Compendium

on

Gradient Materials

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Preface
of the 2nd edition of January 2016

This is the second edition of the compendium following the first one from Oct. 2015. Apart from updating it and making minor corrections and improvements, three major parts have been added, namely

- at the end of Chapt. 1 the balance equations and boundary conditions for the case of third-order continua;
- Chapt. 3 describes the general *N*-th order material;
- Chapt. 5 investigates isotropic hexadics which appear in the second-order linear elasticity theory.

This compendium is in large parts a compilation of already published articles, which are partly paraphrased, in others modified, or extended, and brought into a unified notation.

It is the intention to present various results on gradient materials in a unified manner. It is meant as a working material which everybody may freely use. Whenever an improvement or a correction or a useful comment can be made, the compendium will be up-dated. So all users should *make sure to always use the latest version*.

What is the difference between this compendium and an ordinary scientific book? First of all, this is a non-profit project, just to serve scientific progress. So everybody has free access and can download it at any time. By our format, we are much less restricted by the usual rules of publication policies. We can up-date our compendium at any time.

The compendium is also not meant to be read like a book, linearly from page one till the last page. Instead, we tried to make the chapters self-contained. So it should be possible to just pick out one chapter, without having studied all the foregoing ones. In some cases this leads to repetitions and redundancy.

All researchers in the field are invited to contribute to this compendium. For this purpose please contact the editor. All comments and suggestions to improve this compendiums are also highly welcome at the same address (albrecht.bertram@ovgu.de).

The compendium is organized as follows.

In a first part we deal with balance laws for gradient materials. It will be demonstrated how the laws of motion apply to higher-order materials, and what the boundary conditions look like. This approach is based on the *Principle of Virtual Power* (PVP) as a continuous and linear extension of the power functional. It is compared with the procedure by CAUCHY, who started with forces as primitive concepts.

After having provided the balances, one needs constitutive laws. These are considered here for elastic and elastoplastic materials. Since there are large differences between a (geometrically)

linear theory and a finite theory, we present two frameworks for constitutive modeling, which can be read independently, since all concepts are introduced there right from the beginning.

In both cases, the full thermodynamic setting is exposed so that the restrictions by the dissipation inequality can be studied.

In linear elasticity of gradient materials, new stiffness tensors appear, the interpretation of which still needs more investigation. For the isotropic case, we added some results on hexadics.

Further, the concept of internal constraints is extended to include gradient effects, both in the mechanical and in the thermodynamical setting.

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List of Notations

Sets and spaces

$\mathscr{R}$	space of real numbers
$\mathscr{E}ucl$	three-dimensional EUCLIDean space
$\mathscr{B}$	body (manifold)
$\mathscr{B}_t, \mathscr{B}_0 \subset \mathscr{E}ucl$	domain of the body in the current and reference placement
$\partial\mathscr{B}_t, \partial\mathscr{B}_0 \subset \mathscr{E}ucl$	surface of the body in the current and reference placement
$\mathscr{V}^3$	three-dimensional space of vectors (EUCLIDean shifters)
$\mathscr{SV}$	space of all vector fields on $\mathscr{B}_t$ called *virtual velocities*
$\mathscr{D}yad$	space of linear mappings from $\mathscr{V}^3$ to $\mathscr{V}^3$ (*2*nd-order tensors or dyadics)
$\mathscr{I}nv$	set of invertible dyadics (general linear group)
$\mathscr{O}rth$	set of orthogonal dyadics (general orthogonal group)
$\mathscr{P}sym$	set of symmetric and positive-definite dyadics
$\mathscr{S}ym$	space of symmetric dyadics
$\mathscr{S}kw$	space of antisymmetric or skew dyadics
$\mathscr{U}nim$	set of *2*nd-order tensors with determinant ± 1 (general unimodular group)
$\mathscr{T}riad$	space of all triadics with right subsymmetry
$\mathscr{L}in\mathscr{C}omb$	$= \mathscr{D}yad \times \mathscr{T}riad$
$\mathscr{C}onf$	$= \mathscr{P}sym \times \mathscr{T}riad$
$\mathscr{I}nv\mathscr{C}omb$	$= \mathscr{I}nv \times \mathscr{T}riad$
$\mathscr{U}nim\mathscr{C}omb$	$= \mathscr{U}nim \times \mathscr{T}riad$

A superimposed $+$ at a dyadic set such as $\mathscr{I}nv^+$ means: with positive determinant.
$\mathscr{R}^+$ denotes the positive reals.

Variables and Fields

a	$\in \mathscr{V}^3$	acceleration
b	$\in \mathscr{V}^3$	spec. body force

$\mathbf{b}_{\text{gen}} \quad \in \mathscr{V}^3$	spec. generalized body force
$\mathbf{B} = \mathbf{F}\ \mathbf{F}^T \in \mathscr{P}_{sym}$	left CAUCHY-GREEN tensor
$c = \theta\, \partial_\theta \eta_e \in \mathscr{R}$	specific heat
$\mathbf{C} = \mathbf{F}^T\ \mathbf{F} \in \mathscr{P}_{sym}$	right CAUCHY-GREEN tensor
$\boldsymbol{C}$	linear elasticity operator
dm	mass element
dA , dA_0	surface element in the current and reference placement
dV , dV_0	volume element in the current and reference placement
$\mathbf{d}_O \quad \in \mathscr{V}^3$	angular momentum with respect to the point O
$\mathbf{D} \quad \in \mathscr{S}_{ym}$	rate of stretching tensor
$\mathbf{E} \quad \in \mathscr{S}_{ym}$	linear strain tensor
$\mathbf{E}^{\text{G}} = ½\ (\mathbf{C} - \mathbf{I}) \in \mathscr{S}_{ym}$	GREEN´s strain tensor
$\mathbf{f} \quad \in \mathscr{V}^3$	force
$\mathbf{F} \quad \in \mathscr{I}_{nv}^{+}$	deformation gradient
$\mathbf{g}$, $\mathbf{g}_0 \quad \in \mathscr{V}^3$	spatial and material temperature gradient
$\boldsymbol{G} = \mathbf{F} \circ J^{-1} \boldsymbol{H} \in \mathscr{T}_{riad}$	spatial hyperstress triadic
$\boldsymbol{H} = \mathbf{F}^{-1} \circ J\, \boldsymbol{G} \in \mathscr{T}_{riad}$	material hyperstress triadic
$\mathbf{I} \quad \in \mathscr{P}_{sym}$	second-order identity
$K \quad \in \mathscr{R}$	kinetic energy
$\boldsymbol{K} = \mathbf{F}^{-1} \cdot \mathit{Grad}\, \mathbf{F} \in \mathscr{T}_{riad}$	configuration tensor (triadic)
$\mathbf{L} \quad \in \mathscr{D}_{yad}$	velocity gradient
$\mathbf{m}_O \quad \in \mathscr{V}^3$	torque with respect to O
$\boldsymbol{M} = \mathit{grad}\, \mathbf{E} \in \mathscr{T}_{riad}$	
$\mathbf{p} \quad \in \mathscr{V}^3$	linear momentum
$\mathbf{q}$, $\mathbf{q}_0 \quad \in \mathscr{V}^3$	heat flux in the current placement and in the reference placement
$Q \quad \in \mathscr{R}$	heat supply
$\mathbf{n} \quad \in \mathscr{V}^3$	outer surface normal
$\mathbf{R} = -\rho\, \partial_{\mathbf{E}_e} \eta_e \in \mathscr{S}_{ym}$	2nd-order stress-temperature tensor
$\boldsymbol{R} = -\rho\, \partial_{\mathbf{M}_e} \eta_e \in \mathscr{T}_{riad}$	3rd-order stress-temperature tensor

$\mathbf{S} = \mathbf{F}^{-1} * J\,\mathbf{T}$	$\in \mathscr{Sym}$	PIOLA-KIRCHHOFF stress tensor
t	$\in \mathscr{R}$	time
$\mathbf{t}$	$\in \mathscr{V}^3$	traction
$\mathbf{T}$	$\in \mathscr{Sym}$	CAUCHY´s stress tensor
$\overset{(i)}{\boldsymbol{T}}$		ith hyperstress tensor
$\overset{(i)}{\boldsymbol{U}} := grad^i\,\mathbf{u}$		ith displacement gradient
$\mathbf{u}$	$\in \mathscr{V}^3$	displacement
$\mathbf{v}$	$\in \mathscr{V}^3$	velocity
w	$\in \mathscr{R}$	elastic energy
$\mathbf{W}$	$\in \mathscr{Skw}$	spin tensor
$\mathbf{x}$, $\mathbf{x}_0$	$\in \mathscr{V}^3$	position vector in current and reference placement
$\mathbf{Z}$		hardening variables

Greek letters

$\partial_{\mathbf{x}} f$		partial derivative of a function f with respect to some variable $\mathbf{x}$
δ		virtual
Δ		LAPLACE operator
ε	$\in \mathscr{R}$	internal energy
$\boldsymbol{\chi}$	$\in \mathscr{V}^3$	motion
φ		yield criterion
κ , κ_0		current and reference placement
λ	$\in \mathscr{R}$	plastic parameter
η	$\in \mathscr{R}$	specific entropy
Π_e , π_e	$\in \mathscr{R}$	external power (global and specific)
Π_i , π_i	$\in \mathscr{R}$	internal or stress power (global and specific)
ρ , ρ_0	$\in \mathscr{R}$	density in current and reference placement
$\psi = \varepsilon - \theta\,\eta$	$\in \mathscr{R}$	free HELMHOLTZ energy
$\boldsymbol{\omega}$	$\in \mathscr{V}^3$	angular momentum
θ , θ_0	$\in \mathscr{R}$	temperature, reference temperature

Symbols

∇, ∇_0	nabla in current and reference placement
$\otimes$	tensor product
$*$	RAYLEIGH product
$\circ$	pull-back or push-forward operation
$< , >$	inner product of hyper-vectors in Chapt. 3.

Introduction

Classical mechanics are based on EULER´s equations of motion, *i.e*., the balance of linear momentum and of angular momentum. These equations combine kinematical quantities like momenta with dynamic quantities like forces and torques. While the kinematical quantities are directly measurable within geometry and chronometry, the dynamic ones do not have this property. Forces are not visible, audible, tangible, etc., or how LAGRANGE (1788, p.1) expressed it:

> *On entend, en général, par force ou puissance la cause, quelle qu´elle soit, qui imprime ou tend à imprimer du mouvement au corps auquel on la suppose appliquée,*

and very similar also LAPLACE (1799, p. 4)

> *La nature de cette modification singulière, en vertu de laquelle un corps est transporté d'un lieu dans un autre, est et sera toujours inconnue; on l´a désignée sous le nom de* ***force****; on ne peut déterminer que ses effets et les loix de son action.*

This is surely the reason why it took so long in the history of mechanics to develop the concepts of forces (STEVIN 1586), gravitation (HOOKE and NEWTON et al. around 1680), distributed forces (EULER et al. 18th century), and stresses (CAUCHY 1823).

In principle, a precise introduction of the dynamic quantities is rather controversial. Is NEWTON´s law the definition of force as mass times acceleration? Then this law would be a triviality, which can neither be verified nor falsified. Or do we understand forces as primitive concepts, which would also make them "untouchable"?

For EULER and CAUCHY and many others it was natural to distribute forces into the categories of contact forces and of volumetric forces. After additional assumptions, CAUCHY could then introduce the stress tensor to determine the traction vector on the surface of the body.

The overwhelming success of this approach has at least two reasons. Firstly, it was the most simple approach to take contact actions on the surface into account. And secondly, by his stress concept already a great majority of effects can be described reasonably well.

However, there are certain effects in mechanics which cannot be described by a CAUCHY continuum. Whenever size effects appear, a theory which allows for internal length scales is needed. From the beginning of the 20th century, a variety of non-classical theories has been suggested to overcome the shortcomings of the CAUCHY continuum. The COSSERATs added 1909 micro-rotations and micro-torques to the continuum concepts and thus created the *polar media*. This was later broadened to not only introduce micro-rotations, but also micro-deformations, leading to *micromorphic* theories (ERINGEN 1999).

Another approach is that of considering higher gradients of the displacements. This line was initiated by KORTEWEG (1901) for fluids and by TOUPIN (1962), GREEN/ RIVLIN (1964a and b), and MINDLIN (1965) within elasticity, and later induced also for plasticity. The appealing feature of these theories is that no new kinematical concepts like COSSERATs´ micro spins had to be invented, since only the higher derivatives of the classical displacements are considered.

In the present work we will exclusively deal with this latter class of theories. We will see how the CAUCHY continuum is imbedded in gradient theories as one particular step in an infinite cascade of higher-order theories. The higher this order is, the more effects can we describe, at the cost of an enormous growth of variables and equations. It seems that nature does not tell us how far we have to go, but leaves us to choose a theory of some particular order which allows us to model the effects of our concern, and avoiding to complicate the theory where it is not necessary.

Tensor Notations

In general, $\overset{\langle k \rangle}{\boldsymbol{T}}$ denotes a tensor of k-th-order. As exceptions, we denote vectors (first-order tensors) eventually by bold small letters like $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, second-order tensors or dyadics by bold capital letters like $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and third-order tensors or triadics like $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$.

For every contraction between tensors we put one dot. More exactly, the ***P*-fold contraction** of a K-fold tensor product $\mathbf{v}_1 \otimes ... \otimes \mathbf{v}_K$ with an M-fold tensor product $\mathbf{x}_1 \otimes ... \otimes \mathbf{x}_M$ for $K \geq P \leq M$ is the $(K+M-2P)$-fold tensor product

$$(\mathbf{v}_1 \otimes ... \otimes \mathbf{v}_K) \cdot ... \cdot (\mathbf{x}_1 \otimes ... \otimes \mathbf{x}_M)$$

$$= \mathbf{v}_1 \otimes ... \otimes \mathbf{v}_{K-P} \otimes \mathbf{x}_{P+1} \otimes ... \otimes \mathbf{x}_M \, (\mathbf{v}_{K-P+1} \cdot \mathbf{x}_1) \, (\mathbf{v}_{K-P+2} \cdot \mathbf{x}_2) \, ... \, (\mathbf{v}_K \cdot \mathbf{x}_P) \, . \tag{0.1}$$

wherein " $\cdot ... \cdot$ " stands for P contraction dots. For better visuability, we will eventually arrange these contraction dots in groups with idential meaning. These notions can be immediately and uniquely extended from tensor products to higher-order tensors.

The invariants of a dyadic $\mathbf{T}$ are denoted by $I_{\mathbf{T}} = tr\, \mathbf{T}$, $II_{\mathbf{T}}$, and $III_{\mathbf{T}} = det\, \mathbf{T}$.

For a dyadic $\mathbf{T}$, the symmetric part is $sym(\mathbf{T})$, the skew part is $skw(\mathbf{T})$, and the axial vector of the skew part of $\mathbf{T}$ is noted as $axi(\mathbf{T})$. The latter is defined by its action on an arbitrary vector $\mathbf{v}$ according to

$$skw(\mathbf{T}) \cdot \mathbf{v} = axi(\mathbf{T}) \times \mathbf{v} \, . \tag{0.2}$$

While a second-order tensor $\mathbf{T}$ has a unique transpose $\mathbf{T}^T$, a third-order tensor or triadic has more than one. We will mainly need the right sub-transpose $\boldsymbol{A}^t$ which gives for the components with respect to an orthonormal vector basis $(A^t)_{ijk} = A_{ikj}$. If a triadic is symmetric with respect to this particular transposition, we call it **right subsymmetric**. The **left subsymmetry** is then $A_{ijk} = A_{jik}$. The dimension of the space of all tridadics is $3^3 = 27$. If one of the subsymmetries is assumed it is only 18.

For two triadics we obtain then

$$\boldsymbol{A}^t \cdots \boldsymbol{B} = \boldsymbol{A} \cdots \boldsymbol{B}^t . \tag{0.3}$$

Very helpful for higher-order tensors is the RAYLEIGH product. It maps all basis vectors of a tensor simultaneously without changing its components. To be more precise, let $\overset{\langle k \rangle}{\boldsymbol{C}}$ be a tensor of kth-order ($k \geq 1$) and $\mathbf{T}$ a dyadic. Then the RAYLEIGH product between them is defined as

$$\mathbf{T} * \overset{\langle k \rangle}{\boldsymbol{C}} = \mathbf{T} * (C^{ik \dots l} \, \mathbf{r}_i \otimes \mathbf{r}_k \otimes ... \otimes \mathbf{r}_l)$$

$$:= C^{ik \dots l} \, (\mathbf{T} \cdot \mathbf{r}_i) \otimes (\mathbf{T} \cdot \mathbf{r}_k) \otimes ... \otimes (\mathbf{T} \cdot \mathbf{r}_l) \, . \tag{0.4}$$

Of course, the result does not depend on the choice of the basis. If $\mathbf{T}$ is proper orthogonal, then the product is a rotation of $\overset{\langle k \rangle}{\boldsymbol{C}}$.

