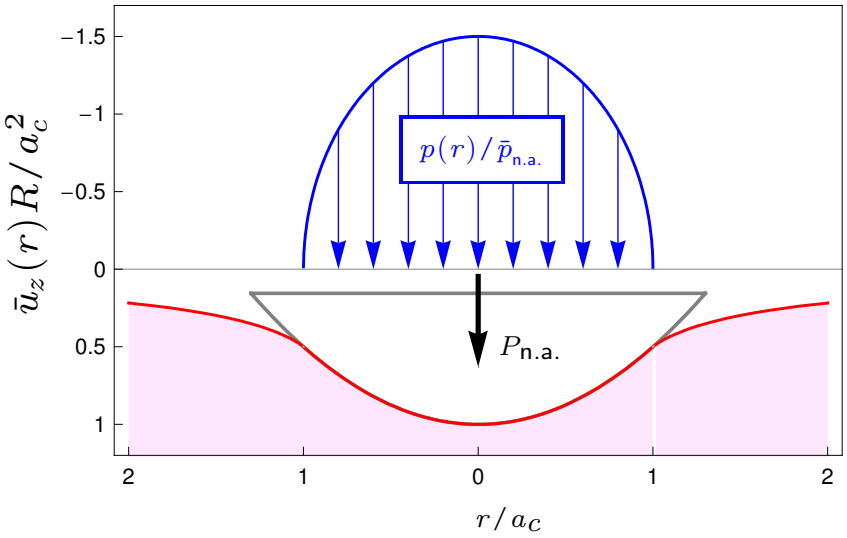
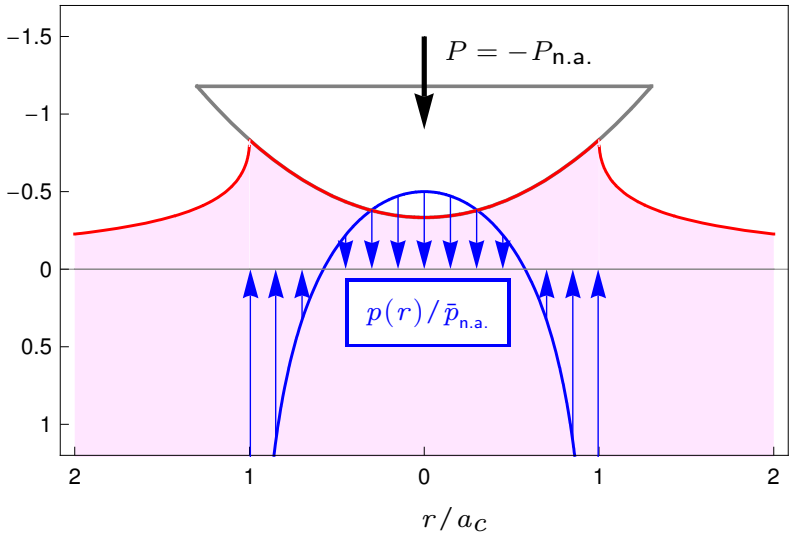
Mapping of generalized JKR-theory

Keynote:

The contact with adhesion arises from the contact without adhesion plus a rigid body translation! (JOHNSON, KENDALL, ROBERTS, 1971)





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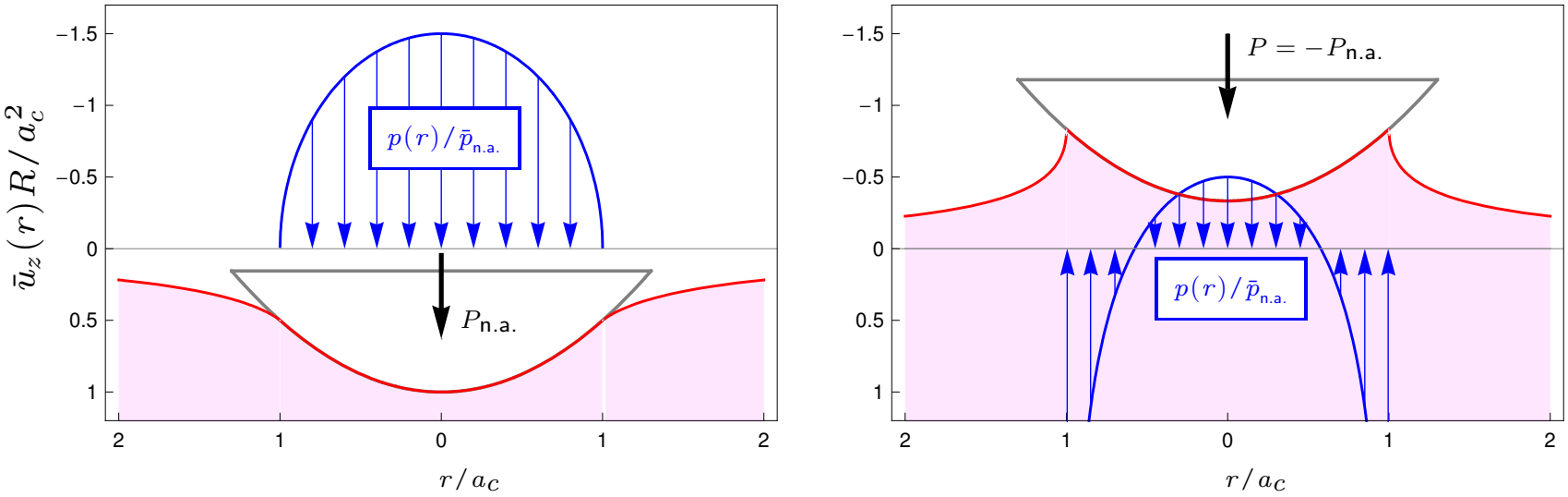

Immediate Conclusion

The contact with adhesion can be mapped exactly by a 1D-model!

Structure of unloading part resembles loading by a flat cylindrical punch

$$\left.\begin{aligned} \delta &= \delta_{\text{n.a.}} - \sqrt{\frac{2a\pi\bar{\gamma}}{\tilde{E}}} \\ P &= P_{\text{n.a.}} - \sqrt{8\pi\tilde{E}a^3\bar{\gamma}} \end{aligned}\right\} \Longrightarrow P_{\text{n.a.}} - P = 2\tilde{E}a\left(\delta_{\text{n.a.}} - \delta\right)$$