For $k \equiv 1$ the RAYLEIGH product coincides with a linear mapping

(0.5) $$\mathbf{T} * \mathbf{c} = \mathbf{T} \cdot \mathbf{c},$$

and for $k \equiv 2$ we obtain

(0.6) $$\mathbf{T} * \mathbf{C} = \mathbf{T} \cdot \mathbf{C} \cdot \mathbf{T}^T.$$

The RAYLEIGH product is associative in the left factor

(0.7) $$\mathbf{S} * (\mathbf{T} * \overset{\langle k \rangle}{\boldsymbol{C}}) = (\mathbf{S} \cdot \mathbf{T}) * \overset{\langle k \rangle}{\boldsymbol{C}}$$

and distributive in the right one. In fact, if $\overset{\langle k \rangle}{\boldsymbol{C}}$ and $\overset{\langle n \rangle}{\boldsymbol{D}}$ are tensors of arbitrary order, then we have

(0.8) $$\mathbf{T} * (\overset{\langle k \rangle}{\boldsymbol{C}} \otimes \overset{\langle n \rangle}{\boldsymbol{D}}) = (\mathbf{T} * \overset{\langle k \rangle}{\boldsymbol{C}}) \otimes (\mathbf{T} * \overset{\langle n \rangle}{\boldsymbol{D}})$$

for all dyadics $\mathbf{T}$. This does not hold, if we would replace the tensor product by an arbitrary contraction, unless $\mathbf{T}$ is orthogonal.

In this product, the second-order identity tensor also gives the identity mapping

(0.9) $$\mathbf{I} * \overset{\langle k \rangle}{\boldsymbol{C}} = \overset{\langle k \rangle}{\boldsymbol{C}}.$$

The inversion for an invertible dyadic $\mathbf{T}$ is done by

(0.10) $$\mathbf{T} * (\mathbf{T}^{-1} * \overset{\langle k \rangle}{\boldsymbol{C}}) = \overset{\langle k \rangle}{\boldsymbol{C}} = \mathbf{T}^{-1} * (\mathbf{T} * \overset{\langle k \rangle}{\boldsymbol{C}}).$$

The RAYLEIGH product commutes with the contraction with the inverse in the following sense

(0.11) $$\mathbf{T}^{-1} \cdot (\mathbf{T} * \overset{\langle k \rangle}{\boldsymbol{C}}) = \mathbf{T} * (\mathbf{T}^{-1} \cdot \overset{\langle k \rangle}{\boldsymbol{C}}).$$

For two second-order tensors $\mathbf{A}$ (invertible) and $\mathbf{B}$ and a higher-order tensor $\overset{\langle k \rangle}{\boldsymbol{C}}$ we obtain the rule

(0.12) $$\mathbf{B} \cdot \mathbf{A}^{-1} \cdot (\mathbf{A} * \overset{\langle k \rangle}{\boldsymbol{C}}) = \mathbf{A} * (\mathbf{A}^{-1} \cdot \mathbf{B} \cdot \overset{\langle k \rangle}{\boldsymbol{C}}).$$

For the *k*-fold scalar product of two arbitrary *k*th-order tensors we get

(0.13) $$(\mathbf{T} * \overset{\langle k \rangle}{\boldsymbol{C}}) \cdot \ldots \cdot \overset{\langle k \rangle}{\boldsymbol{D}} = \overset{\langle k \rangle}{\boldsymbol{C}} \cdot \ldots \cdot (\mathbf{T}^T * \overset{\langle k \rangle}{\boldsymbol{D}}).$$

The RAYLEIGH product acts on a simple triadic like

(0.14) $$\begin{aligned} \mathbf{T} * (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) &= (\mathbf{T} \cdot \mathbf{a}) \otimes (\mathbf{T} \cdot \mathbf{b}) \otimes (\mathbf{T} \cdot \mathbf{c}) \\ &= (\mathbf{T} \cdot \mathbf{a}) \otimes (\mathbf{T} \cdot \mathbf{b}) \otimes \mathbf{c} \cdot \mathbf{T}^T \\ &= \mathbf{T} \cdot (\mathbf{a} \otimes \mathbf{c} \otimes \mathbf{b} \cdot \mathbf{T}^T)^t \cdot \mathbf{T}^T \end{aligned}$$

and analogously on a triadic $\boldsymbol{A}$

(0.15) $$\mathbf{T} * \boldsymbol{A} = \mathbf{T} \cdot (\boldsymbol{A}^t \cdot \mathbf{T}^T)^t \cdot \mathbf{T}^T.$$

Besides the RAYLEIGH product, we will need another product between an invertible dyadic $\mathbf{T}$ and a triadic $\boldsymbol{A}$ denoted by

(0.16) $$\mathbf{T} \circ \boldsymbol{A} := A_{ijk} (\mathbf{T}^{-T} \cdot \mathbf{e}_i) \otimes (\mathbf{T} \cdot \mathbf{e}_j) \otimes (\mathbf{T} \cdot \mathbf{e}_k)$$

$$= \mathbf{T}^{-T} \cdot [\boldsymbol{A}^t \cdot \mathbf{T}^T]^t \cdot \mathbf{T}^T$$

with (0.15) $$= \mathbf{T} * (\mathbf{T}^{-1} \cdot \mathbf{T}^{-T} \cdot \boldsymbol{A})$$

with (0.12) $$= \mathbf{T}^{-T} \cdot \mathbf{T}^{-1} \cdot (\mathbf{T} * \boldsymbol{A}) .$$

The following rules hold for this product.

(0.17) $$(\mathbf{T} \circ \boldsymbol{A}) \cdots \boldsymbol{B} = \boldsymbol{A} \cdots (\mathbf{T}^T \circ \boldsymbol{B})$$

for all dyadics $\mathbf{T}$ and all triadics $\boldsymbol{A}$ and $\boldsymbol{B}$. The second-order identity tensor also gives the identity mapping

(0.18) $$\mathbf{I} \circ \boldsymbol{A} = \boldsymbol{A}$$

and the inversion is done by

(0.19) $$\mathbf{T} \circ (\mathbf{T}^{-1} \circ \boldsymbol{A}) = \boldsymbol{A} .$$

Furthermore, the product is associative

(0.20) $$\mathbf{S} \circ (\mathbf{T} \circ \boldsymbol{A}) = (\mathbf{S} \cdot \mathbf{T}) \circ \boldsymbol{A}$$

for all dyadics $\mathbf{S}$ and $\mathbf{T}$ and triadics $\boldsymbol{A}$.

For the case of $\mathbf{T}$ being orthogonal, this transformation coincides with the RAYLEIGH product.

We denote an arbitrary basis by $\{\mathbf{r}_i\}$ and its dual by $\{\mathbf{r}^i\}$. In particular, such bases occur as the natural bases induced by a coordinate system $\{\varphi^i\}$ and then written as $\{\mathbf{r}_{\varphi i}\}$ and $\{\mathbf{r}_\varphi{}^i\}$. An orthonormal vector basis is written as $\{\mathbf{e}_i\}$.

Kinematics

We denote by

$\mathscr{E}ucl$	the three-dimensional EUCLIDean space
$\mathscr{V}^3$	the three-dimensional vector space of EUCLIDean shifters
$\mathscr{B}$	the body (manifold)
$\mathscr{B}_0 \subset \mathscr{E}ucl$	the domain of the body in the reference placement with surface $\partial\mathscr{B}_0$
$\mathscr{B}_t \subset \mathscr{E}ucl$	the domain of the body in the current placement with surface $\partial\mathscr{B}_t$

A **placement** of the body $\mathscr{B}$ at a time t is an embedding (bijective)

$$\kappa(\bullet, t) : \mathscr{B} \rightarrow \mathscr{B}_t \subset \mathscr{E}ucl .$$

While the time t runs through a finite time interval $\mathscr{I} := [0, t_e]$, the parameterized sequence of placements defines a **motion** of the body.

It is rather customary (although not necessary) to introduce a (time- and observer-independent) **reference placement**

$$\kappa_0 : \mathscr{B} \rightarrow \mathscr{B}_0 \subset \mathscr{E}ucl \tag{0.21}$$

which is also assumed to be bijective. The composition of its inverse with a motion

$$\chi(\bullet, t) := \kappa(\kappa_0^{-1}(\bullet), t) : \mathscr{B}_0 \rightarrow \mathscr{B}_t \subset \mathscr{E}ucl \tag{0.22}$$

is the EUCLIDean description of a motion of the body, being for all times t a bijection between the regions $\mathscr{B}_0$ and $\mathscr{B}_t$ of the EUCLIDean space. Practically, one would describe this function either by coordinates or by position vectors. The latter leads to a mapping

$$\boldsymbol{\chi} : \mathscr{V}^3 \times \mathscr{I} \rightarrow \mathscr{V}^3 \tag{0.23}$$

or

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{x}_0, t) \tag{0.24}$$

with the position vector of a material point $\mathbf{x}_0$ in the reference placement and $\mathbf{x}$ in the current placement.

The volume and surface elements in the current placement are dV and dA, and in the reference placement dV_0 and dA_0, respectively. The mass element is dm for which a distinction of the two placements is not necessary because of mass conservation. The mass densities in the two placements are denoted by ρ and ρ_0, respectively. The nabla operator in the reference placement is denoted as ∇_0, and simply ∇ in the current placement. If we consider tensor fields of any order, we have at least three choices of representations.

1.) We can introduce the field as a mapping from the body manifold

(0.25) $\phi_I : \mathscr{B} \rightarrow \underline{\mathscr{Lin}}$

where $\underline{\mathscr{Lin}}$ stands for the particular tensor space. This can be done by convected coordinates and is called **intrinsic description**.

2.) We can introduce the field as a mapping from the region $\mathscr{B}_0 \subset \mathscr{Eucl}$ which the body occupies in the reference placement

(0.26) $\phi_L : \mathscr{B}_0 \rightarrow \underline{\mathscr{Lin}}$.

This can be done by material coordinates and is called material or **LAGRANGEan description**.

3.) We can introduce the field as a mapping from the region $\mathscr{B}_t$ which the body occupies in the reference placement

(0.27) $\phi_E : \mathscr{B}_t \rightarrow \underline{\mathscr{Lin}}$.

This can be done by spatial coordinates and is called spatial or **EULERean description**.

By the motion (0.21) - (0.22) we are able to uniquely transform any of these descriptions into any other.

When using gradients or differentials of such fields, we have to take care of these different choices. So *grad*, *div* and *curl* are related to the gradient, divergence, and curl operation respectively in the current placement, while *Grad*, *Div*, and *Curl* are related to the same operations in the reference placement.

The (material) gradient of the motion (0.24) with respect to the first argument is the deformation gradient

(0.28) $\mathbf{F}(\mathbf{x}_0 , t) = Grad\, \boldsymbol{\chi}(\mathbf{x}_0 , t) \in \mathscr{Inv}^+$.

The transformations of the nablas in the reference and the current placement is after the chain rule

(0.29) $\nabla_0 = \nabla \cdot \mathbf{F} = \mathbf{F}^T \cdot \nabla$ and $\nabla = \nabla_0 \cdot \mathbf{F}^{-1} = \mathbf{F}^{-T} \cdot \nabla_0$.

The linear strain tensor is

(0.30) $\mathbf{E} := \frac{1}{2} (\mathbf{F} + \mathbf{F}^T - 2\, \mathbf{I}) \in \mathscr{Sym}$.

The velocity field is the first partial time derivative

(0.31) $\mathbf{v}(\mathbf{x}_0 , t) = \boldsymbol{\chi}(\mathbf{x}_0 , t)^{\bullet}$

and the acceleration field the second time derivative

(0.32) $\mathbf{a}(\mathbf{x}_0 , t) = \boldsymbol{\chi}(\mathbf{x}_0 , t)^{\bullet\bullet}$

of the motion (0.24). The (spatial) velocity gradient is

(0.33) $\mathbf{L} = grad\, \mathbf{v} = grad\, \boldsymbol{\chi}^{\bullet} \in \mathscr{Dyad}$

which can be decomposed into its symmetric part $\mathbf{D}$ and its skew part $\mathbf{W}$ as

(0.34) $\mathbf{L} = \mathbf{D} + \mathbf{W}$.

The time derivative of the deformation gradient is related to the velocity gradient by

(0.35) $\mathbf{L} = \mathbf{F}^{\bullet} \cdot \mathbf{F}^{-1}$.

The right CAUCHY-GREEN tensor is defined as

(0.36) $\mathbf{C} := \mathbf{F}^{T} \cdot \mathbf{F} \in \mathscr{P}_{sym}$

and the left CAUCHY-GREEN tensor

(0.37) $\mathbf{B} := \mathbf{F} \cdot \mathbf{F}^{T} \in \mathscr{P}_{sym}$

and GREEN´s strain tensor

(0.38) $\mathbf{E}^{\mathrm{G}} := ½ (\mathbf{C} - \mathbf{I}) \in \mathscr{S}_{ym}$

such that

(0.39) $\mathbf{E}^{\mathrm{G}\bullet} = ½ \mathbf{C}^{\bullet} = \mathbf{F}^{T} * \mathbf{D} \in \mathscr{S}_{ym}$.

1. Balance Laws

The prominent fundamental aim of mechanics is to decide whether a given motion of a body is (dynamically) admissible or not. Such a decision is based on certain axioms, which can later be reformulated by equivalent statements. For the selection of such axioms, different choices exist. The classical way to do this is based on what we call today EULER's equations of motion.

It is the intention of this chapter to introduce a series of (equivalent) sets of balance equations or equivalent formulations, which altogether assure the dynamic admissibility of a motion. Constitutive equations are beyond the scope of this chapter.

1.1 Method of EULER and CAUCHY

In the following we repeat the essentials of the procedure used by EULER (for fluids) and by CAUCHY (1823, 1828) (for the so-called *CAUCHY continuum*). It is based on a series of axioms or assumptions, which we will number as A1, A2 etc.

(A1) The starting point is the *Principle of Cuts* after which we can cut every body out of the universe, and substitute the mechanical influence of the outer world by (solely) applying forces or enforced displacements.

(A2) Such forces induce torques in the usual way, which are linked to a turning point. If we change this turning point, we have to add moments after VARIGNON´s principle.

(A3) In continuum mechanics, such forces are understood as resulting from distributed forces acting either on the interior of the body as *body forces* like, *e.g.*, gravity, or on the surface as *contact forces* such as the air pressure. Line or point forces have not been included by CAUCHY.

(A4) Both types of forces are understood as continuously distributed, so that force densities exist. The density of the body force per unit mass is the *specific body force field* $\mathbf{b}$, that of the contact force is the *traction field* $\mathbf{t}$, so that the resulting force acting on the body can be determined by

$$\mathbf{f} = \int_{\mathscr{B}_t} \mathbf{b}\, dm + \int_{\partial\mathscr{B}_t} \mathbf{t}\, dA \tag{1.1}$$

and the resulting torque

$$\mathbf{m}_O = \int_{\mathscr{B}_t} \mathbf{x}_O \times \mathbf{b}\, dm + \int_{\partial\mathscr{B}_t} \mathbf{x}_O \times \mathbf{t}\, dA \tag{1.2}$$

with $\mathbf{x}_O$ being the position vector with respect to some turning point O in space. While the specific body force $\mathbf{b}$ is assumed to be given by a gravitational law or the like, the surface tractions are induced by dynamic (or NEUMANN) boundary conditions, or as reactions to DIRICHLET boundary conditions.

Further, it is assumed that two balance laws hold, which we call **EULER´s laws of motion**, namely

- **(A5)** the **balance of linear momentum**

(1.3) $$\mathbf{p}^{\bullet} = \mathbf{f}$$

with the linear momentum

(1.4) $$\mathbf{p} := \int_{\mathscr{B}_t} \mathbf{v}\, dm$$

- **(A6)** and the **balance of angular momentum**

(1.5) $$\mathbf{d}_O{}^{\bullet} = \mathbf{m}_O$$

with the angular momentum

(1.6) $$\mathbf{d}_O = \int_{\mathscr{B}_t} \mathbf{x}_O \times \mathbf{v}\, dm$$

with respect to some fixed point O.

The two balance laws can be brought into the form using (1.1) and (1.2)

(1.7) $$\int_{\mathscr{B}_t} \mathbf{a}\, dm = \int_{\mathscr{B}_t} \mathbf{b}\, dm + \int_{\partial\mathscr{B}_t} \mathbf{t}\, dA$$

(1.8) $$\int_{\mathscr{B}_t} \mathbf{x}_O \times \mathbf{a}\, dm = \int_{\mathscr{B}_t} \mathbf{x}_O \times \mathbf{b}\, dm + \int_{\partial\mathscr{B}_t} \mathbf{x}_O \times \mathbf{t}\, dA\,.$$

The (dynamical) admissibility of a motion of a body is then understood as the validity of these two laws.

Axiom 1.1 (EULER´s laws of motion)
A motion of a body is admissible if and only if the balance of linear momentum (1.7) *and the balance of angular momentum* (1.8) *hold during the motion.*

We will need one further assumption which seems to be rather natural, but is by no means trivial.

Axiom 1.2 (Compatibility Assumption)
Let χ be an admissible motion of a body $\mathscr{B}_0$, and $\mathscr{B}_1 \subset \mathscr{B}_0$ a subbody of $\mathscr{B}_0$. Then the restriction of χ to $\mathscr{B}_1$ is an admissible motion of $\mathscr{B}_1$.

As a consequence, the inverse statement is then also true.

Let χ be a motion of a body $\mathscr{B}_0$, and $\mathscr{B}_1 \subset \mathscr{B}_0$ a subbody of $\mathscr{B}_0$. If the restriction of χ to $\mathscr{B}_1$ is not an admissible motion of $\mathscr{B}_1$, then χ cannot be admissible for $\mathscr{B}_0$.

The next step of CAUCHY (1823, 1828) for the introduction of stresses is the famous tetrahedron argument, which can be found in every book on continuum mechanics. It uses the balance of linear momentum (1.7) and continuity under some limit processes and the fundamental assumption **(A7)** that the tractions depend on the particular surface for different cuts only through their orientations. Then CAUCHY could demonstrate that the traction vector $\mathbf{t}$ is a *linear* function of the normal $\mathbf{n}$ to the tangent plane in that particular point (**Theorem of CAUCHY**), which gives rise to the introduction of CAUCHY´s stress tensor $\mathbf{T}$, thus

$$\mathbf{t}(\mathbf{x}\,,\,t\,,\,\mathbf{n}) \;=\; \mathbf{T}(\mathbf{x}\,,\,t)\cdot\mathbf{n}(\mathbf{x}\,,\,t)\,. \tag{1.9}$$

By this equation, the tractions on a surface point are related to the stress tensor. In the case of dynamic or NEUMANN boundary conditions, the tractions are prescribed on the boundary

$$\mathbf{t}_{pres}(\mathbf{x}\,,\,t) \;=\; \mathbf{T}(\mathbf{x}\,,\,t)\cdot\mathbf{n}\,. \tag{1.10}$$

If we determine by (1.9) the resulting contact force, we can apply the divergence theorem to transform the surface integral into a volume integral

$$\int\limits_{\partial\mathscr{B}_t} \mathbf{t}\,dA \;=\; \int\limits_{\partial\mathscr{B}_t} \mathbf{T}\cdot\mathbf{n}\,dA \;=\; \int\limits_{\mathscr{B}_t} div\,\mathbf{T}\,dV. \tag{1.11}$$

Inserting this into the balance of linear momentum (1.7) gives

$$\int\limits_{\mathscr{B}_t} div\,\mathbf{T}\,dV \;+\; \int\limits_{\mathscr{B}_t} \mathbf{b}\,dm \;=\; \int\limits_{\mathscr{B}_t} \mathbf{a}\,dm\,. \tag{1.12}$$

and into the balance of angular momentum (1.8)

$$\int\limits_{\mathscr{B}_t} (\mathbf{x}_O \times div\,\mathbf{T}\,dV \;+ 2\,axi\,\mathbf{T})\,dV \;+\; \int\limits_{\mathscr{B}_t} \mathbf{x}_O\times\mathbf{b}\,dm \;=\; \int\limits_{\mathscr{B}_t} \mathbf{x}_O\times\mathbf{a}\,dm \tag{1.13}$$

being valid for the body if the motion is admissible. Here we used the integral transformation

$$\int\limits_{\partial\mathscr{B}_t} \mathbf{x}_O\times\mathbf{T}\cdot\mathbf{n}\,dA \;=\; \int\limits_{\mathscr{B}_t} (\mathbf{x}_O\times\mathbf{T})\cdot\nabla\,dV \;=\; \int\limits_{\mathscr{B}_t} [\mathbf{x}_O\times(\mathbf{T}\cdot\nabla) + 2\,axi\,\mathbf{T}]\,dV.$$

After the above Compatibility Axiom 1.2, the same motion shall also be admissible for each subbody. This would be fulfilled if we take the restrictions of the fields $\mathbf{T}$ and $\mathbf{b}$ to the subbody (for $\mathbf{a}$ it follows already from the definition (0.31)). So in what follows we will automatically use these restrictions without further mentioning.

Then for smooth arguments we obtain the **local balance of linear momentum** or **1st law of CAUCHY** (1823)

$$div\,\mathbf{T} + \rho\,\mathbf{b} \;=\; \rho\,\mathbf{a}\,. \tag{1.14}$$

By use of the two (*sic*!) equations of motion, one can show the symmetry of the stress tensor, sometimes labeled **BOLTZMANN´s axiom** or **2nd law of CAUCHY**

(1.15) $$\mathbf{T} = \mathbf{T}^T.$$

So we can state the following

Theorem 1.1 (CAUCHY´s laws)
A motion of a body is admissible if and only if CAUCHY´s laws (1.14) *and* (1.15) *hold everywhere in the body during the motion.*

Later in the history of mechanics, the assumption A1 has been extended by also introducing distributed torques which do not result from forces (*polar media*). In such a case, the stress tensor is not symmetric and the *balance of angular momentum* takes the form of a typical balance equation. Such media are called *constrained COSSERAT media* or *KOITER* (1964) *media*. If one, moreover, considers rotations as additional kinematical degrees of freedom other than *curl* $\mathbf{v}$, one obtains a *COSSERAT* (1909) *medium*. In what follows, however, we will restrict our concern to non-polar media.

Let K be an integer, the value of which will be specified later. We will call any K-times differentiable vector field (test function) on the body in the current placement $\mathscr{B}_t$ a **virtual velocity field**. All such fields form the **space of virtual velocities** denoted by $\delta\mathscr{V}$. It always contains the current (real) velocity field $\mathbf{v}$ as a distinguished member. It also contains the set of constant vector fields, which can be identified with the three-dimensional vector space of the EUCLIDean shifters $\mathscr{V}^3$ in a natural way.

In analogy to the velocity field, we will use the following notations for the virtual fields:

(1.16)
$$\begin{aligned} \delta\mathbf{L} &:= \mathit{grad}\,\delta\mathbf{v} \\ \delta\mathbf{W} &:= \tfrac{1}{2}(\mathit{grad}\,\delta\mathbf{v} - \mathit{grad}^T\delta\mathbf{v}) = \mathit{skw}\,\delta\mathbf{L} \\ \delta\mathbf{D} &:= \tfrac{1}{2}(\mathit{grad}\,\delta\mathbf{v} + \mathit{grad}^T\delta\mathbf{v}) = \mathit{sym}\,\delta\mathbf{L} \\ \delta\boldsymbol{\omega} &:= \mathit{axi}(\delta\mathbf{W}) = \mathit{axi}(\delta\mathbf{L}) . \end{aligned}$$

If we multiply (1.14) by an arbitrary virtual velocity field $\delta\mathbf{v} \in \delta\mathscr{V}$

$$\mathit{div}\,\mathbf{T}\cdot\delta\mathbf{v} + \rho(\mathbf{b}-\mathbf{a})\cdot\delta\mathbf{v} = 0$$

and add (or subtract) the term

$$\mathbf{T}\cdot\cdot\,\delta\mathbf{W} = 2\,\mathit{axi}\,\mathbf{T}\cdot\delta\boldsymbol{\omega}$$

then the result is obviously zero if (1.14) and (1.15) hold. We now integrate over the volume and obtain the following version of the PVP.

Theorem 1.2 (volumetric form of the principle of virtual power)
A motion of a body is admissible if and only if

(1.17) $$\int_{\mathscr{B}_t} [\mathit{div}\,\mathbf{T}\cdot\delta\mathbf{v} + \rho(\mathbf{b}-\mathbf{a})\cdot\delta\mathbf{v} + \mathbf{T}\cdot\cdot\,\delta\mathbf{W}]\,dV = 0$$

holds during the motion for arbitrary fields $\delta\mathbf{v} \in \delta\mathscr{V}$.

Proof. We choose for the virtual velocity field an arbitrary constant field $\delta\mathbf{v}_0 \in \delta\mathscr{V}$, so that the gradient of it is zero everywhere. Then the above equations gives

$$\{ \int_{\mathscr{B}_t} [div(\mathbf{T}) + \rho\,(\mathbf{b} - \mathbf{a})]\, dV \} \cdot \delta\mathbf{v}_0 = 0 \tag{1.18}$$

and by the arbitrariness of $\delta\mathbf{v}_0$

$$\int_{\mathscr{B}_t} [div(\mathbf{T}) + \rho\,(\mathbf{b} - \mathbf{a})]\, dV = \mathbf{o}\,. \tag{1.19}$$

This must hold also for every subbody after the Compatibility Axiom 1.2, so that the integrand must be zero everywhere in the volume, which gives (1.14).

As a second choice for $\delta\mathbf{v} \in \delta\mathscr{V}$ we choose a rotational field $\delta\boldsymbol{\omega} \times \mathbf{x} \in \delta\mathscr{V}$ with an arbitrary constant vector $\delta\boldsymbol{\omega}$ and the position vector $\mathbf{x}$ with respect to some arbitrary point of reference. Then the skew tensor field $\delta\mathbf{W}$ is arbitrary but constant

$$\int_{\mathscr{B}_t} \mathbf{T} \cdot\cdot\, \delta\mathbf{W}\, dV = 0 = \int_{\mathscr{B}_t} 2\, axi\, \mathbf{T} \cdot \delta\boldsymbol{\omega}\, dV \qquad \text{with } \delta\boldsymbol{\omega} := axi(\delta\mathbf{W})\,. \tag{1.20}$$

By a similar reasoning as before we can conclude that the integrand must be zero for any skew $\delta\mathbf{W}$ so that the stress tensor must be symmetric everywhere in the body, which gives (1.15). So we have that CAUCHY´s laws hold everywhere in the body. On the other hand, the validity of these laws gives immediately the equation in the theorem, *q.e.d.*

We apply the divergence theorem and the product rule to the equation (1.17) from the above theorem

$$0 = \int_{\mathscr{B}_t} [div\, \mathbf{T} \cdot \delta\mathbf{v} + \rho\,(\mathbf{b} - \mathbf{a}) \cdot \delta\mathbf{v} + \mathbf{T} \cdot\cdot\, \delta\mathbf{W}]\, dV$$

$$= \int_{\mathscr{B}_t} [div\, \mathbf{T} \cdot \delta\mathbf{v} + \rho\,(\mathbf{b} - \mathbf{a}) \cdot \delta\mathbf{v} + \mathbf{T} \cdot\cdot\, \delta\mathbf{L} - \mathbf{T} \cdot\cdot\, \delta\mathbf{D}]\, dV$$

$$= \int_{\mathscr{B}_t} [\rho\,(\mathbf{b} - \mathbf{a}) \cdot \delta\mathbf{v} - \mathbf{T} \cdot\cdot\, \delta\mathbf{D}]\, dV + \int_{\partial\mathscr{B}_t} \mathbf{t} \cdot \delta\mathbf{v}\, dA \tag{1.21}$$

since the divergence theorem applies as

$$\int_{\mathscr{B}_t} \mathbf{T} \cdot\cdot\, grad\, \delta\mathbf{v}\, dV = \int_{\partial\mathscr{B}_t} (\mathbf{T} \cdot \mathbf{n}) \cdot \delta\mathbf{v}\, dA - \int_{\mathscr{B}_t} div\, \mathbf{T} \cdot \delta\mathbf{v}\, dm\,. \tag{1.22}$$

This gives the

Theorem 1.3 (global PVP)
A motion of a body is admissible if and only if

(1.23) $$\int_{\mathscr{B}_t} \mathbf{b} \cdot \delta\mathbf{v}\, dm + \int_{\partial\mathscr{B}_t} \mathbf{t} \cdot \delta\mathbf{v}\, dA = \int_{\mathscr{B}_t} \mathbf{a} \cdot \delta\mathbf{v}\, dm + \int_{\mathscr{B}_t} \mathbf{T} \cdot\cdot\, \delta\mathbf{D}\, dV$$

holds during the motion for all virtual velocity fields $\delta\mathbf{v} \in \delta\mathscr{V}$.

If we insert in (1.23) the (real) velocity field $\mathbf{v}$ for $\delta\mathbf{v}$, then we obtain the mechanical work balance, which is only a necessary condition for the motion to be admissible.

Theorem 1.4 (global work balance)
For every admissible motion of a body, the work balance

(1.24) $$\Pi_e = \Pi_i + K^{\bullet}$$

holds during the motion with the **power of the forces**

(1.25) $$\Pi_e := \int_{\mathscr{B}_t} \mathbf{b} \cdot \mathbf{v}\, dm + \int_{\partial\mathscr{B}_t} \mathbf{t} \cdot \mathbf{v}\, dA\,,$$

the **stress power**

(1.26) $$\Pi_i := \int_{\mathscr{B}_t} \mathbf{T} \cdot\cdot\, \mathbf{D}\, dV$$

and the **kinetic energy**

(1.27) $$K := \tfrac{1}{2} \int_{\mathscr{B}_t} \mathbf{v} \cdot \mathbf{v}\, dm\,.$$

Changes of Observers

Since all of our variables, both kinematical and dynamic ones, depend on observers, we have to consider changes of observers. Such changes contain a temporal and a spatial transformation. The temporal transformation is just a time shift of the reference point of time. Since we mainly deal here with differences of time, this transformation does not matter and will not be further mentioned.

The spatial transformation induced by a change of observer, however, does matter. It is given by the **EUCLIDean transformation** which transforms the position vector $\mathbf{x}(P, t)$ of a particle or a material point P at a time t for one observer into that of some other observer indicated by an upper asterisk, (not to be confused with the RAYLEIGH product (0.4))

$$\mathbf{x}^*(P, t) = \mathbf{Q}(t) \cdot \mathbf{x}(P, t) + \mathbf{c}(t) \tag{1.28}$$

by a time-dependent vector $\mathbf{c}(t) \in \mathscr{V}^3$ and a time-dependent proper orthogonal tensor $\mathbf{Q}(t) \in \mathscr{O}rth^+$, both of which are determined solely by the two observers, but are independent of the motion.

Sometimes the question is raised wether $\mathbf{Q}(t)$ could also be improper, *i.e.*, with negative determinant. In this case, the EUCLIDean transformation would also change the orientation. Since the orientation of the EUCLIDean space is not an intrinsic property but rather a convention or a definition, there would be no loss of generality if all observers would agree to one orientation, so that $\mathbf{Q}(t)$ was proper orthogonal. This will be further on assumed.

The EUCLIDean transformation forms a group under composition, and determines the transformations of all kinematical variables as group actions.

If we transform fields, we have to take into account that the dependence on the locus expressed by $\mathbf{x}_0$ in the LAGRANGEan representation is invariant under observer changes, while the position vector $\mathbf{x}$ in the EULERean description has to be transformed according to (1.28). When we will further on compare field variables in the EULERean description, it is without further mentioning understood that such a transformation of the spatial variables for the different observer is made.

We will call a tensorial variable $\mathbf{T}$ of any order **objective** if it transforms like

$$\mathbf{T}^* = \mathbf{Q} * \mathbf{T} \tag{1.29}$$

and **invariant** if

$$\mathbf{T}^* = \mathbf{T} . \tag{1.30}$$

For scalars (as zeroth-order tensors) both definitions coincide.