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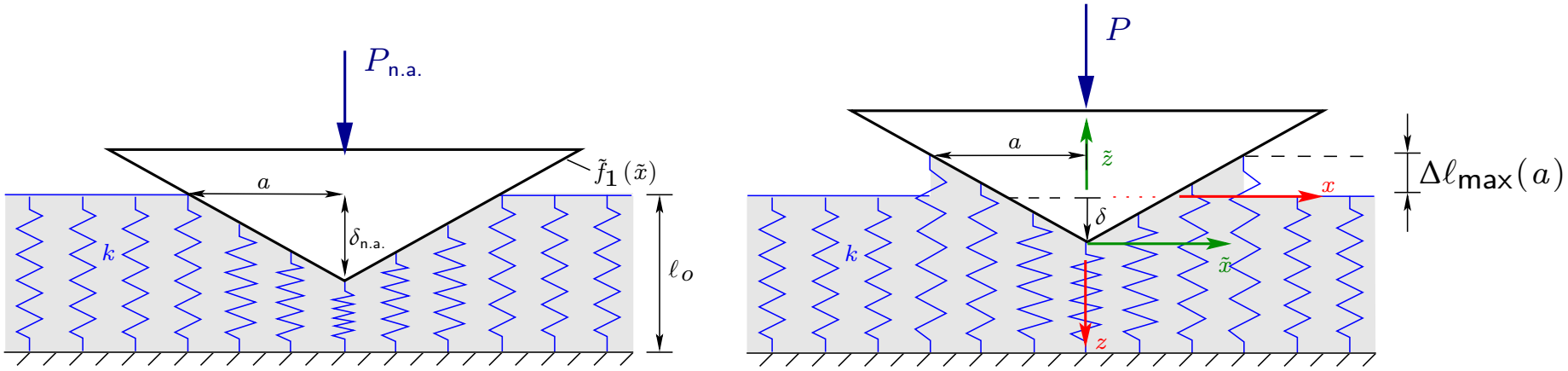
Loading und pull-off in 1D-model

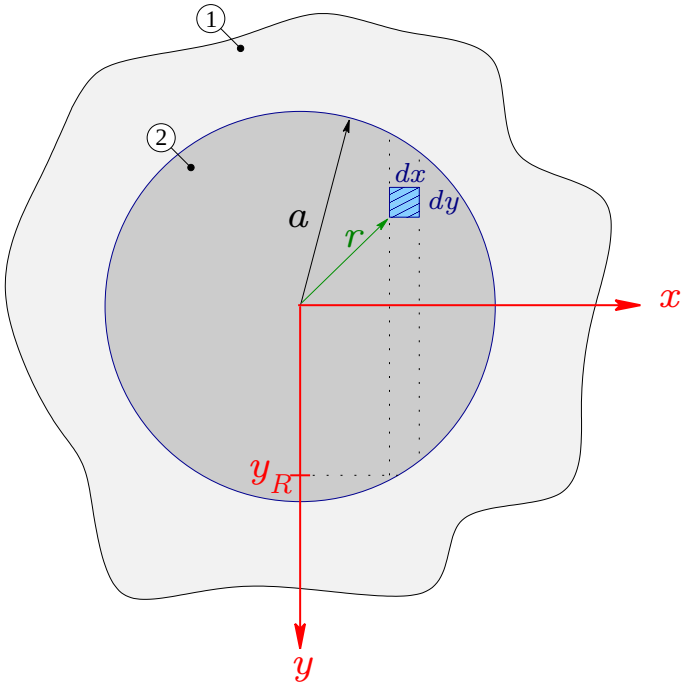


Pull-off condition: $\Delta\ell_{\text{max}}(a) = \sqrt{\frac{2\pi\bar{\gamma}}{\tilde{E}}\,a}$

Outline

WEBER-Transformation

Conversion of axisymmetric pressure distribution on two-dimensional ones by projection (WEBER, 1940)

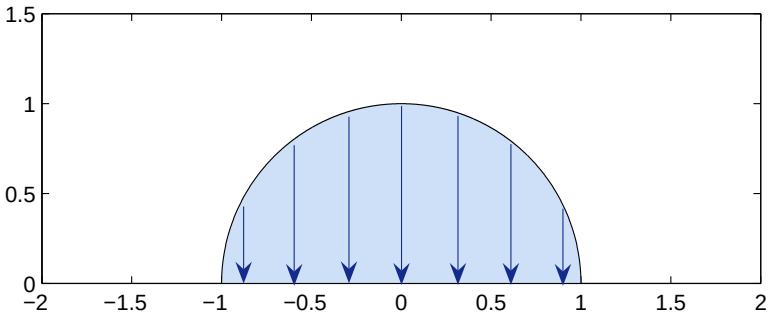


① Surface of elastic half-space
② Loaded circular area

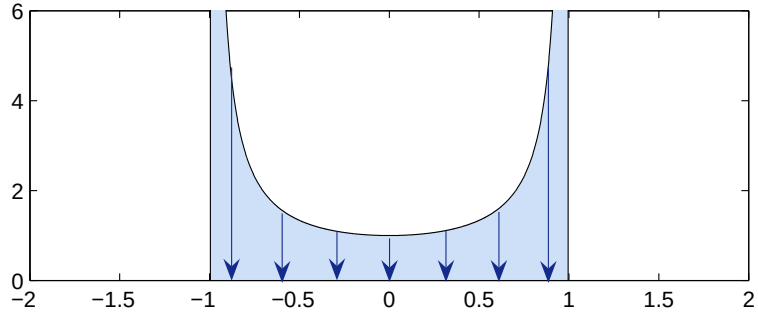
$$\begin{aligned}
\mathcal{W}\left[p\left(r\right);x\right] &:= \int_{-y_R}^{y_R} p\left(r\right)\,dy \\
&= 2\int_{0}^{\sqrt{a^2-x^2}} p\left(r\right)\,dy \\
&= 2\int_{x}^{a} \frac{rp\left(r\right)}{\sqrt{r^2-x^2}}\,dr \\
&\equiv \underline{\underline{\sqrt{2\pi}\mathcal{A}_2\left[rp\left(r\right);x\right]}}
\end{aligned}$$

Axisymmetr. pressure distribution

Hertz: $$\frac{p_{3D}(r)}{p_o} = \sqrt{1-\left(\frac{r}{a}\right)^2}$$



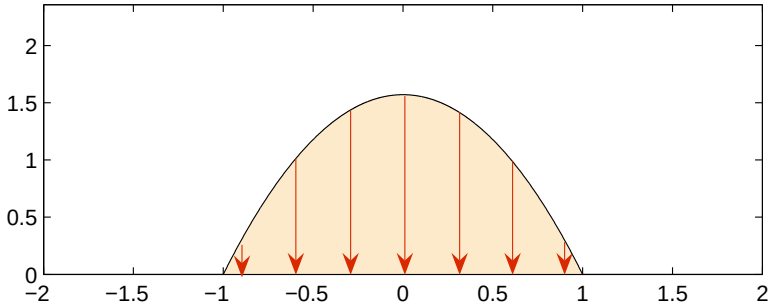
Flat punch: $$\frac{p_{3D}(r)}{p_o} = \frac{1}{\sqrt{1-\left(\frac{r}{a}\right)^2}}$$