Nabla transforms after the chain rule as

$$\nabla = \nabla^* \cdot \mathbf{Q} \quad \Leftrightarrow \quad \nabla^* = \nabla \cdot \mathbf{Q}^T \tag{1.31}$$

and the gradient of some field as

$$grad\,(\bullet) = grad^*(\bullet) \cdot \mathbf{Q} \quad \Leftrightarrow \quad grad^*(\bullet) = grad\,(\bullet) \cdot \mathbf{Q}^T. \tag{1.32}$$

The action of the EUCLIDean transformation on the motion χ of the body during a closed time interval is

$$\chi^*(\mathbf{x}_0, t) = \mathbf{Q}(t) \cdot \chi(\mathbf{x}_0, t) + \mathbf{c}(t) . \tag{1.33}$$

The transformation of the velocity field results as

$$\begin{aligned} \mathbf{v}^* &= \mathbf{Q} \cdot \chi(\mathbf{x}_0, t)^\bullet + \mathbf{Q}^\bullet \cdot \chi(\mathbf{x}_0, t) + \mathbf{c}^\bullet \\ &= \mathbf{Q} \cdot \mathbf{v} + \mathbf{Q}^\bullet \cdot \mathbf{Q}^T \cdot (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet \\ &= \mathbf{Q} \cdot \mathbf{v} + \boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet \end{aligned} \tag{1.34}$$

where the angular velocity $\boldsymbol{\omega}$ is the axial vector of the skew tensor $\mathbf{Q}^\bullet \cdot \mathbf{Q}^T$

$$\boldsymbol{\omega} := axi(\mathbf{Q}^\bullet \cdot \mathbf{Q}^T) , \tag{1.35}$$

both of which depend on the time, but not on the locus. The terms $\boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet$ are sometimes called **distributors**. They also result in an identical form for the velocity field of a rigid body motion.

The acceleration transforms as

$$\mathbf{a}^* = \mathbf{Q} \cdot \mathbf{a} + \mathbf{c}^{\bullet\bullet} + \boldsymbol{\omega}^\bullet \times (\mathbf{x}^* - \mathbf{c}) + \boldsymbol{\omega} \times [(\mathbf{x}^* - \mathbf{c}) \times \boldsymbol{\omega}] + 2\, \boldsymbol{\omega} \times (\mathbf{v}^* - \mathbf{c}^\bullet) \tag{1.36}$$

as a sum of the relative acceleration, the translational acceleration, the angular acceleration, the centripetal acceleration, and the CORIOLIS acceleration, respectively.

We also find that neither $\mathbf{L}$ nor $\mathbf{W}$ are objective

$$\mathbf{L}^* = \mathbf{Q} * \mathbf{L} + \mathbf{Q}^\bullet \cdot \mathbf{Q}^T \tag{1.37}$$

$$\mathbf{W}^* = \mathbf{Q} * \mathbf{W} + \mathbf{Q}^\bullet \cdot \mathbf{Q}^T \tag{1.38}$$

because of the additional skew parts $\mathbf{Q}^\bullet \cdot \mathbf{Q}^T$, while the symmetric part $\mathbf{D}$ is objective

$$\mathbf{D}^* = \mathbf{Q} * \mathbf{D} , \tag{1.39}$$

and so are all higher velocity gradients from the second one onwards.

The question arises about the transformations of the dynamic quantities. Usually, one assumes **(A8)** objectivity of not only the resulting force but also for each part of it:

$$\begin{aligned} \mathbf{f}^* &= \mathbf{Q} \cdot \mathbf{f} \\ \mathbf{b}^* &= \mathbf{Q} \cdot \mathbf{b} \\ \mathbf{t}^* &= \mathbf{Q} \cdot \mathbf{t} . \end{aligned} \tag{1.40}$$

Then we can show that the power of the forces is objective (and invariant) $\Pi_e^* = \Pi_e$ as well as the stress power $\Pi_i^* = \Pi_i$, and so is the stress tensor as already denoted in (1.29).

EULER´s laws of motion have a kinematical side and a dynamic side. The dynamic side is objective after A8, while the kinematical side is not objective due to the transformations of the acceleration (1.36). Consequently, these fundamental equations do not hold for all observers, but only for those (*inertial observers*) for which the acceleration transforms like an objective vector ($\boldsymbol{\omega} \equiv \mathbf{o}$ and $\mathbf{c}^\bullet \equiv \mathbf{o}$). Such changes of observers are called GALILEIan transformations.

This situation is, of course, not satisfying. One can remove this shortcoming by relaxing A8 by assuming that the **generalized body forces** $\mathbf{b}_{\text{gen}} := \mathbf{b} - \mathbf{a}$ are objective **(A8')**, which contain the specific body force and the inertial force[1]. This would mean that the fields $\mathbf{t}$ and $\mathbf{T}$ and $\mathbf{b}_{\text{gen}}$ are objective, but not $\mathbf{b}$. If we do this, both sides of EULER´s equations are objective, and they hold for every observer. So by sacrificing the objectivity of forces and torques in our concept, we gain the freedom of choice of the observer, since we are no longer restricted to inertial observers. Hence, *all the following statements hold for arbitrary observers*, whether inertial or not[2].

Under this assumption A8' we can prove the following

Theorem 1.5 (principle of invariance of the power)
A motion of a body is admissible if and only if

$$\int_{\mathscr{B}_t} \mathbf{b}_{\text{gen}} \cdot \mathbf{v}\, dm + \int_{\partial\mathscr{B}_t} \mathbf{t} \cdot \mathbf{v}\, dA - \int_{\mathscr{B}_t} \mathbf{T} \cdot\cdot\, \mathbf{D}\, dV \tag{1.41}$$

is invariant under all changes of observer during the motion.

Proof. We first show that the invariance of (1.41) leads to the balance of linear momentum (1.7) and of angular momentum (1.8). For that purpose, we determine (1.41) for some other observer using A8' and the transformation of the velocities (1.34)

$$\begin{aligned} &\int_{\mathscr{B}_t} (\mathbf{Q} \cdot \mathbf{b}_{\text{gen}}) \cdot [\mathbf{Q} \cdot \mathbf{v} + \boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet]\, dm \\ &+ \int_{\partial\mathscr{B}_t} (\mathbf{Q} \cdot \mathbf{t}) \cdot [\mathbf{Q} \cdot \mathbf{v} + \boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet]\, dA \\ &- \int_{\mathscr{B}_t} (\mathbf{Q} \cdot \mathbf{T} \cdot \mathbf{Q}^T) \cdot\cdot\, \mathit{sym\ grad}^{\,*} [\mathbf{Q} \cdot \mathbf{v} + \boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet]\, dV\,. \end{aligned} \tag{1.42}$$

The difference with (1.41) with respect to the other observer is

$$\begin{aligned} &\int_{\mathscr{B}_t} (\mathbf{Q} \cdot \mathbf{b}_{\text{gen}}) \cdot [\boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet]\, dm + \int_{\partial\mathscr{B}_t} (\mathbf{Q} \cdot \mathbf{t}) \cdot [\boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet]\, dA \\ &- \int_{\mathscr{B}_t} (\mathbf{Q} \cdot \mathbf{T} \cdot \mathbf{Q}^T) \cdot\cdot\, \mathit{sym\ grad}^{\,*} [\boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet]\, dV\,. \end{aligned} \tag{1.43}$$

[1] CAUCHY (1823) calls it *force accélératrice.*
[2] Of course, this is not the only way to proceed. One could also introduce the concept of an intertial observer, and would then obtain the objectivity of forces only with respect to GALIILEIan transformations.

This must be zero, if (1.41) is invariant for arbitrary observers and, hence, arbitrary vectors $\boldsymbol{\omega}$, $\mathbf{c}$, and $\mathbf{c}^\bullet$. For $\boldsymbol{\omega} \equiv \mathbf{o}$ and $\mathbf{Q} \equiv \mathbf{I}$ and a constant (in space) arbitrary vector $\mathbf{c}^\bullet$ we obtain necessarily

$$\int_{\mathscr{B}_t} \mathbf{b}_{\text{gen}}\, dm + \int_{\partial\mathscr{B}_t} \mathbf{t}\, dA = \mathbf{o} \tag{1.44}$$

i.e., the balance of linear momentum (1.7). For $\mathbf{c} \equiv \mathbf{o}$, $\mathbf{c}^\bullet \equiv \mathbf{o}$ and $\mathbf{Q} \equiv \mathbf{I}$, the rest is

$$0 = \int_{\mathscr{B}_t} \mathbf{b}_{\text{gen}} \cdot \boldsymbol{\omega} \times \mathbf{x}^*\, dm + \int_{\partial\mathscr{B}_t} \mathbf{t} \cdot \boldsymbol{\omega} \times \mathbf{x}^*\, dA - \int_{\mathscr{B}_t} (\mathbf{T} \cdot\cdot\, sym\, grad^*(\boldsymbol{\omega} \times \mathbf{x}^*)\, dV \tag{1.45}$$

$$= \boldsymbol{\omega} \cdot \left[\int_{\mathscr{B}_t} \mathbf{x}^* \times \mathbf{b}_{\text{gen}}\, dm + \int_{\partial\mathscr{B}_t} \mathbf{x}^* \times \mathbf{t}\, dA \right]$$

since

$$grad^*(\boldsymbol{\omega} \times \mathbf{x}^*) = \boldsymbol{\omega} \times \mathbf{x}^* \otimes \nabla^* = \boldsymbol{\omega} \times \mathbf{I}$$

which is skew. (1.54) holds for arbitrary $\boldsymbol{\omega}$ if and only if the balance of angular momentum (1.8) is fulfilled. The other direction of the proof is straight forward; *q.e.d.*

By the same rationale, we can reformulate the volumetric form of the principle of virtual power.

Theorem 1.6 (principle of invariance of global power)
A motion of a body is admissible if and only if

$$\int_{\mathscr{B}_t} [(div\, \mathbf{T} + \rho\, \mathbf{b}_{\text{gen}}) \cdot \mathbf{v} + \mathbf{T} \cdot\cdot\, \mathbf{W}]\, dV \tag{1.46}$$

is invariant for the body and all its subbodies under all changes of observer during the motion.

Instead of using the skew part of the velocity gradient $\mathbf{W}$ in the last term, we can also use the complete gradient $\mathbf{L}$ since both transform in the same way after (1.38).

Note that the integral (1.46) in the previous theorem is not necessarily zero, but only under rigid body motions, *i.e.*, under *distributors*.

The same principle holds also for the local form of the power.

Theorem 1.7 (principle of invariance of local power)
A motion of a body is admissible if and only if the power

$$(div\, \mathbf{T} + \rho\, \mathbf{b}_{\text{gen}}) \cdot \mathbf{v} + \mathbf{T} \cdot\cdot\, \mathbf{W} \tag{1.47}$$

is everywhere invariant under all changes of observer during the motion.

We also see that both principles can be extended to the virtual power by the declaration that the virtual velocities transform like velocities (1.34) under change of observer

$$\delta\mathbf{v}^* = \mathbf{Q} \cdot \delta\mathbf{v} + \mathbf{Q}^\bullet \cdot \mathbf{Q}^T \cdot (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet \tag{1.48}$$

$$= \mathbf{Q} \cdot \delta\mathbf{v} + \boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet$$

or inversely

$$\delta\mathbf{v} = \mathbf{Q}^T \cdot [\delta\mathbf{v}^* - \mathbf{Q}^\bullet \cdot \mathbf{Q}^T \cdot (\mathbf{x}^* - \mathbf{c}) - \mathbf{c}^\bullet]$$
$$= \mathbf{Q}^T \cdot [\delta\mathbf{v}^* - \boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) - \mathbf{c}^\bullet]$$

with $\boldsymbol{\omega}$ being the axial vector of the skew tensor $\mathbf{Q}^\bullet \cdot \mathbf{Q}^T$.

This transformation obviously forms a bijection between the spaces $\delta\mathscr{V}$ and $\delta\mathscr{V}^*$ of the virtual velocities for different observers. If $\delta\mathbf{v}^*$ is *K*-times differentiable, then so is $\delta\mathbf{v}^*$, and *vice versa*.

We can then extend the Principle of Invariance of the Power to all virtual velocity fields.

Theorem 1.8 (global form of the principle of invariance of the virtual power)
A motion of a body is admissible if and only if

(1.49) $$\int_{\mathscr{B}_t} \mathbf{b}_{\text{gen}} \cdot \delta\mathbf{v}\, dm + \int_{\partial\mathscr{B}_t} \mathbf{t} \cdot \delta\mathbf{v}\, dA - \int_{\mathscr{B}_t} \mathbf{T} \cdot\cdot\, \delta\mathbf{D}\, dV$$

is invariant under changes of observer during the motion for all $\delta\mathbf{v} \in \delta\mathscr{V}$.

The same extension applies to the principle of invariance of the global power (1.46).

Theorem 1.9 (volumetric form of the principle of invariance of the virtual power)
A motion of a body is admissible if and only if the integral

(1.50) $$\int_{\mathscr{B}_t} [(div\, \mathbf{T} + \rho\, \mathbf{b}_{\text{gen}}) \cdot \delta\mathbf{v} + \mathbf{T} \cdot\cdot\, \delta\mathbf{W}]\, dV$$

is invariant under changes of observer for the body during the motion for all $\delta\mathbf{v} \in \delta\mathscr{V}$.

For the last term $\mathbf{T} \cdot\cdot\, \delta\mathbf{W}$ in the forgoing Theorem one could also use $\mathbf{T} \cdot\cdot\, \delta\mathbf{L}$ since only the skew part of the deformation gradient can be freely varied by observer changes after (1.38).

1.2 Mechanics based on the Principle of Virtual Power

This chapter is partly identical with

> Bertram, A.; Forest, S.: *Mechanics based on an objective power functional.* Techn. Mechanik **27**,1, 1-17 (2007)

In the previous chapter we have seen that CAUCHY´s method is based on a series of assumptions and leads to a rather specific theory. In contrast, we will show in this chapter that a procedure based on an invariance requirement of the power functional is more elegant and leads to a broader theory which might also include higher gradients.

The principle of virtual power (PVP) has gained much popularity because of its versatility to the inclusion of non-classical effects. Already the COSSERATs used it as the starting point for their inclusion of additional rotations.

In the sequel, this challenging task will be solved by a method suggested and used by TRUESDELL/ TOUPIN (1960), GERMAIN (1972, 1973a, 1973b), MAUGIN (1980), BERTRAM (1983, 1989), TROSTEL (1993), BERTRAM/ FOREST (2007), DELL´ISOLA/ SEPPECHER (2011), LIDSTRÖM (2012), DEL PIERO (2009, 2014), and many others.

In GREEN/ RIVLIN (1964a) the equations of motion for a classical continuum have been derived by a similar invariance requirement, however, not of the virtual power, but rather of the thermodynamical energy balance (first law of thermodynamics).

PODIO-GUIDUGLI (2009) enlarged the principle of virtual power to include the thermodynamical fields and, thus, producing the entropy balance.

Our approach here has been widely inspired by the seminal papers of GERMAIN (1972, 1973a and b), who based different non-classical continuum theories on the principle of virtual power. However, certain differences between his and the present approach should be mentioned.

(*i*) GERMAIN´s starting point is the principle of virtual power (PVP), while that of the present theory is the power itself. As the attribute *virtual* already indicates, is the PVP in a certain sense unreal, while the power balance applies only to real processes. It is the aim of the present theory to reflect this fact and to show what the real kernel of the PVP essentially is.

(*ii*) GERMAIN distinguishes from the outset between internal and external contributions to the power. This distinction is not made at this stage in the present text, since the forms of both types of contributions are identical.

(*iii*) GERMAIN introduces from the outset volumetric and surface parts of the (virtual) power, while we start exclusively with volumetric parts, which can be later transformed into those acting on the surface of the body.

In contrast to the preceding chapter, we do not take here forces as primitive concepts and the assumptions A1 - A8 of the method used by EULER and CAUCHY. Instead we start with the (total) *power* of a body as a primitive concept. All kinematical concepts remain the same as

before, but all dynamic concepts will be introduced completely anew. The *Compatibility Axiom* 1.2 is also assumed further on. In addition, we will postulate two more axioms.

Axiom 1.3 (principle of determinism)
For each body there exists a power functional Π with respect to some observer that assigns to any motion of the body the (total) **power** $\Pi(\chi)$ *that the body currently (i.e., at the end of this motion) produces.*
It is zero, whenever the current velocity field is zero for all points of the body.

In modern continuum mechanics it is usually assumed that variables like the stresses depend on the motion of the body in the past until the presence. Such dependencies on *processes* are called (*process*) *functionals*. In the older literature[3] often (semi-infinite) *histories* are considered, which led however to some conceptual problems[4].

In mathematics, a functional is in most cases understood as a mapping from (infinite dimensional) function spaces into finite dimensional spaces. For the stress power the label *functional* is substantiated in two ways. Firstly, since it is an integral over fields defined on the body or its surface, as we will see in the sequel. And secondly, since it contains variables like the stresses which can be determined by a process functional.

The integral (1.46) from Theorem 1.6 would be one candidate for such a power with all dynamic variables being functionals of the motion. But there are other candidates as we will see later.

Note that both the motion and the power functional will in general depend on the observer. For the motion this dependence is specified by the EUCLIDean transformation (1.33). For the power it will be done by the following *Objectivity Axiom*.

Axiom 1.4 (principle of objectivity of the power)
A motion of the body is admissible if and only if the power is objective under all changes of observer

(1.51) $$\Pi^*(\chi^*) = \Pi(\chi)$$

at all times during the motion.

Such an axiom has been used by NOLL (1959, 1963) to derive the balance of linear and angular momentum. However, NOLL´s starting point is different from ours. He starts with an objective force as a primitive concept and defines the power, while we do it here *vice versa*.

We are not assuming that the power functional $\Pi(\chi)$ for a given body was unique. But we do assume that every choice of it gives us the same answer to the question whether some motion is admissible or not after Axiom 1.4.

One is tempted to state that the power is linear in the current velocity field (*i.e.*, at the end of the motion), and often reads such a statement. However, because of regularity, the velocity field at the end of some motion is determined by the motion. So we cannot vary the velocity field without varying the motion, and such a statement makes no sense.

[3] see, *e.g.*, TRUESDELL/ NOLL (1965)
[4] see BERTRAM (2014) on different approaches to material theory

The remedy to overcome this problem is to introduce a *virtual power functional* as a continuous and linear extension of the power functional.

To make this more precise, we first have to endow the space of virtual velocities $\delta\mathscr{V}$ with a linear and a topological structure, in order to give properties of functions like linearity and continuity a meaning.

The linear operations of such fields are introduced point-wise, as usual.

The topological structure, however, is non-trivial, since we work in function spaces (with infinite dimension). We will assume further on that all virtual velocity fields are K-times piecewise differentiable in space for some $K \geq 0$, which shall be specified later. A topological structure on $\delta\mathscr{V}$ is introduced by the SOBOLEV **(2,*K*)-norm**

$$|\delta\mathbf{v}|^K := \sqrt{\int_{\mathscr{B}_t} [|\delta\mathbf{v}(\mathbf{x})|^2 + |grad\, \delta\mathbf{v}(\mathbf{x})|^2 + ... + |grad^K\, \delta\mathbf{v}(\mathbf{x})|^2]\, dV} \tag{1.52}$$

for all $\delta\mathbf{v} \in \delta\mathscr{V}$. Here $grad^K$ means the K-fold EULERean gradient of the field $\delta\mathbf{v}$

$$grad^K \delta\mathbf{v} := \delta\mathbf{v} \otimes \nabla^K = \delta\mathbf{v} \otimes \nabla \otimes ... \otimes \nabla \quad (K\text{-times}). \tag{1.53}$$

Note that the norm on the left-hand side of (1.52) is a norm on a vector field, *i.e.*, in a functional space, while the norms on the right-hand side are the usual FROBENIUS norms of finite dimensional vector and tensor spaces evaluated point-wise.

This norm makes a topological vector space out of $\delta\mathscr{V}$, which is contained in the SOBOLEV space $\mathscr{W}^{2,K}$.

One might argue that the terms in the above norm have all different dimensions. This can be cured by using appropriate factors for every term. However, such factors do not alter the induced topology and are, hence, omitted.

The current velocity field shall always be contained in $\delta\mathscr{V}$. This assumption restricts the regularity of the velocity fields in a way that is not appropriate for certain purposes. Shock waves, shear bands, and other localizations will require weaker regularities in order to also allow for non-smooth fields with singularities. For the present context, however, we will not include such behavior for the sake of simplicity and clearness.

The action of the group of EUCLIDean transformations on the space of virtual velocities (1.48) preserves the topology of the spaces $\delta\mathscr{V}$, *i.e.*, if some $\delta\mathbf{v}$ is continuous, then its image under (1.48) $\delta\mathbf{v}^*$ is also continuous, and the same holds for the differentiability.

We are now able to introduce the *virtual power* as a continuous extension of the *power* being linear in the virtual velocities.

Definition 1.1 (virtual power)
For a given motion $\boldsymbol{\chi}$ of a body with respect to some observer, the **virtual power functional** is a functional $\delta\Pi(\boldsymbol{\chi}, \delta\mathbf{v})$ of the motion and of the virtual velocity field such that for each motion

$$\delta\Pi(\boldsymbol{\chi}, \bullet) : \delta\mathscr{V} \to \mathscr{R}$$

(P1) is continuous and linear

(P2) and extends the power functional Π, *i.e.*,

(1.54) $\delta\Pi(\chi, \mathbf{v}) = \Pi(\chi)$

for the current velocity field $\mathbf{v}$

(P3) it transforms like the power functional, *i.e.*, for all observers we have

(1.55) $\delta\Pi^*(\chi^*, \delta\mathbf{v}^*) - \delta\Pi(\chi, \delta\mathbf{v}) = \Pi^*(\chi^*) - \Pi(\chi)$

if χ is admissible and transformed after (1.33) and $\delta\mathbf{v}$ are transformed like $\mathbf{v}$ after (1.48).

Such an extension is by no means unique. This non-uniqueness will influence all the derived concepts like forces, stresses, etc., but will have no influence on the distinction between admissible and non-admissible motions, as we will see later.

Using (1.48) we can bring (1.55) into the form

(1.56) $\delta\Pi^*(\chi^*, \delta\mathbf{v}^*) = \delta\Pi(\chi, \mathbf{Q}^T \cdot \delta\mathbf{v}^*) + \Pi^*(\chi^*) - \Pi(\chi) - \delta\Pi(\chi, \mathbf{Q}^T \cdot [\boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) - \mathbf{c}^\bullet])$

or inversely

(1.57) $\delta\Pi^*(\chi^*, \mathbf{Q} \cdot \delta\mathbf{v} + \mathbf{Q}^\bullet \cdot \mathbf{Q}^T \cdot (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet) - \delta\Pi(\chi, \delta\mathbf{v}) = \Pi^*(\chi^*) - \Pi(\chi)$

and by the linearity of the virtual power functional (P1)

(1.58) $\delta\Pi^*(\chi^*, \mathbf{Q} \cdot \delta\mathbf{v}) - \delta\Pi(\chi, \delta\mathbf{v})$
$= \Pi^*(\chi^*) - \Pi(\chi) - \delta\Pi^*(\chi^*, \mathbf{Q}^\bullet \cdot \mathbf{Q}^T \cdot (\mathbf{x}^* - \mathbf{c})) - \delta\Pi^*(\chi^*, \mathbf{c}^\bullet)$ for all $\delta\mathbf{v} \in \delta\mathcal{V}$.

Regarding the dependencies upon $\delta\mathbf{v}$, the right-hand side of this equation is constant. The only linear function that equals a constant, is the zero function. Thus,

(1.59) $\delta\Pi^*(\chi^*, \mathbf{Q} \cdot \delta\mathbf{v}) = \delta\Pi(\chi, \delta\mathbf{v})$ for all $\delta\mathbf{v} \in \delta\mathcal{V}$.

The remaining parts of equation (1.58) give

(1.60) $\Pi^*(\chi^*) - \Pi(\chi) = \delta\Pi^*(\chi^*, \mathbf{Q}^\bullet \cdot \mathbf{Q}^T \cdot (\mathbf{x}^* - \mathbf{c})) + \delta\Pi^*(\chi, \mathbf{c}^\bullet)$.

Forces and Torques

By the linearity of the virtual power, there exist after the RIESZ representation theorem (here applied to a finite-dimensional vector space with inner product[5]) two motion- and time-dependent vectors (not vector *fields*) $\underline{\mathbf{f}} \in \mathcal{V}^3$ and $\underline{\mathbf{m}}_O \in \mathcal{V}^3$ for every observer which give the virtual power for an arbitrary spatially constant translational field $\delta\mathbf{v} \equiv \delta\mathbf{v}_o \in \delta\mathcal{V}$ (here identified with $\mathcal{V}^3$)

(1.61) $\delta\Pi(\chi, \delta\mathbf{v}_o) = \underline{\mathbf{f}} \cdot \delta\mathbf{v}_o$

and for an arbitrary rotational field $\delta\mathbf{v} \equiv \mathbf{Q}^\bullet \cdot \mathbf{Q}^T \cdot \mathbf{x} = \delta\boldsymbol{\omega} \times \mathbf{x} \in \delta\mathcal{V}$

[5] We identify the set of all constant fields on the body with the three-dimensional EUCLIDean vector space $\mathcal{V}^3$.

(1.62) $$\delta\Pi(\boldsymbol{\chi}, \delta\boldsymbol{\omega} \times \mathbf{x}) = \underline{\mathbf{m}}_O \cdot \delta\boldsymbol{\omega}$$

such that the virtual power for a virtual velocity field resulting from a rigid body motion is

(1.63) $$\delta\Pi(\boldsymbol{\chi}, \delta\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{c})) + \delta\Pi(\boldsymbol{\chi}, \mathbf{c}^\bullet) = \underline{\mathbf{f}} \cdot \delta\mathbf{v}_o + \underline{\mathbf{m}}_O \cdot \delta\boldsymbol{\omega}$$

with the relative virtual velocity of the reference point O (which need not be a material point)

(1.64) $$\delta\mathbf{v}_o = \mathbf{c}^\bullet - \delta\boldsymbol{\omega} \times \mathbf{c} = \mathbf{c}^\bullet - \mathbf{Q}^\bullet \cdot \mathbf{Q}^T \cdot \mathbf{c}.$$

We will call the vector $\underline{\mathbf{f}}$ the (resultant**) generalized**[6] **force** and the vector $\underline{\mathbf{m}}_O$ the (resultant**) generalized torque** of the body induced by the virtual power functional $\delta\Pi$. These generalized forces and torques also contain inertial forces. If we subtract the inertial forces, *i.e.*, negative momenta rates, we obtain the (resultant) **force**

(1.65) $$\mathbf{f} := \underline{\mathbf{f}} + \mathbf{p}^\bullet \quad \in \mathscr{V}^3$$

and analogously the (resultant**) torque** with respect to O

(1.66) $$\mathbf{m}_O := \underline{\mathbf{m}}_O + \mathbf{d}_O^\bullet \quad \in \mathscr{V}^3$$

with the linear momentum (1.4) and the angular of momentum (1.6) as before.

As we introduced the forces and moments by definitions and not as primitive concepts, we can derive their properties without need of further assumptions[7].

The generalized forces and moments are observer-dependent functionals of the motion, like the power itself. The observer-dependence of the generalized forces and torques will be clarified by the next theorem.

Theorem 1.10 (objectivity of forces and torques)
The generalized force and the generalized moment are objective vectors

(1.67) $$\underline{\mathbf{f}}^* = \mathbf{Q} \cdot \underline{\mathbf{f}} \quad \textit{and} \quad \underline{\mathbf{m}}_O^* = \mathbf{Q} \cdot \underline{\mathbf{m}}_O .$$

Proof. We evaluate (1.56) for the field $\delta\mathbf{v} \equiv \mathbf{a} \times \mathbf{x} + \mathbf{b}$ with two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$

(1.68) $$\begin{aligned} &\Pi^*(\boldsymbol{\chi}^*) - \Pi(\boldsymbol{\chi}) \\ &= \delta\Pi^*(\boldsymbol{\chi}^*, \delta\mathbf{v}^*) - \delta\Pi(\boldsymbol{\chi}, \delta\mathbf{v}) \\ \text{with (1.34)} \quad &= \delta\Pi^*(\boldsymbol{\chi}^*, \mathbf{Q} \cdot (\mathbf{a} \times \mathbf{x} + \mathbf{b}) + \boldsymbol{\omega} \times (\mathbf{x}^* - \mathbf{c}) + \mathbf{c}^\bullet) - \delta\Pi(\boldsymbol{\chi}, \mathbf{a} \times \mathbf{x} + \mathbf{b}) \\ &= \underline{\mathbf{f}}^* \cdot (\mathbf{Q} \cdot \mathbf{b}) + \underline{\mathbf{m}}_O^* \cdot (\mathbf{Q} \cdot \mathbf{a}) + \underline{\mathbf{f}}^* \cdot \mathbf{c}^\bullet + \underline{\mathbf{m}}_O^* \cdot \boldsymbol{\omega} - \underline{\mathbf{f}} \cdot \mathbf{b} - \underline{\mathbf{m}}_O \cdot \mathbf{a} \\ &= (\mathbf{Q}^T \cdot \underline{\mathbf{f}}^* - \underline{\mathbf{f}}) \cdot \mathbf{b} + (\mathbf{Q}^T \cdot \underline{\mathbf{m}}_O^* - \underline{\mathbf{m}}_O) \cdot \mathbf{a} + \underline{\mathbf{f}}^* \cdot \mathbf{c}^\bullet + \underline{\mathbf{m}}_O^* \cdot \boldsymbol{\omega} \end{aligned}$$

since

$$\mathbf{Q} \cdot (\mathbf{a} \times \mathbf{x}) = (\mathbf{Q} \cdot \mathbf{a}) \times (\mathbf{Q} \cdot \mathbf{x}) = (\mathbf{Q} \cdot \mathbf{a}) \times \mathbf{x}^*$$

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[6] some authors call the *apparent forces*, while HAMEL prefers the name *verlorene Kräfte* (lost forces)

[7] For the introduction of forces and stresses see also SEGEV (1986) and SEGEV/ DE BOTTON (1991)

with (1.28) after an appropriate choice of the point of reference for the second observer (with $\mathbf{c} \equiv \mathbf{o}$). By the arbitrariness of $\mathbf{a}$ and $\mathbf{b}$ we conclude the objectivity of the two vectors; *q.e.d.*

After the above definition (1.65), the forces do not depend on a point of reference, while the moments do so (through the position vector). This dependence is specified by the following theorem originally due to VARIGNON.

Theorem 1.11 (VARIGNON´s principle)
The generalized torque depends on the point of reference after

$$\underline{\mathbf{m}}_{O'} = \underline{\mathbf{m}}_O + \vec{O'O} \times \underline{\mathbf{f}} \tag{1.69}$$

with $\vec{O'O}$ being the position vector of the second point of reference with respect to the first.

Proof. We use the equation of the position vectors $\mathbf{x}_O = \mathbf{x}_{O'} + \vec{OO'}$, so that

$$\mathbf{a} \times \mathbf{x}_O = \mathbf{a} \times \mathbf{x}_{O'} + \mathbf{a} \times \vec{OO'} \tag{1.70}$$

holds for arbitrary vectors $\mathbf{a}$, and

$$\begin{aligned} &\delta\Pi(\boldsymbol{\chi}, \mathbf{a} \times \mathbf{x}_O) = \underline{\mathbf{m}}_O \cdot \mathbf{a} \\ &= \delta\Pi(\boldsymbol{\chi}, \mathbf{a} \times \mathbf{x}_{O'} + \mathbf{a} \times \vec{OO'}) = \underline{\mathbf{f}} \cdot \mathbf{a} \times \vec{OO'} + \underline{\mathbf{m}}_{O'} \cdot \mathbf{a} \\ &= \vec{OO'} \times \underline{\mathbf{f}} \cdot \mathbf{a} + \underline{\mathbf{m}}_{O'} \cdot \mathbf{a} = (\vec{OO'} \times \underline{\mathbf{f}} + \underline{\mathbf{m}}_{O'}) \cdot \mathbf{a} \end{aligned} \tag{1.71}$$

which leads with $\vec{OO'} = -\vec{O'O}$ to VARIGNON´s formula; *q.e.d.*

By the definition of the torques (1.66), VARIGNON´s principle holds analogously for the torques

$$\mathbf{m}_{O'} = \mathbf{m}_O + \vec{O'O} \times \mathbf{f}. \tag{1.72}$$

The next results are direct consequences of Axiom 1.4 and equations (1.60) and (1.63), with which we see that the difference of the power for the two observers vanishes if and only if $\underline{\mathbf{f}} = \mathbf{o}$ and $\underline{\mathbf{m}}_O = \mathbf{o}$.