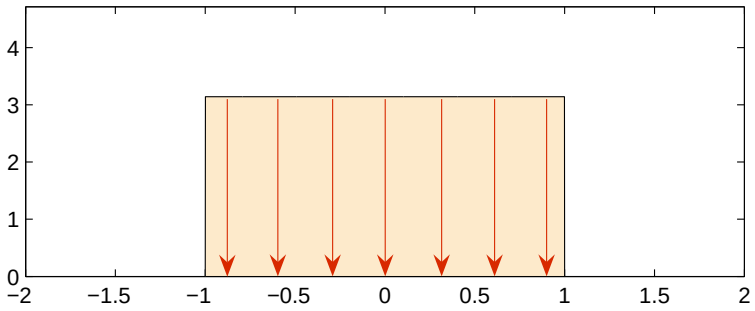
r/a

Weber-transform (line load)

$$\frac{\mathcal{W}\left[p_{3D}(r)\,;\,x\right]}{p_o\,a} = \frac{\pi}{2}\left(1-\left(\frac{x}{a}\right)^2\right)$$



$$\frac{\mathcal{W}\left[p_{3D}(r)\,;\,x\right]}{p_o\,a} = \pi$$



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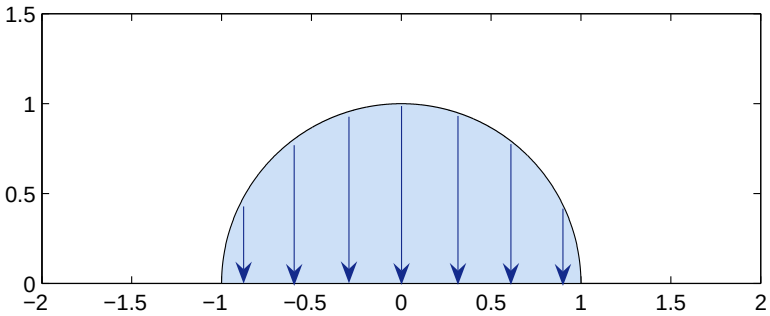

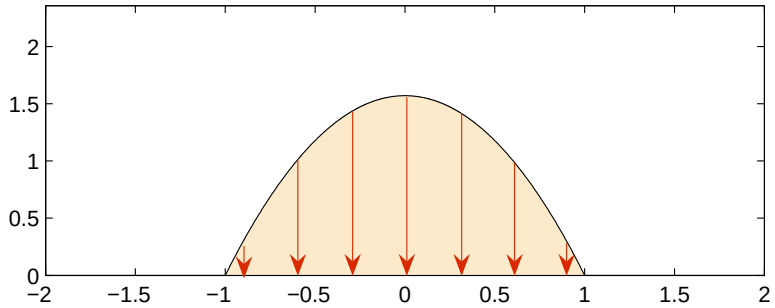

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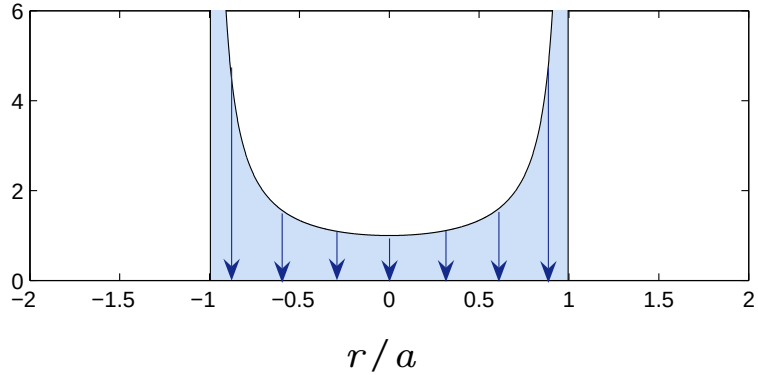
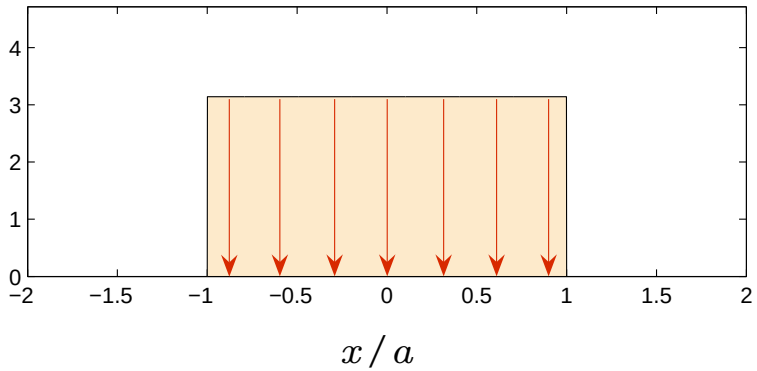




3D-pressure calculable from 1D-spring-pressure by inverse Weber-tranform

Alternative, applicable to even exponents

Expansion of real pressure distribution ...

... in series of special Legendre polynomials (G.IA. POPOV, 1962; JAFFAR, 2004)

$$p(\bar{r}) = \frac{1}{\sqrt{1-\bar{r}^2}} \sum_{n=0}^{N} k_{2n} L_{2n}\left(\sqrt{1-\bar{r}^2}\right)$$

$$u_z(\bar{r}, z=0) = \frac{\pi a (1-\nu)}{2G} \sum_{n=0}^{N} k_{2n} \left[L_{2n}(0)\right]^2 L_{2n}\left(\sqrt{1-\bar{r}^2}\right)$$

Expansion of displacements of one-dimensional model ...

... in series of special root elements $s^m := \left(1-\tilde{x}^2\right)^{m/2}$ (HESS, 2007)

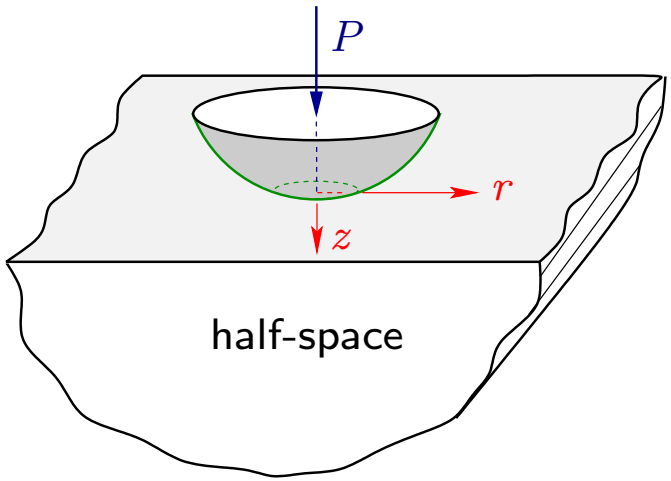
$$\tilde{u}_z(s,0) = \sum_{k=1}^{K} \tilde{g}_k(s) = \sum_{k=1}^{K} e_k s^k$$

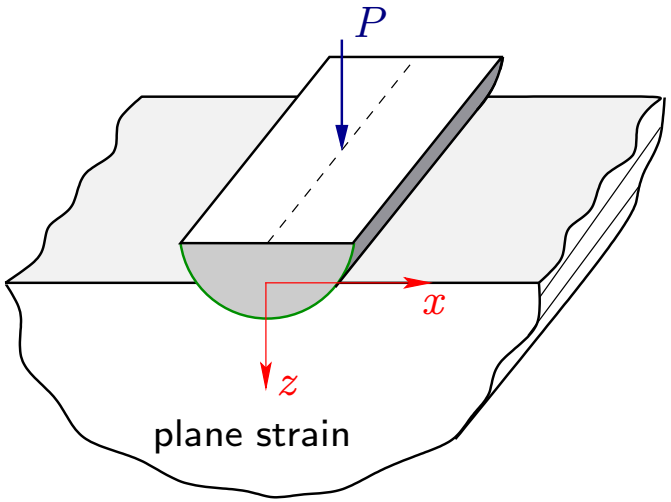
$$p(s) = \frac{\tilde{E}}{\pi a} \frac{1}{s} \sum_{k=1}^{K} \varkappa_k \tilde{g}_k(s) = \frac{\tilde{E}}{\pi a} \frac{1}{s} \sum_{k=1}^{K} \varkappa_k e_k s^k$$

"3D-pressure equal to 1D-spring-pressure divided by s", but weighting necessary!