Theorem 1.12 (EULER´s laws of motion)
A motion of the body is admissible if and only if the laws of motion

$$\mathbf{f} = \mathbf{p}^{\bullet} \tag{1.73}$$

$$\mathbf{m}_O = \mathbf{d}_O^{\bullet} \tag{1.74}$$

hold for the body for one observer (and hence for all) with respect to one point of reference O (and hence for all) during the motion.

The following statement is a direct consequence of the foregoing theorem.

Theorem 1.13 (global version of PVP)
A motion of the body is dynamically admissible if and only if the balance of virtual power

(1.75) $$\underline{\mathbf{f}} \cdot \delta\mathbf{v}_o + \underline{\underline{\mathbf{m}}}_O \cdot \delta\boldsymbol{\omega} = 0$$

holds for the body for all vectors $\delta\mathbf{v}_o$ *and* $\delta\boldsymbol{\omega} \in \mathscr{V}$ *for one observer (and hence for all) during the motion.*

(1.75) is not identical with the virtual power functional $\delta\Pi$, but only contains its essential parts for the distinction between admissible and non-admissible processes in the sense of Axiom 1.4.

With these laws we are already able to completely describe the dynamics of rigid bodies for assigned forces and moments. For deformable bodies, however, a field formulation of these concepts is needed, which will be given in the next section.

Field Equations

In order to obtain a field formulation of the power functional, we make use of the RIESZ representation theorem of linear continuous functionals on topological vector spaces[8].

Theorem 1.14 (field formulation of the *virtual* power)

For each observer there exist $K+1$ *time-dependent tensor fields* $\overset{\langle i \rangle}{\boldsymbol{T}}$ *of order* $i = 1, \dots, K+1$ *such that*

$$\delta\Pi(\boldsymbol{\chi}, \delta\mathbf{v}) = \int_{\mathscr{B}_t} (\overset{\langle 1 \rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v} + \overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot \operatorname{grad} \delta\mathbf{v} + \dots + \overset{\langle K+1 \rangle}{\boldsymbol{T}} \cdot\dots\cdot \operatorname{grad}^K \delta\mathbf{v})\, dV \quad \forall\, \delta\mathbf{v} \in \delta\mathscr{V}. \tag{1.76}$$

Again, the integral (1.50) written as

$$\int_{\mathscr{B}_t} [(\operatorname{div} \overset{\langle 2 \rangle}{\boldsymbol{T}} + \rho\, \mathbf{b}_{\text{gen}}) \cdot \delta\mathbf{v} + \overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot \operatorname{grad} \delta\mathbf{v}]\, dV$$

would be one candidate for such a representation of the virtual power.

The dynamic variables $\overset{\langle 1 \rangle}{\boldsymbol{T}}$, $\overset{\langle 2 \rangle}{\boldsymbol{T}}$, ..., $\overset{\langle K+1 \rangle}{\boldsymbol{T}}$ are in each material point still functionals of the motion $\boldsymbol{\chi}$, but do not depend on the virtual velocity. These functionals must be further specified by material laws. This task is, however, beyond the scope of the present chapter.

By (1.54) we obtain the same representation for the power

$$\Pi(\boldsymbol{\chi}) = \int_{\mathscr{B}_t} (\overset{\langle 1 \rangle}{\boldsymbol{T}} \cdot \mathbf{v} + \overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot \operatorname{grad} \mathbf{v} + \dots + \overset{\langle K+1 \rangle}{\boldsymbol{T}} \cdot\dots\cdot \operatorname{grad}^K \mathbf{v})\, dV. \tag{1.77}$$

The question arises whether the fields of the dynamic variables can be restricted to subbodies or not. This question is addressed in the following theorem.

Theorem 1.15 (localization of the power)
Let the power functional be given in the form (1.77)

$$\Pi(\boldsymbol{\chi}) = \int_{\mathscr{B}_0} (\overset{\langle 1 \rangle}{\boldsymbol{T}} \cdot \mathbf{v} + \overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot \operatorname{grad} \mathbf{v} + \dots + \overset{\langle K+1 \rangle}{\boldsymbol{T}} \cdot\dots\cdot \operatorname{grad}^K \mathbf{v})\, dV \tag{1.77}$$

for a body $\mathscr{B}_0$, and let $\mathscr{B}_1 \subset \mathscr{B}_0$ be a subbody of $\mathscr{B}_0$. *Then the integral*

$$\Pi(\boldsymbol{\chi}) = \int_{\mathscr{B}_1} (\overset{\langle 1 \rangle}{\boldsymbol{T}} \cdot \mathbf{v} + \overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot \operatorname{grad} \mathbf{v} + \dots + \overset{\langle K+1 \rangle}{\boldsymbol{T}} \cdot\dots\cdot \operatorname{grad}^K \mathbf{v})\, dV \tag{1.78}$$

[8] see, *e.g.*, ADAMS (1975) p. 48

with the same dynamic fields $\overset{\langle 1\rangle}{\boldsymbol{T}}$, $\overset{\langle 2\rangle}{\boldsymbol{T}}$, ... , $\overset{\langle K+1\rangle}{\boldsymbol{T}}$ *restricted to the domain of the subbody is a power functional for* $\mathscr{B}_1$.

Proof. We compose the body by a set of subbodies in the following way. Let $\mathscr{N}$ be an index set and $\{ \mathscr{B}_i , i \in \mathscr{N}\}$ a collection of subbodies which cover $\mathscr{B}_0$, *i.e.*,

$$\bigcup_{\mathscr{N}} \mathscr{B}_i = \mathscr{B}_0 \qquad \text{and} \qquad \bigcap_{\mathscr{N}} \mathscr{B}_i = \text{null set.}$$

Let $\Pi_i(\boldsymbol{\chi})$ be the (local) power functional of some $\mathscr{B}_i$, which has a representation as an integral (1.77). Then we can construct a (global) power functional $\Pi_0(\boldsymbol{\chi})$ for $\mathscr{B}_0$ by composing the local power functionals $\Pi_i(\boldsymbol{\chi})$ by simply taking the dynamic fields $\overset{\langle 1\rangle}{\boldsymbol{T}}$, $\overset{\langle 2\rangle}{\boldsymbol{T}}$, ... , $\overset{\langle K+1\rangle}{\boldsymbol{T}}$ at some point from the particular local power functional $\Pi_i(\boldsymbol{\chi})$ in this region.

In fact, after the Compatibility Assumption 1.2 we know that $\Pi_0(\boldsymbol{\chi})$ is objective if and only if all $\Pi_i(\boldsymbol{\chi})$ are objective. Therefore, a motion $\boldsymbol{\chi}$ for $\mathscr{B}_0$ is admissible if and only if its restrictions to all subbodies is also admissible.

Since we do not demand uniqueness of the power functional, this $\Pi_0(\boldsymbol{\chi})$ would always be a choice for a power functional that gives us a correct answer to the question whether some motion is admissible or not.

This construction of $\Pi_0(\boldsymbol{\chi})$ works for every collection of subbodies. So for every subbody $\mathscr{B}_1$ one can find such a collection that contains $\mathscr{B}_1$; *q.e.d.*

This means that whenever two bodies share a common part, then in this part the integrands of the power-functionals can be taken as identical. This is typical property of densities, and the integrand of (1.76) is in fact the **virtual power density**.

The same holds if we replace the power by the virtual power and the velocity by the virtual velocity in the above theorem.

Theorem 1.16 (localization of the virtual power)
Let the virtual power functional be given in the form (1.76)

$$\delta\Pi(\boldsymbol{\chi} , \delta\mathbf{v}) = \int_{\mathscr{B}_0} (\overset{\langle 1\rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v} + \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot \operatorname{grad}\delta\mathbf{v} + ... + \overset{\langle K+1\rangle}{\boldsymbol{T}} \cdot ... \cdot \operatorname{grad}^K \delta\mathbf{v})\, dV \tag{1.76}$$

for a body $\mathscr{B}_0$, and let $\mathscr{B}_1 \subset \mathscr{B}_0$ be a subbody of $\mathscr{B}_0$. *Then the integral*

$$\delta\Pi(\boldsymbol{\chi} , \delta\mathbf{v}) = \int_{\mathscr{B}_1} (\overset{\langle 1\rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v} + \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot \operatorname{grad}\delta\mathbf{v} + ... + \overset{\langle K+1\rangle}{\boldsymbol{T}} \cdot ... \cdot \operatorname{grad}^K \delta\mathbf{v})\, dV \tag{1.79}$$

with the same dynamic fields $\overset{\langle 1\rangle}{\boldsymbol{T}}$, $\overset{\langle 2\rangle}{\boldsymbol{T}}$, ... , $\overset{\langle K+1\rangle}{\boldsymbol{T}}$ *restricted to the domain of the subbody is a virtual power functional for* $\mathscr{B}_1$.

We will next consider the transformation behavior of the dynamic fields under change of observer.

Theorem 1.17 (transformations of dynamic fields)

The fields of the dynamic variables $\overset{\langle i \rangle}{\boldsymbol{T}}$ *, i = 1, ... , K+1, are objective under change of observer*

(1.80) $$\overset{\langle i \rangle}{\boldsymbol{T}}{}^{*} = \mathbf{Q} * \overset{\langle i \rangle}{\boldsymbol{T}} .$$

Proof. With (1.31) we get

(1.81) $$grad^{*}(\mathbf{Q} \cdot \delta\mathbf{v}) = \mathbf{Q} \cdot grad^{*}\, \delta\mathbf{v} = \mathbf{Q} \cdot (grad\, \delta\mathbf{v}) \cdot \mathbf{Q}^{T} = \mathbf{Q} * grad\, \delta\mathbf{v}$$

$\forall\ \delta\mathbf{v} \in \delta\mathscr{V}$, and more generally for higher gradients, by the use of the RAYLEIGH product $*$ (0.4) (which shall not be confused with the upper asterisk indicating the change of observer)

(1.82) $$grad^{*\,K}(\mathbf{Q} \cdot \delta\mathbf{v}) = \mathbf{Q} * grad^{K}\, \delta\mathbf{v} \qquad \forall\ \delta\mathbf{v} \in \delta\mathscr{V}.$$

By (1.59) we have for all motions

(1.83) $$\delta\Pi^{*}(\boldsymbol{\chi}^{*}, \mathbf{Q} \cdot \delta\mathbf{v})$$

$$= \int_{\mathscr{B}_t} [\overset{\langle 1 \rangle}{\boldsymbol{T}}{}^{*} \cdot (\mathbf{Q} * \delta\mathbf{v}) + \overset{\langle 2 \rangle}{\boldsymbol{T}}{}^{*} \cdot\cdot\, (\mathbf{Q} * grad\, \delta\mathbf{v}) + ... + \overset{\langle K+1 \rangle}{\boldsymbol{T}}{}^{*} \cdot ... \cdot (\mathbf{Q} * grad^{K}\, \delta\mathbf{v})]\, dV$$

by (0.13) $$= \int_{\mathscr{B}_t} [(\mathbf{Q}^{T} * \overset{\langle 1 \rangle}{\boldsymbol{T}}{}^{*}) \cdot \delta\mathbf{v} + (\mathbf{Q}^{T} * \overset{\langle 2 \rangle}{\boldsymbol{T}}{}^{*}) \cdot\cdot\, grad\, \delta\mathbf{v} + ... + (\mathbf{Q}^{T} * \overset{\langle K+1 \rangle}{\boldsymbol{T}}{}^{*}) \cdot ... \cdot grad^{K}\, \delta\mathbf{v}]\, dV$$

$$= \delta\Pi(\boldsymbol{\chi}, \delta\mathbf{v})$$

$$= \int_{\mathscr{B}_t} [\overset{\langle 1 \rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v} + \overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot\, grad\, \delta\mathbf{v} + ... + \overset{\langle K+1 \rangle}{\boldsymbol{T}} \cdot ... \cdot grad^{K}\, \delta\mathbf{v}]\, dV \qquad \forall\ \delta\mathbf{v} \in \delta\mathscr{V}.$$

A comparison in the arbitrary fields $\delta\mathbf{v} \in \delta\mathscr{V}$ leads to the desired result; *q.e.d.*

By the definition of the generalized force (1.61), we have for any observer

(1.84) $$\delta\Pi(\boldsymbol{\chi}, \delta\mathbf{v}_o) = \underline{\mathbf{f}} \cdot \delta\mathbf{v}_o = \int_{\mathscr{B}_t} \overset{\langle 1 \rangle}{\boldsymbol{T}}(\mathbf{x}) \cdot \delta\mathbf{v}_o\, dV = \int_{\mathscr{B}_t} \overset{\langle 1 \rangle}{\boldsymbol{T}}(\mathbf{x})\, dV \cdot \delta\mathbf{v}_o$$

and obtain the representation

(1.85) $$\underline{\mathbf{f}} = \int_{\mathscr{B}_t} \overset{\langle 1 \rangle}{\boldsymbol{T}}(\mathbf{x})\, dV.$$

Analogously, the definition of the generalized torque (1.62)

(1.86) $$\delta\Pi(\boldsymbol{\chi}, \delta\boldsymbol{\omega} \times \mathbf{x}) = \underline{\mathbf{m}}_O \cdot \delta\boldsymbol{\omega}$$

$$= \int_{\mathscr{B}_t} [\overset{\langle 1 \rangle}{\boldsymbol{T}} \cdot \delta\boldsymbol{\omega} \times \mathbf{x} + \overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot\, grad\, (\delta\boldsymbol{\omega} \times \mathbf{x})]\, dV$$

$$= \int_{\mathscr{B}_t} (\mathbf{x} \times \overset{\langle 1 \rangle}{\boldsymbol{T}} + 2\, axi\, \overset{\langle 2 \rangle}{\boldsymbol{T}})\, dV \cdot \delta\boldsymbol{\omega}$$

with $axi\, \overset{\langle 2 \rangle}{\boldsymbol{T}}$ being the axial vector of $\overset{\langle 2 \rangle}{\boldsymbol{T}}$ leads to the representation

$$\underline{\mathbf{m}}_O = \int_{\mathscr{B}_t} (\mathbf{x} \times \overset{\langle 1 \rangle}{\boldsymbol{T}} + 2\, axi\, \overset{\langle 2 \rangle}{\boldsymbol{T}})\, dV. \tag{1.87}$$

Here we used the following rules. For the position vector $\mathbf{x}$ and any constant vector $\delta\boldsymbol{\omega}$, one finds

$$grad\,(\delta\boldsymbol{\omega} \times \mathbf{x}) = (\delta\boldsymbol{\omega} \times \mathbf{x}) \otimes \nabla = \delta\boldsymbol{\omega} \times (\mathbf{x} \otimes \nabla) = \delta\boldsymbol{\omega} \times \mathbf{I} \tag{1.88}$$

which is antisymmetric. Thus

$$axi\, grad\,(\delta\boldsymbol{\omega} \times \mathbf{x}) = axi\,(\delta\boldsymbol{\omega} \times \mathbf{I}) = \delta\boldsymbol{\omega}$$

and, consequently,

$$\begin{aligned} &\overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot\, grad(\delta\boldsymbol{\omega} \times \mathbf{x}) = skw\, \overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot\, grad(\delta\boldsymbol{\omega} \times \mathbf{x}) \\ &= 2\, axi\, \overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot axi\, grad\,(\delta\boldsymbol{\omega} \times \mathbf{x}) = 2\, axi(\overset{\langle 2 \rangle}{\boldsymbol{T}}) \cdot \delta\boldsymbol{\omega}\,. \end{aligned} \tag{1.89}$$

The force is then after (1.65)

$$\mathbf{f} = \int_{\mathscr{B}_t} (\overset{\langle 1 \rangle}{\boldsymbol{T}} + \rho\, \mathbf{x}^{\bullet\bullet})\, dV \tag{1.90}$$

and the torque after (1.66)

$$\mathbf{m}_O = \int_{\mathscr{B}_t} [\mathbf{x} \times (\overset{\langle 1 \rangle}{\boldsymbol{T}} + \rho\, \mathbf{x}^{\bullet\bullet}) + 2\, axi\, \overset{\langle 2 \rangle}{\boldsymbol{T}}]\, dV. \tag{1.91}$$

By Theorem 1.17 we know that $\overset{\langle 1 \rangle}{\boldsymbol{T}}$ and $axi\, \overset{\langle 2 \rangle}{\boldsymbol{T}}$ are objective vector fields, but not $\mathbf{f}$ and $\mathbf{m}_O$.