Outline

Forminvariances





Bipotential equation

$$\triangle\triangle L(r,z) = 0$$

$$2G\mathbf{u} = 2(1-\nu)\triangle L\mathbf{e}_z - \nabla\nabla\cdot L\mathbf{e}_z$$

L LOVE displacement function

$$\triangle\triangle\Phi(x,z) = 0$$

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial z^2}, \sigma_{zz} = \frac{\partial^2\Phi}{\partial x^2}, \tau_{xz} = -\frac{\partial^2\Phi}{\partial x\partial z}$$

Φ AIRY stress function

Elastic half-space - axisymmetry

$$\begin{bmatrix} 2G\mathcal{H}_1\left[u_r\left(r,z\right);\xi\right] \\ 2G\mathcal{H}_0\left[u_z\left(r,z\right);\xi\right] \\ \mathcal{H}_1\left[\tau_{rz}\left(r,z\right);\xi\right] \\ \mathcal{H}_0\left[\sigma_{zz}\left(r,z\right);\xi\right] \end{bmatrix} = \begin{bmatrix} \xi^{-1}\left(-1+2\nu\right)+z \\ 2\xi^{-1}\left(1-\nu\right)+z \\ -\xi z \\ -1-\xi z \end{bmatrix} \bar{p}_0\left(\xi\right) e^{-\xi z} + \begin{bmatrix} 2\xi^{-1}\left(1-\nu\right)-z \\ -\xi^{-1}\left(1-2\nu\right)-z \\ -1+\xi z \\ \xi z \end{bmatrix} \bar{q}_0\left(\xi\right) e^{-\xi z}$$

Elastic half-space - state of plane strain

$$\begin{bmatrix} 2G\mathcal{F}_s[u_x^p\left(x,z\right);\lambda] \\ 2G\mathcal{F}_c[u_z^p\left(x,z\right);\lambda] \\ \mathcal{F}_s[\tau_{xz}^p\left(x,z\right);\lambda] \\ \mathcal{F}_c[\sigma_{zz}^p\left(x,z\right);\lambda] \end{bmatrix} = \begin{bmatrix} -\lambda^{-1}\left(1-2\nu\right)+z \\ 2\lambda^{-1}\left(1-\nu\right)+z \\ -\lambda z \\ -1-\lambda z \end{bmatrix} p_c^p\left(\lambda\right) e^{-\lambda z} + \begin{bmatrix} 2\lambda^{-1}\left(1-\nu\right)-z \\ -\lambda^{-1}\left(1-2\nu\right)-z \\ -1+\lambda z \\ \lambda z \end{bmatrix} q_s^p\left(\lambda\right) e^{-\lambda z}$$

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Approach: Boundary stresses equal

in frequency space: $\bar{p}_0\left(\xi\right) \stackrel{!}{=} b\, p_c^p\left(\xi\right) \qquad \bar{q}_0\left(\xi\right) \stackrel{!}{=} b\, q_s^p\left(\xi\right)$

Correspondence principle – part I –

$$u_x^p(x,z) = b^{-1} x \mathcal{A}_2 \left[u_r(r,z);\, x\right] \qquad u_r(r,z) = b\, \mathcal{A}_2^{-1} \left[x^{-1} u_x^p(x,z);\, r\right]$$

$$u_z^p(x,z) = b^{-1} \mathcal{A}_2 \left[r u_z(r,z);\, x\right] \qquad u_z(r,z) = b\, r^{-1} \mathcal{A}_2^{-1} \left[u_z^p(x,z);\, r\right]$$

$$\tau_{xz}^p(x,z) = b^{-1} x \mathcal{A}_2 \left[\tau_{rz}(r,z);\, x\right] \qquad \tau_{rz}(r,z) = b\, \mathcal{A}_2^{-1} \left[x^{-1} \tau_{xz}^p(x,z);\, r\right]$$

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$$\sigma_{zz}^p(x,z) = b^{-1} \mathcal{A}_2 [r \sigma_{zz}(r,z); x] \qquad \sigma_{zz}(r,z) = b\, r^{-1} \mathcal{A}_2^{-1} [\sigma_{zz}^p(x,z); r]$$

Conclusion

- *By means of correspondence principle the conversion between internal fields of axisymmetry and state of plane strain/stress is possible!*
- *Simple Connection to 1D-model possible, because of* $\sigma_{zz}^p(x,0) = b^{-1} \mathcal{A}_2 [r \sigma_{zz}(r,0); x] \overset{!}{=} \frac{1}{\sqrt{2\pi}} p_{\mathsf{F}}(x)$: *Spring-pressure divided by* $\sqrt{2\pi}$ *equal to plane boundary load.*

Complete algorithm of reduction

Complete algorithm

1. Indentation of 1D-spring-bed with modified punch $\Longrightarrow P$-δ-a-relationship exact
2. Loading of half-plane by spring-pressure (divided by $\sqrt{2\pi}$)
3. Detection of plane internal fields
4. Inverse ABEL-transform of plane fields $\Longrightarrow$ All axisymmetric internal fields exact

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- **Disadvantage of correspondence principle**
  Inverse Abel-transform of plane fields!

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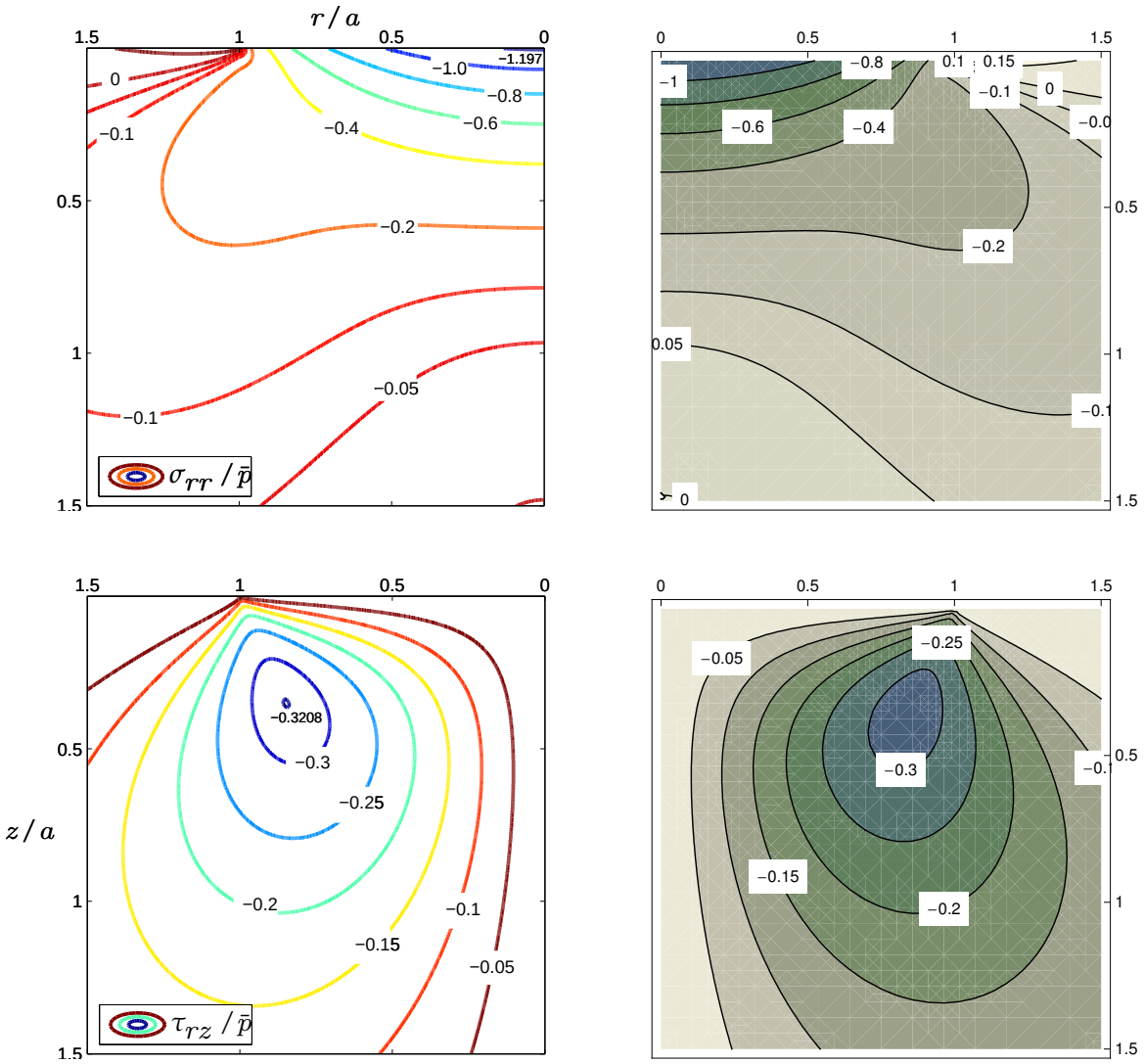
- **Disadvantage of correspondence principle**
  Inverse Abel-transform of plane fields!
- **Advantage**
  Discretization of two- instead of three-dimensional system!

Outline

Contact of HERTZ and HUBER 1904



Left
Analytical solution of HERTZ and HUBER

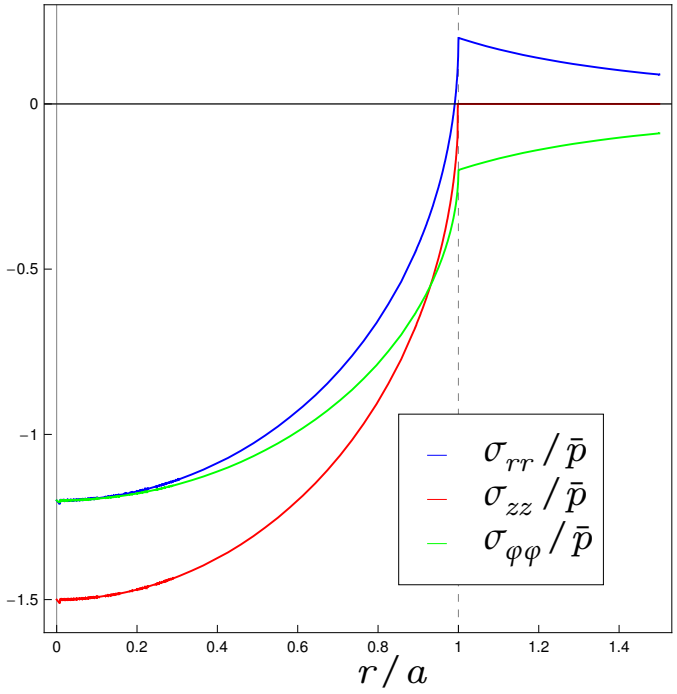
Right
Numerical results with the upgrading reduction method

Here
Plane fields by FOURIER-Integrals produced by MATHEMATICA

In principle
reduction method independant of numerical discretization method!

Charakteristica of HERTZian contact

Stress distribution at surface ($z = 0$)



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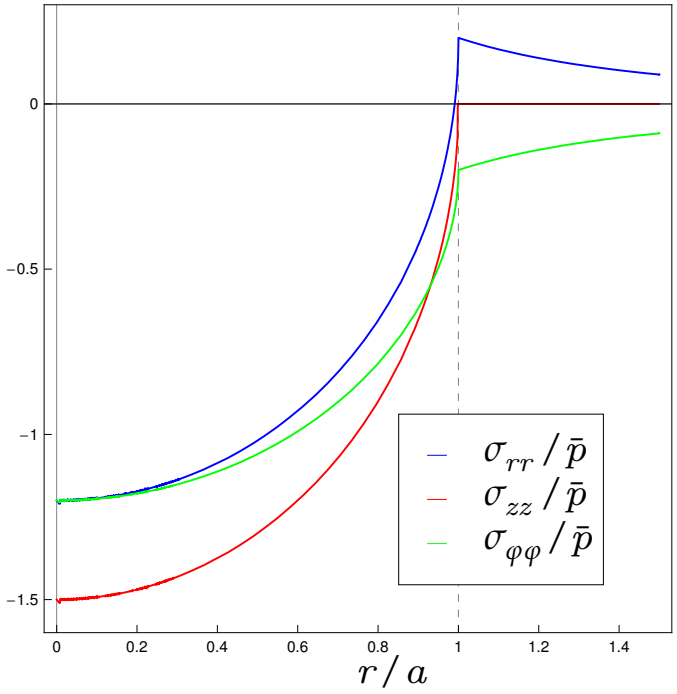
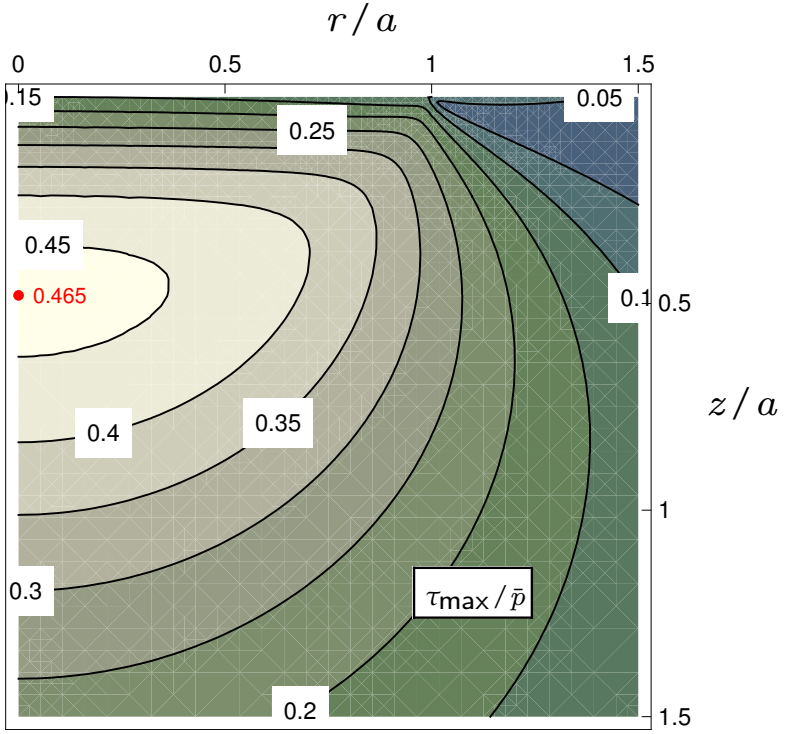