If we substitute these representations into the EULER´s laws of motion (1.73) and (1.74), we obtain their local forms.

Theorem 1.18 (local form of the laws of motion)
A motion of the body is dynamically admissible if and only if the local form of the laws of motion hold

$$\overset{\langle 1 \rangle}{\boldsymbol{T}} = \mathbf{o} \tag{1.92}$$

$$axi\, \overset{\langle 2 \rangle}{\boldsymbol{T}} = \mathbf{o} \tag{1.93}$$

almost everywhere in the body during the motion.

The other higher-order stress tensors $sym(\overset{\langle 2\rangle}{\boldsymbol{T}})$, $\overset{\langle 3\rangle}{\boldsymbol{T}}$, ... , $\overset{\langle K+1\rangle}{\boldsymbol{T}}$ do not directly enter the laws of motion. This, however, does not mean that such quantities cannot play a useful role in mechanical theories, as we will see in what follows.

The next theorem is a stronger version of the PVP of the last section.

Theorem 1.19 (integral version of PVP)
A motion of the body is dynamically admissible if and only if the balance of virtual power holds in the form

$$\int_{\mathscr{B}_t} (\overset{\langle 1\rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v} + axi\ \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot curl\ \delta\mathbf{v})\, dV = 0 \tag{1.94}$$

for all virtual velocity fields $\delta\mathbf{v} \in \delta\mathscr{V}$ *for one observer (and hence for all) during the motion.*

Proof. We multiply the local laws of motion (1.92) and (1.93) by arbitrary vectors $\delta\mathbf{v}$ and $2\,\delta\boldsymbol{\omega}$. Then

$$\overset{\langle 1\rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v} + axi\ \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot 2\,\delta\boldsymbol{\omega} = 0 \tag{1.95}$$

if and only if (1.92) and (1.93) hold, *i.e.*, if the motion is admissible. If we interpret $\delta\mathbf{v}$ as the local value of some virtual velocity field $\delta\mathbf{v} \in \delta\mathscr{V}$, and $2\,\delta\boldsymbol{\omega}$ as the local value of its curl, then we obtain the above form of the balance of virtual power as a necessary condition for a motion to be dynamically admissible.

On the other hand, if (1.94) holds for the body, then after Theorem 1.16 it would also hold for all subbodies. So the integrand must vanish everywhere in $\mathscr{B}_t$ if it is continuous, which gives (1.92) and (1.93). By Theorem 1.18 the motion must then be admissible; *q.e.d.*

In what follows we will derive a number of alternative forms of the forces and torques, which are altogether equivalent to those of (1.90) and (1.91). The divergence theorem gives

$$\int_{\mathscr{B}_t} \overset{\langle K+1\rangle}{\boldsymbol{T}} \cdot ... \cdot grad^K\, \delta\mathbf{v}\, dV \tag{1.96}$$

$$= \int_{\partial\mathscr{B}_t} (\overset{\langle K+1\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot ... \cdot grad^{K-1}\, \delta\mathbf{v}\, dA - \int_{\mathscr{B}_t} div\, \overset{\langle K+1\rangle}{\boldsymbol{T}} \cdot ... \cdot grad^{K-1}\, \delta\mathbf{v}\, dV$$

for any $K \geq 1$. In particular we achieve for

$K \equiv 1$:

$$\int_{\mathscr{B}_t} \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot\, grad\ \delta\mathbf{v}\, dV = \int_{\partial\mathscr{B}_t} (\overset{\langle 2\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot \delta\mathbf{v}\, dA - \int_{\mathscr{B}_t} div\, \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v}\, dV \tag{1.97}$$

$K \equiv 2$:

(1.98) $$\int_{\mathscr{B}_t} \overset{\langle 3\rangle}{\boldsymbol{T}} \cdots grad^2\, \delta\mathbf{v}\, dV = \int_{\partial\mathscr{B}_t} (\overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot\, grad\, \delta\mathbf{v}\, dA - \int_{\mathscr{B}_t} div \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot\cdot\, grad\, \delta\mathbf{v}\, dV$$

$$= \int_{\partial\mathscr{B}_t} [(\overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot\, grad\, \delta\mathbf{v} - (div \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot \delta\mathbf{v}]\, dA + \int_{\mathscr{B}_t} div^2 \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v}\, dV$$

$K \equiv 3$:

(1.99) $$\int_{\mathscr{B}_t} \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot\cdot\cdot grad^3\, \delta\mathbf{v}\, dV = \int_{\partial\mathscr{B}_t} (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad^2\, \delta\mathbf{v}\, dA - \int_{\mathscr{B}_t} div \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots grad^2\, \delta\mathbf{v}\, dV$$

$$= \int_{\partial\mathscr{B}_t} [(\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad^2\, \delta\mathbf{v}\, dA - (div \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot\, grad\, \delta\mathbf{v} + (div^2 \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot \delta\mathbf{v}]\, dA$$

$$- \int_{\mathscr{B}_t} div^3 \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v}\, dV$$

We substitute this into (1.76). For an arbitrary K it gives

$$\delta\Pi(\boldsymbol{\chi}, \delta\mathbf{v}) = \int_{\mathscr{B}_t} \sum_{j=1}^{k+1} [(-1)^{j-1} div^{j-1} \overset{\langle j\rangle}{\boldsymbol{T}}] \cdot \delta\mathbf{v}\, dV$$

(1.100) $$+ \int_{\partial\mathscr{B}_t} \sum_{j=2}^{k+1} [(-1)^j div^{j-2} \overset{\langle j\rangle}{\boldsymbol{T}}) \cdot \mathbf{n})] \cdot \delta\mathbf{v}\, dA$$

$$\ldots$$

$$+ \int_{\partial\mathscr{B}_t} (-1)^{K+1} (\overset{\langle K+1\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\ldots\cdot grad^{K-1}\, \delta\mathbf{v}\, dA\,.$$

For a constant $\delta\mathbf{v}_o \in \delta\mathscr{V}$ this gives the force in the form

(1.101) $$\mathbf{f} = \int_{\mathscr{B}_t} [\rho\, \mathbf{x}^{\bullet\bullet} + \overset{\langle 1\rangle}{\boldsymbol{T}} - div \overset{\langle 2\rangle}{\boldsymbol{T}} + div^2 \overset{\langle 3\rangle}{\boldsymbol{T}} - \ldots + (-1)^K div^K \overset{\langle K+1\rangle}{\boldsymbol{T}}]\, dV$$

$$+ \int_{\partial\mathscr{B}_t} [\overset{\langle 2\rangle}{\boldsymbol{T}} - div \overset{\langle 3\rangle}{\boldsymbol{T}} + \ldots - (-1)^K div^{K-1} \overset{\langle K+1\rangle}{\boldsymbol{T}}] \cdot \mathbf{n}\, dA$$

and for $\delta\mathbf{v} \equiv \delta\boldsymbol{\omega} \times \mathbf{x} \in \delta\mathscr{V}$ this gives the torque

(1.102) $$\mathbf{m}_O = \int_{\mathscr{B}_t} \mathbf{x} \times [\rho\, \mathbf{x}^{\bullet\bullet} + \overset{\langle 1\rangle}{\boldsymbol{T}} - div \overset{\langle 2\rangle}{\boldsymbol{T}} + div^2 \overset{\langle 3\rangle}{\boldsymbol{T}} - \ldots + (-1)^K div^K \overset{\langle K+1\rangle}{\boldsymbol{T}}]\, dV$$

$$+ \int\limits_{\partial \mathscr{B}_t} \left(\mathbf{x} \times [\overset{\langle 2 \rangle}{\boldsymbol{T}} - div \overset{\langle 3 \rangle}{\boldsymbol{T}} + ... - (-1)^K div^{K-1} \overset{\langle K+1 \rangle}{\boldsymbol{T}}] \cdot \mathbf{n} \right.$$

$$\left. - 2\, axi \{ [\overset{\langle 3 \rangle}{\boldsymbol{T}} - div \overset{\langle 4 \rangle}{\boldsymbol{T}} + ... - (-1)^{K-1} div^{K-2} \overset{\langle K+1 \rangle}{\boldsymbol{T}}] \cdot \mathbf{n} \right) dA \, .$$

So by incrementing K, we can produce a catalogue of infinitely many theories, which are altogether equivalent with respect to the question of whether a motion is dynamically admissible or not. They are, however, not equivalent in their possibilities of the material modeling.

In what follows, we will specify these findings for the cases $K \equiv 0$, 1, and 2. In each case we obtain a theory that gives reasonable results for a certain class of problems.

Historically, the case $K \equiv 0$ dates back to the *Principia* of NEWTON (1687), the case $K \equiv 1$ to CAUCHY (1823), and the case $K \equiv 2$ to the pioneers of gradient theories like TOUPIN (1962) and MINDLIN (1965), including time laps of approximately one and a half century.

Example: NEWTON continuum ($K \equiv 0$)

The most simple, but by no means trivial case is that of $K \equiv 0$. This gives for the virtual power (1.76) the only remainder

$$\delta\Pi(\boldsymbol{\chi}, \delta\mathbf{v}) = \int_{\mathscr{B}_t} \overset{\langle 1 \rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v} \, dV \tag{1.103}$$

and for the total power

$$\Pi(\boldsymbol{\chi}) = \int_{\mathscr{B}_t} \overset{\langle 1 \rangle}{\boldsymbol{T}} \cdot \mathbf{v} \, dV . \tag{1.104}$$

The force remains in the form of (1.90)

$$\mathbf{f} = \int_{\mathscr{B}_t} (\overset{\langle 1 \rangle}{\boldsymbol{T}} + \rho \, \mathbf{x}^{\bullet\bullet}) \, dV \tag{1.105}$$

while the torque becomes after (1.91)

$$\mathbf{m}_O = \int_{\mathscr{B}_t} \mathbf{x} \times (\overset{\langle 1 \rangle}{\boldsymbol{T}} + \rho \, \mathbf{x}^{\bullet\bullet}) \, dV. \tag{1.106}$$

The equations of motion are in this case

$$\overset{\langle 1 \rangle}{\boldsymbol{T}} = \mathbf{o} \tag{1.107}$$

$$\mathbf{x} \times \overset{\langle 1 \rangle}{\boldsymbol{T}} = \mathbf{o} . \tag{1.108}$$

Here, the balance of linear momentum obviously includes the balance of angular momentum.

Such a situation is known for so-called *mass points*, *i.e.*, bodies that are (*i*) not in direct contact, and (*ii*) only interact by central (body) forces. The latter can be probably assumed if the diameter of the bodies is small compared to the distances between the bodies, as we have it if we consider the motion of the planets around the central star under mutual gravitation. To obtain KEPLER´s elliptic orbits it is sufficient to solve the linear momentum balance named **NEWTON´s law** for the usual gravitational law. The balance of angular momentum becomes trivial, since no torque acts with respect to the centers of mass of the mass points, so that *conservation of angular momentum* holds.

We see that even this extreme case $K \equiv 0$ gives a reasonable theory that applies well under certain circumstances.

Example: CAUCHY continuum ($K \equiv 1$) (simple materials)

In order to show the relation to the classical theory, we take $K \equiv 1$. Then (1.76) reduces to

$$\delta\Pi(\boldsymbol{\chi}\,,\,\delta\mathbf{v}) \;=\; \int\limits_{\mathscr{B}_t} (\overset{\langle 1\rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v} + \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot\, grad\,\delta\mathbf{v})\, dV. \tag{1.109}$$

By applying the divergence theorem (1.97), we obtain alternatively

$$\delta\Pi(\boldsymbol{\chi}\,,\,\delta\mathbf{v}) \;=\; \int\limits_{\mathscr{B}_t} (\overset{\langle 1\rangle}{\boldsymbol{T}} - div\, \overset{\langle 2\rangle}{\boldsymbol{T}}) \cdot \delta\mathbf{v}\, dV + \int\limits_{\partial\mathscr{B}_t} (\overset{\langle 2\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot \delta\mathbf{v}\, dA\,. \tag{1.110}$$

For a constant field $\delta\mathbf{v}_o$ this gives the force after (1.65) in two versions

$$\mathbf{f} = \int\limits_{\mathscr{B}_t} (\overset{\langle 1\rangle}{\boldsymbol{T}} + \rho\,\mathbf{x}^{\bullet\bullet})\, dV = \int\limits_{\mathscr{B}_t} (\overset{\langle 1\rangle}{\boldsymbol{T}} - div\, \overset{\langle 2\rangle}{\boldsymbol{T}} + \rho\,\mathbf{x}^{\bullet\bullet})\, dV + \int\limits_{\partial\mathscr{B}_t} \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot \mathbf{n}\, dA \tag{1.111}$$

and for $\delta\mathbf{v} \equiv \delta\boldsymbol{\omega} \times \mathbf{x}$ the torque after (1.66) also in two versions

$$\mathbf{m}_O = \int\limits_{\mathscr{B}_t} [\mathbf{x} \times (\overset{\langle 2\rangle}{\boldsymbol{T}} + \rho\,\mathbf{x}^{\bullet\bullet}) + 2\, axi\, \overset{\langle 2\rangle}{\boldsymbol{T}}]\, dV \tag{1.112}$$

$$= \int\limits_{\mathscr{B}_t} [\mathbf{x} \times (\overset{\langle 1\rangle}{\boldsymbol{T}} - div\, \overset{\langle 2\rangle}{\boldsymbol{T}} + \rho\,\mathbf{x}^{\bullet\bullet})\, dV + \int\limits_{\partial\mathscr{B}_t} \mathbf{x} \times \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot \mathbf{n}\, dA\,.$$

Usually the force acting in the interior of the body is named **specific body force**

$$\mathbf{b} \;:=\; (\overset{\langle 1\rangle}{\boldsymbol{T}} - div\, \overset{\langle 2\rangle}{\boldsymbol{T}})/\rho + \mathbf{x}^{\bullet\bullet} \tag{1.113}$$

and that acting on its surface is the **traction** field

$$\mathbf{t} \;=\; \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot \mathbf{n} \tag{1.114}$$

so that the forces and torques are

$$\mathbf{f} = \int\limits_{\mathscr{B}_t} \mathbf{b}\, dm + \int\limits_{\partial\mathscr{B}_t} \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot \mathbf{n}\, dA \tag{1.115}$$

$$\mathbf{m}_O = \int\limits_{\mathscr{B}_t} \mathbf{x} \times \mathbf{b}\, dm \;+ \int\limits_{\partial\mathscr{B}_t} \mathbf{x} \times \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot \mathbf{n}\, dA \tag{1.116}$$

which are the forms known from the CAUCHY continuum (1.7) and (1.8). The equations of motion are the classical CAUCHY laws (1.14) and (1.15)

$$div\, \overset{\langle 2\rangle}{\boldsymbol{T}} + \mathbf{b}\rho \;=\; \mathbf{x}^{\bullet\bullet}\rho \tag{1.117}$$

$$\text{axi}\ \overset{\langle 2\rangle}{\boldsymbol{T}} = \mathbf{o} \quad \Leftrightarrow \quad \overset{\langle 2\rangle}{\boldsymbol{T}} = \overset{\langle 2\rangle}{\boldsymbol{T}}{}^{T} \tag{1.118}$$

so that $\overset{\langle 2\rangle}{\boldsymbol{T}}$ can be identified with CAUCHY´s stress tensor.