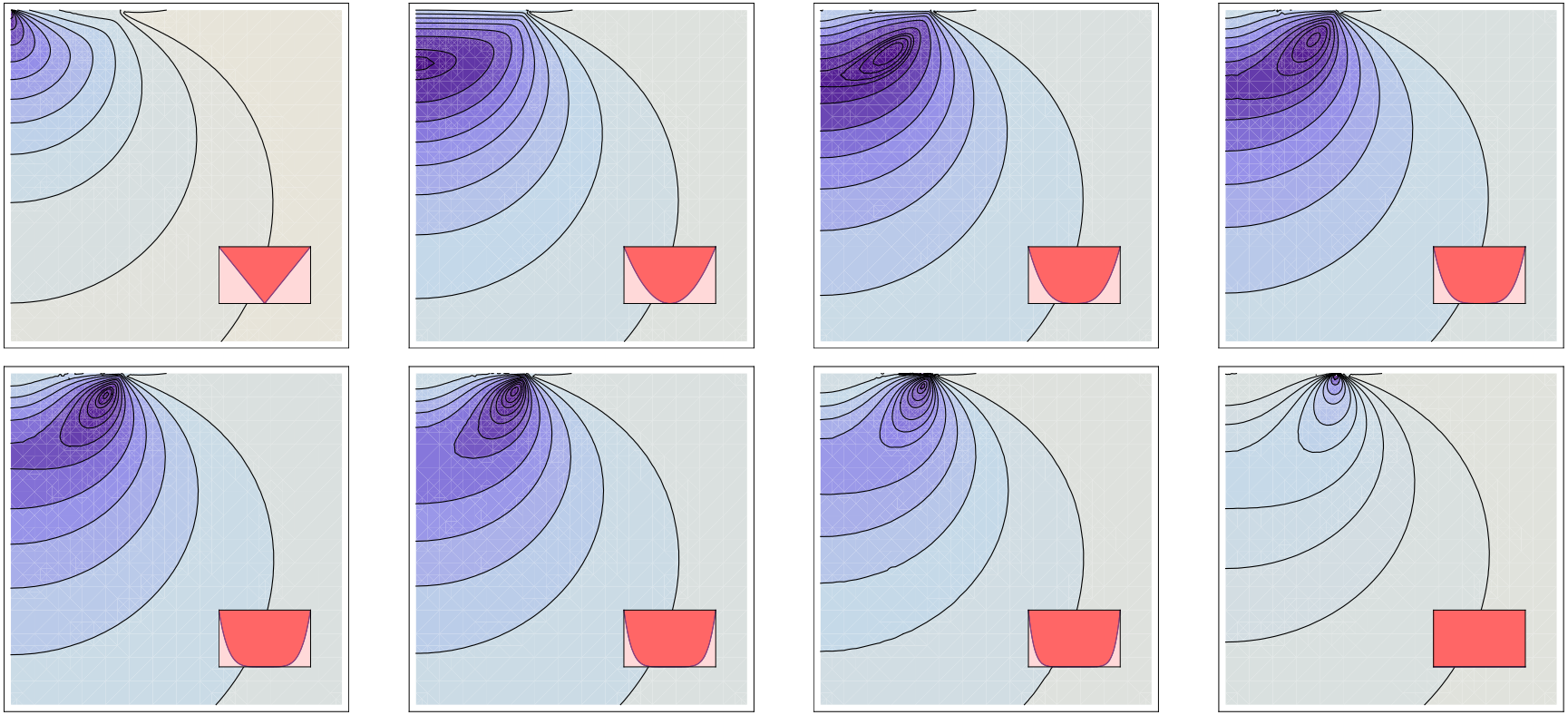
Contours of principal shear stress



Reduction method shows exactly the same results as theory! (Cf. JOHNSON, 1987)

Maximum of principal shear stresses

Influence of punch-profil

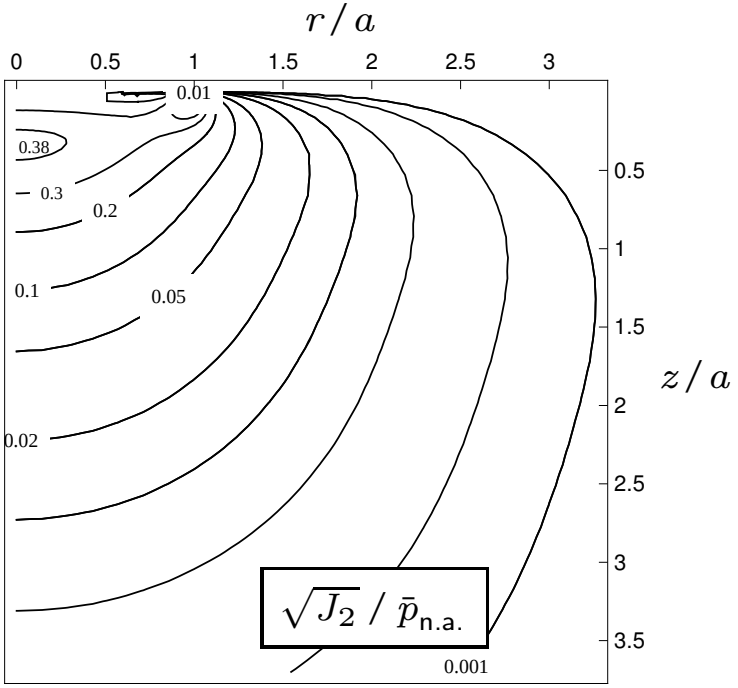


Change of position of maximum principal shear stresses; **results produced with reduction method!**

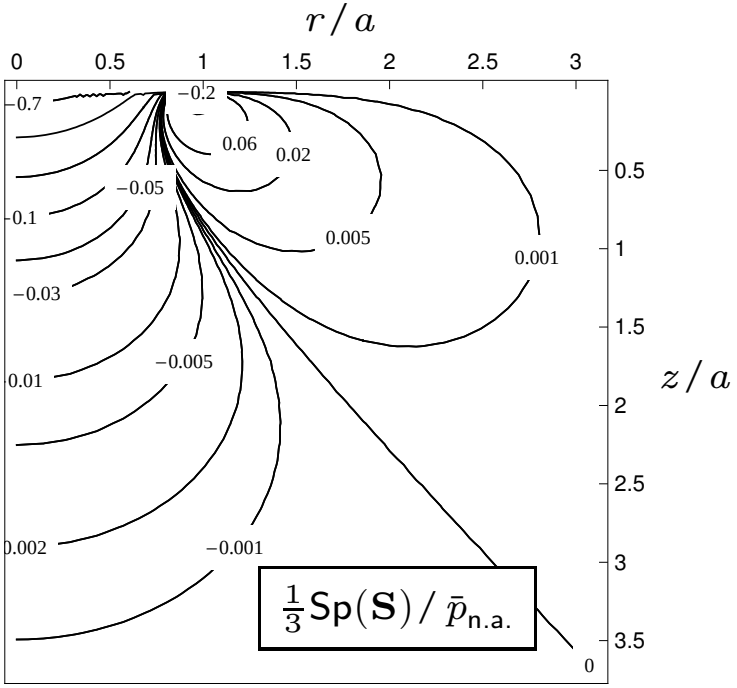
Contact of a sphere with adhesion

... in case of no external force

VON MISES' equivalent stresses



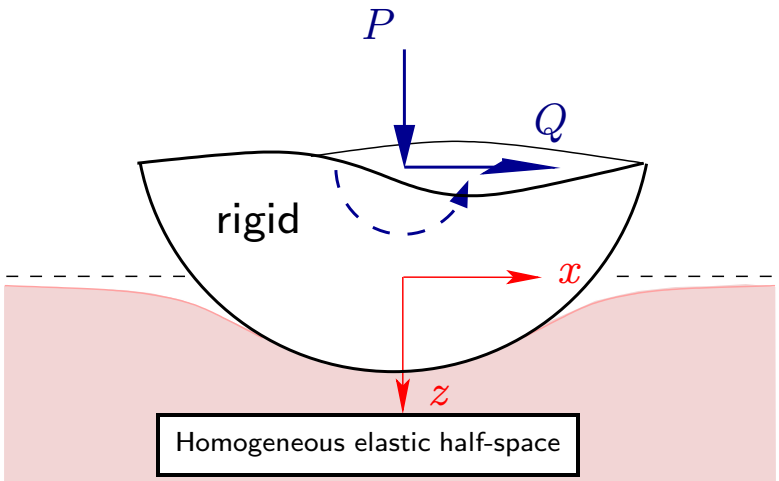
Hydrostatic state of stress



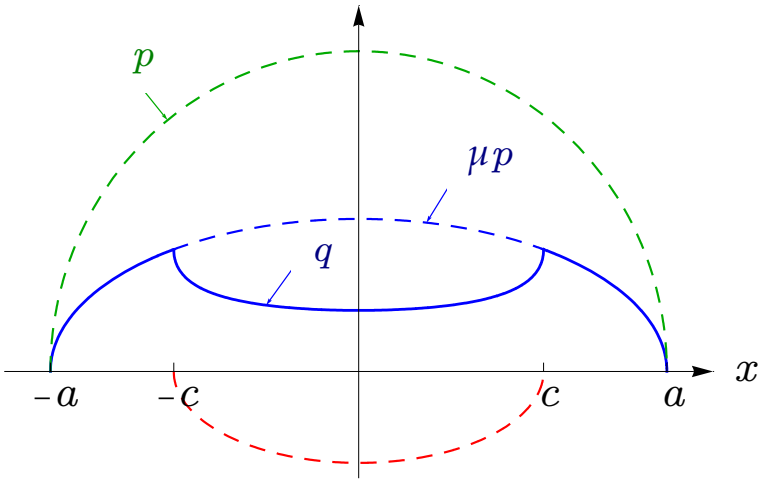
Reduction method leads to exactly the same results as ones of BARQUINS, MAUGIS, (1982, 2000)!

Outline

Tangential contact of sphere – partial slip



Tangential traction (Cattaneo, Mindlin)



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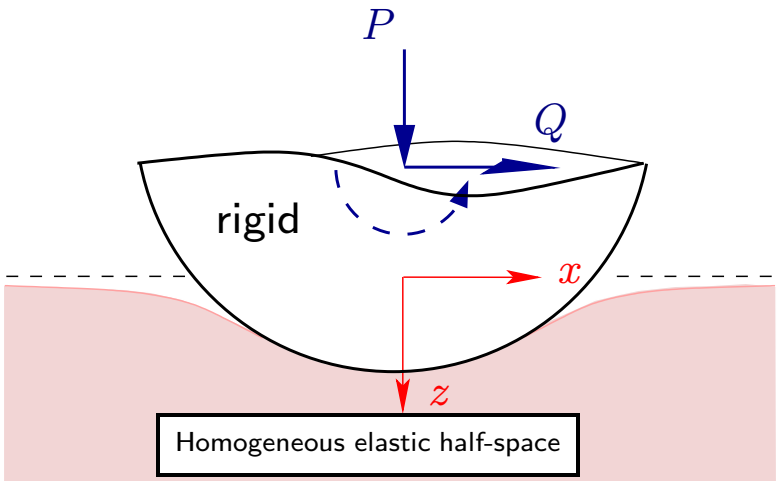

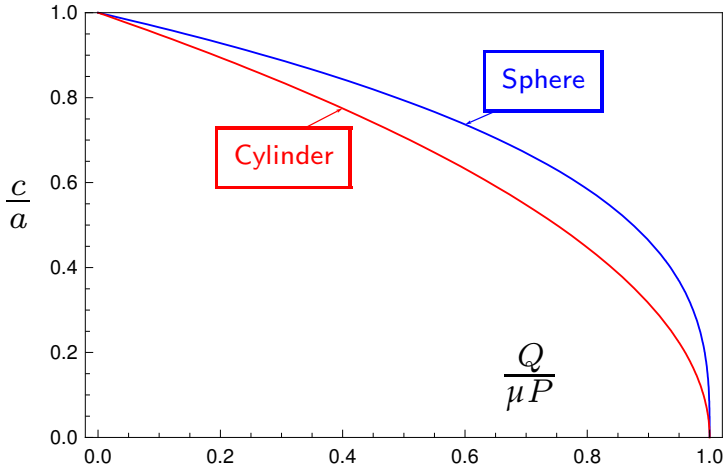

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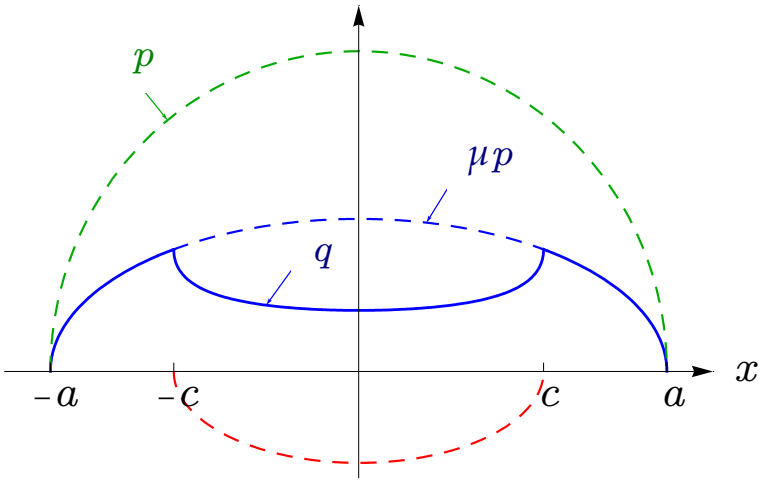


Stick region

Sphere: $$\frac{c}{a} = \left(1 - \frac{Q}{\mu P}\right)^{1/3}$$

Cylinder: $$\frac{c}{a} = \left(1 - \frac{Q}{\mu P}\right)^{1/2}$$

Tang. contact of cylinder and inhomogeneous half-space

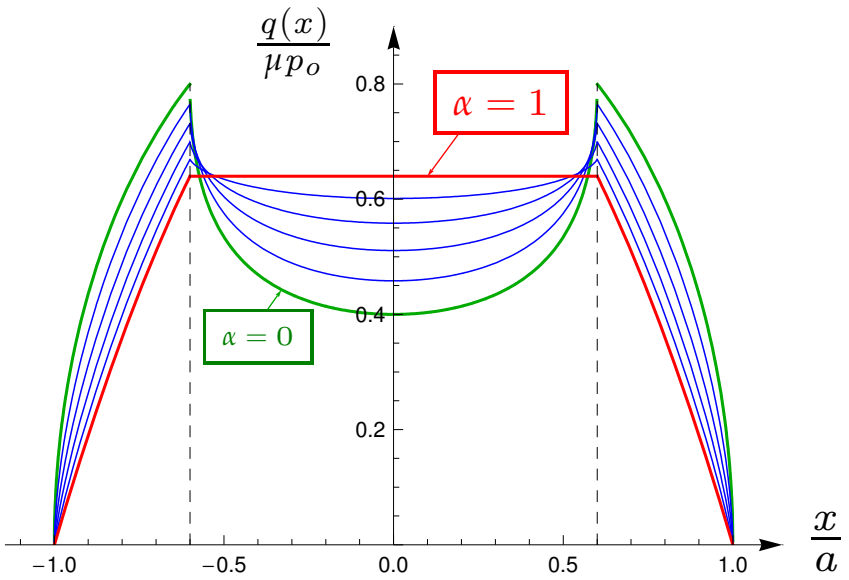
Kind of inhomogenity

$$E(z) = m_E z^{\alpha}, \; 0 < \alpha \leq 1$$

MINDLINs technique leads to

$$q^I(x) = \mu p_o(a) \left[1 - \left(\frac{x}{a}\right)^2\right]^{\frac{1+\alpha}{2}}, \; |x| \leq a$$

$$q^{II}(x) = q_o^{II}(c) \left[1 - \left(\frac{x}{c}\right)^2\right]^{\frac{1+\alpha}{2}}, \; |x| \leq c$$



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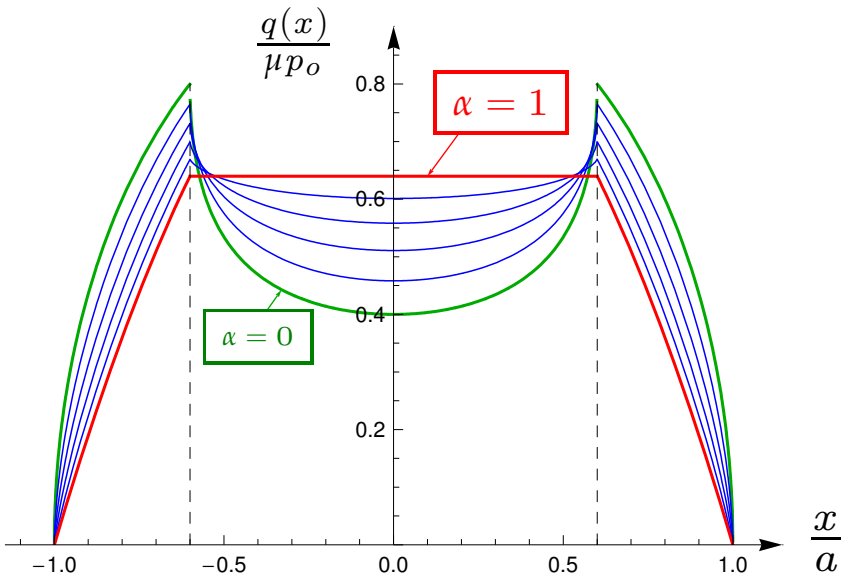

Dimension of stick-region: $\frac{c}{a} = \left(1 - \frac{Q}{\mu P}\right)^{\frac{1}{2+\alpha}}$ for $\alpha = 1 \implies \frac{c}{a} = \left(1 - \frac{Q}{\mu P}\right)^{1/3}$

Conclusion

2D-stick-region agrees with 3D-tangential contact of sphere on homogeneous half-space!

Summary and outlook

Summary

Reduction method provides exactly the same results as in threedimensional case. We showed how to map

- conforming and non-conforming contacts with and without adhesion,
- the pressure distribution and
- the internal fields of axisymmetric half-space.

Summary and outlook

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Reduction method provides exactly the same results as in threedimensional case. We showed how to map

- conforming and non-conforming contacts with and without adhesion,
- the pressure distribution and
- the internal fields of axisymmetric half-space.

Outlook

The reduction method is also applicable to

- inhomogeneous and layered half-space,
- to viscoelastic solids.
- ...

Thank you for your attention!

For further reading

1. T. Geike, V.L. Popov. *Mapping of three-dimensional contact problems into one dimension*. Physical Review E,76:036710, 2007
2. V.L. Popov. *Kontaktmechanik und Reibung. Von der Nanotribologie bis zur Erdbebendynamik,* 2.Auflage. Springer, 2010
3. M. Heß. *Über die exakte Abbildung ausgewählter dreidimensionaler Kontakte auf Systeme mit niedrigerer räumlicher Dimension*. PhD thesis, 2011