Example[9]: Second gradient continuum ($K \equiv 2$)

In this case the virtual power can be brought into one of the following alternative forms

$$\delta\Pi(\boldsymbol{\chi}, \delta\mathbf{v}) = \int_{\mathscr{B}_t} (\overset{\langle 1 \rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v} + \overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot \operatorname{grad} \delta\mathbf{v} + \overset{\langle 3 \rangle}{\boldsymbol{T}} \cdot\cdot\cdot \operatorname{grad}\operatorname{grad} \delta\mathbf{v})\, dV$$

$$= \int_{\mathscr{B}_t} [(\overset{\langle 1 \rangle}{\boldsymbol{T}} - \operatorname{div} \overset{\langle 2 \rangle}{\boldsymbol{T}}) \cdot \delta\mathbf{v} - \operatorname{div} \overset{\langle 3 \rangle}{\boldsymbol{T}} \cdot\cdot \operatorname{grad} \delta\mathbf{v}]\, dV$$

$$+ \int_{\partial\mathscr{B}_t} [(\overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot \delta\mathbf{v} + (\overset{\langle 3 \rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot \operatorname{grad} \delta\mathbf{v}]\, dA \tag{1.119}$$

$$= \int_{\mathscr{B}_t} (\overset{\langle 1 \rangle}{\boldsymbol{T}} - \operatorname{div} \overset{\langle 2 \rangle}{\boldsymbol{T}} + \operatorname{div}\operatorname{div} \overset{\langle 3 \rangle}{\boldsymbol{T}}) \cdot \delta\mathbf{v}\, dV$$

$$+ \int_{\partial\mathscr{B}_t} \{[(\overset{\langle 2 \rangle}{\boldsymbol{T}} - \operatorname{div} \overset{\langle 3 \rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot \delta\mathbf{v} + (\overset{\langle 3 \rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot \operatorname{grad} \delta\mathbf{v}\}\, dA$$

$$= \int_{\mathscr{B}_t} (\overset{\langle 1 \rangle}{\boldsymbol{T}} - \operatorname{div} \overset{\langle 2 \rangle}{\boldsymbol{T}} + \operatorname{div}\operatorname{div} \overset{\langle 3 \rangle}{\boldsymbol{T}}) \cdot \delta\mathbf{v}\, dV$$

$$+ \int_{\partial\mathscr{B}_t} \{[(\overset{\langle 2 \rangle}{\boldsymbol{T}} - \operatorname{div} \overset{\langle 3 \rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot \delta\mathbf{v} - \operatorname{div} (\overset{\langle 3 \rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot \delta\mathbf{v}\}\, dA$$

$$= \int_{\mathscr{B}_t} (\overset{\langle 1 \rangle}{\boldsymbol{T}} - \operatorname{div} \overset{\langle 2 \rangle}{\boldsymbol{T}} + \operatorname{div}\operatorname{div} \overset{\langle 3 \rangle}{\boldsymbol{T}}) \cdot \delta\mathbf{v}\, dV$$

$$+ \int_{\partial\mathscr{B}_t} \{[(\overset{\langle 2 \rangle}{\boldsymbol{T}} - 2 \operatorname{div} \overset{\langle 3 \rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot \delta\mathbf{v} - (\overset{\langle 3 \rangle}{\boldsymbol{T}} \cdot\cdot \operatorname{grad} \mathbf{n}) \cdot \delta\mathbf{v}\}\, dA.$$

These forms can be further reformulated by applying the divergence theorem on the surface terms. Herein we follow the lines of TOUPIN (1962) and MINDLIN (1965)[10]. First we decompose the gradient at surface points into its normal and its tangential part

$$\operatorname{grad} = \operatorname{grad}_n + \operatorname{grad}_t \tag{1.120}$$

[9] This part has been coauthored by CHRISTIAN REIHER.

[10] see also BLEUSTEIN (1967), DILLON/ KRATOCHVIL (1970), TROSTEL (1985), PODIO-GUIDUGLI/ VIANELLO (2013), POLIZZOTTO (2013), JAVILI/ DELL'ISOLA/ STEINMANN (2013)

which corresponds to the natural split of the spatial nabla operator

$$\nabla \;=\; \nabla_n + \nabla_t$$

with $\nabla_n := \mathbf{n} \otimes \mathbf{n} \cdot \nabla = \dfrac{\partial}{\partial x_n}\,\mathbf{n}$ with the outer normal $\mathbf{n}$ to the surface and the normal coordinate x_n. The trace of these gradients is the divergence

$$div \;=\; div_n + div_t \,. \tag{1.121}$$

If we decompose the gradient within the following expression with a vector field $\mathbf{v}$ and a second-order tensor field $\overset{\langle 2\rangle}{\boldsymbol{T}}$ and use the chain rule, we obtain

$$\begin{aligned} \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot\, grad\, \mathbf{v} \;&=\; \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot\, grad_n\, \mathbf{v} + \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot\, grad_t\, \mathbf{v} \\ &=\; \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot\, grad_n\, \mathbf{v} + div_t\,(\mathbf{v} \cdot \overset{\langle 2\rangle}{\boldsymbol{T}}) - (div_t\, \overset{\langle 2\rangle}{\boldsymbol{T}}) \cdot \mathbf{v}\,. \end{aligned} \tag{1.122}$$

Then the surface divergence theorem holds in the form[11]

$$\int_{\partial \mathscr{B}_t} div_t\, \mathbf{u}\, dA \;=\; \int_{\partial \mathscr{B}_t} (div_t\, \mathbf{n})\, (\mathbf{u} \cdot \mathbf{n})\, dA \tag{1.123}$$

for any smooth vector-field $\mathbf{u}$ and a smooth surface without edges or corners[12]. The term $(-div_t\, \mathbf{n})$ stands for twice the mean curvature of the surface. By taking $\mathbf{u} \equiv \mathbf{v} \cdot \overset{\langle 2\rangle}{\boldsymbol{T}}$ (1.123) gives

$$\int_{\partial \mathscr{B}_t} div_t\, (\overset{\langle 2\rangle}{\boldsymbol{T}} \cdot \mathbf{v})\, dA \;=\; \int_{\partial \mathscr{B}_t} (div_t\, \mathbf{n})\, (\overset{\langle 2\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot \mathbf{v}\, dA\,. \tag{1.124}$$

We apply these identities to the following term from the power functional (1.119.3)

$$\begin{aligned} &\int_{\partial \mathscr{B}_t} (\overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot\, grad\, \delta\mathbf{v}\, dA \\ &= \int_{\partial \mathscr{B}_t} \{(\overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot\, grad_n\, \delta\mathbf{v} + div_t\, (\delta\mathbf{v} \cdot \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) - [div_t\, (\overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n})] \cdot \delta\mathbf{v}\}\, dA \\ &= \int_{\partial \mathscr{B}_t} \{(\overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot\, grad_n\, \delta\mathbf{v} \\ &\qquad + [(div_t\, \mathbf{n})\, \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot\cdot\, \mathbf{n} \otimes \mathbf{n} - \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot\cdot\, grad_t\, \mathbf{n} - (div_t\, \overset{\langle 3\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot \delta\mathbf{v}\}\, dA\,. \end{aligned} \tag{1.125}$$

We substitute this into the above expression for the virtual power (1.119.3)

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[11] see BRAND (1947) p. 222

[12] In DELL´ISOLA/ SEPPECHER (1997) also these line integrals on edges and vertices have been considered.

$$\delta\Pi(\boldsymbol{\chi}, \delta\mathbf{v}) = \int_{\mathscr{B}_t} (\overset{\langle 1\rangle}{\boldsymbol{T}} - div\ \overset{\langle 2\rangle}{\boldsymbol{T}} + div\ div \overset{\langle 3\rangle}{\boldsymbol{T}}) \cdot \delta\mathbf{v}\ dV$$

$$(1.126) \qquad + \int_{\partial\mathscr{B}_t} \{[(\overset{\langle 2\rangle}{\boldsymbol{T}} - div_n \overset{\langle 3\rangle}{\boldsymbol{T}} - 2\ div_t \overset{\langle 3\rangle}{\boldsymbol{T}}) \cdot \mathbf{n} + (div_t\ \mathbf{n}) \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot\cdot\ \mathbf{n} \otimes \mathbf{n} - \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot\cdot\ grad_t\ \mathbf{n}] \cdot \delta\mathbf{v}$$

$$+ (\overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot\ grad_n\ \delta\mathbf{v}\}\ dA\ .$$

For a constant $\delta\mathbf{v}$ this gives the generalized force in the form

$$(1.127) \qquad \underline{\mathbf{f}} = \int_{\mathscr{B}_t} (\mathbf{b}_2 - \mathbf{x}^{\bullet\bullet})\ dV + \int_{\partial\mathscr{B}_t} \mathbf{t}_2\ dA$$

with the **specific body force**

$$(1.128) \qquad \mathbf{b}_2 := (\overset{\langle 1\rangle}{\boldsymbol{T}} - div\ \overset{\langle 2\rangle}{\boldsymbol{T}} + div\ div\ \overset{\langle 3\rangle}{\boldsymbol{T}})/\rho + \mathbf{x}^{\bullet\bullet}$$

and the **surface tractions**

$$(1.129) \qquad \mathbf{t}_2 := (\overset{\langle 2\rangle}{\boldsymbol{T}} - div_n \overset{\langle 3\rangle}{\boldsymbol{T}} - 2\ div_t \overset{\langle 3\rangle}{\boldsymbol{T}}) \cdot \mathbf{n} + (div_t\ \mathbf{n}) \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot\cdot\ \mathbf{n} \otimes \mathbf{n} - \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot\cdot\ grad_t\ \mathbf{n}$$

such that the forces are

$$\mathbf{f} = \int_{\mathscr{B}_t} \mathbf{b}_2\ dm + \int_{\partial\mathscr{B}_t} \mathbf{t}_2\ dA\ .$$

For $\delta\mathbf{v} \equiv \delta\boldsymbol{\omega} \times \mathbf{x}$ we obtain for the torque from (1.126) using (1.89)

$$(1.130) \qquad \mathbf{m}_O = \int_{\mathscr{B}_t} \mathbf{x} \times \mathbf{b}_2\ dV + \int_{\partial\mathscr{B}_t} \{\mathbf{x} \times \mathbf{t}_2 + 2\ axi(\overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n})\}\ dA\ .$$

The next theorem is an alternative form of EULER´s laws of motion (1.73), (1.74) using (1.127) and (1.130).

Theorem 1.20 (generalized EULER´s laws of motion)
A motion of the body is dynamically admissible if and only if the laws of motion

$$(1.131) \qquad \int_{\mathscr{B}_t} \mathbf{b}_2\ dm + \int_{\partial\mathscr{B}_t} \mathbf{t}_2\ dA = \int_{\mathscr{B}_t} \mathbf{x}^{\bullet\bullet}\ dm$$

$$(1.132) \qquad \int_{\mathscr{B}_t} \mathbf{x} \times \mathbf{b}_2\ dm + \int_{\partial\mathscr{B}_t} \{\mathbf{x} \times \mathbf{t}_2 + 2\ axi(\overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n})\}\ dA = \int_{\mathscr{B}_t} \mathbf{x} \times \mathbf{x}^{\bullet\bullet}\ dm$$

hold for the body for one observer (and hence for all).

By use of (1.92), (1.93), and (1.128) we obtain the field formulations.

Theorem 1.21 (extended CAUCHY´s laws)
A motion of the body is dynamically admissible if and only if

$$\text{(1.133)} \qquad div\ \overset{\langle 2\rangle}{\boldsymbol{T}} - div\, div \overset{\langle 3\rangle}{\boldsymbol{T}} + \rho\, \mathbf{b}_2 = \rho\, \mathbf{x}^{\bullet\bullet}$$

$$\text{(1.134)} \qquad \overset{\langle 2\rangle}{\boldsymbol{T}} = \overset{\langle 2\rangle}{\boldsymbol{T}}{}^{T}$$

hold everywhere in the body.

We are now able to reformulate the principle of virtual power extending Theorem 1.19 by (1.126).

Theorem 1.22 (integral version of PVP)
A motion of the body is dynamically admissible if and only if the balance of virtual power holds in the form

$$\text{(1.135)} \qquad \int_{\mathscr{B}_t} (\mathbf{b}_2 - \mathbf{x}^{\bullet\bullet}) \cdot \delta\mathbf{v}\, dm + \int_{\partial\mathscr{B}_t} [\mathbf{t}_2 \cdot \delta\mathbf{v} + (\overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot\, grad_n\, \delta\mathbf{v}]\, dA$$

$$= \int_{\mathscr{B}_t} (\overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot\, sym\, grad\, \delta\mathbf{v} + \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot\cdot\cdot\, grad\, grad\, \delta\mathbf{v})\, dV$$

for all vector fields $\delta\mathbf{v} \in \delta\mathscr{V}$ *for one observer (and hence for all).*

Proof. Let us assume that a motion of the body is dynamically admissible so that (1.133) and (1.134) hold. Then

$$\text{(1.136)} \qquad div\ \overset{\langle 2\rangle}{\boldsymbol{T}} - div\, div\ \overset{\langle 3\rangle}{\boldsymbol{T}} + \rho\, (\mathbf{b}_2 - \mathbf{x}^{\bullet\bullet}) = \mathbf{o}$$

holds in almost every point, and also

$$\text{(1.137)} \qquad \{div\ \overset{\langle 2\rangle}{\boldsymbol{T}} - div\, div\ \overset{\langle 3\rangle}{\boldsymbol{T}} + \rho\, (\mathbf{b}_2 - \mathbf{x}^{\bullet\bullet})\} \cdot \delta\mathbf{v} = 0$$

for any field $\delta\mathbf{v} \in \delta\mathscr{V}$. The integral over the body also vanishes

$$\text{(1.138)} \qquad \int_{\mathscr{B}_t} \{div\ \overset{\langle 2\rangle}{\boldsymbol{T}} - div\, div\ \overset{\langle 3\rangle}{\boldsymbol{T}} + \rho\, (\mathbf{b}_2 - \mathbf{x}^{\bullet\bullet})\} \cdot \delta\mathbf{v}\, dm = 0$$

so that we get with (1.119) and (1.125)

$$\text{(1.139)} \qquad \int_{\mathscr{B}_t} \rho\, (\mathbf{b}_2 - \mathbf{x}^{\bullet\bullet}) \cdot \delta\mathbf{v}\, dV + \int_{\partial\mathscr{B}_t} [\mathbf{t}_2 \cdot \delta\mathbf{v} + (\overset{\langle 3\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot\, grad_n\, \delta\mathbf{v}\}\, dA$$

$$= \int_{\mathscr{B}_t} (\overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot\, grad\, \delta\mathbf{v} + \overset{\langle 3\rangle}{\boldsymbol{T}} \cdot\cdot\cdot\, grad\, grad\, \delta\mathbf{v})\, dV\,.$$

Its difference with (1.135) is

(1.140) $$\int_{\mathscr{B}_t} \overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot \, skw\, grad\, \delta\mathbf{v}\, dV$$

using the decomposition

(1.141) $$grad\, \delta\mathbf{v} = sym\, grad\, \delta\mathbf{v} + skw\, grad\, \delta\mathbf{v} .$$

The integrand is zero in almost all points for any field $\delta\mathbf{v} \in \delta\mathscr{V}$ if and only if $\overset{\langle 2 \rangle}{\boldsymbol{T}}$ is symmetric (1.134). The reverse direction of the proof is now easy. If (1.135) holds for any field $\delta\mathbf{v} \in \delta\mathscr{V}$, then for an arbitrary but constant field we obtain (1.132), which completes our proof; *q.e.d.*

From the above theorem we also see the dynamic or **NEUMANN boundary conditions** for the surface tractions which can be prescribed, namely the vector field of the tractions

(1.142) $$\mathbf{t}_{2\,\mathrm{presc}} = \mathbf{t}_2$$

and the tensor field of the double tractions in normal direction in the following form

(1.143) $$\mathbf{s}_{2\,\mathrm{presc}} \otimes \mathbf{n} = \overset{\langle 3 \rangle}{\boldsymbol{T}} \cdot \mathbf{n}$$

since $grad_n\, \delta\mathbf{v}$ has only a normal component in its second entry. So two vector fields $\mathbf{t}_{2\,\mathrm{presc}}$ and $\mathbf{s}_{2\,\mathrm{presc}}$ are needed on the surface of the body.

The **DIRICHLET boundary conditions** are then the description of the displacement field $\mathbf{u}$ and its normal gradient $grad_n\, \mathbf{u}$ on the surface of the body or parts of it.

By identifying $\delta\mathbf{v}$ with the current velocity field $\mathbf{v} \in \delta\mathscr{V}$, we obtain the balance of power.

Theorem 1.23 (global work balance)
If a motion of the body is dynamically admissible then the balance of power states that the power of the external loads equals the change of the kinetic energy plus the stress power

(1.144) $$\int_{\mathscr{B}_t} \mathbf{b}_2 \cdot \mathbf{v}\, dm + \int_{\partial\mathscr{B}_t} [\mathbf{t}_2 \cdot \mathbf{v} + (\overset{\langle 3 \rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot\, grad_n\, \mathbf{v}] dA$$

$$= \left(\int_{\mathscr{B}_t} \tfrac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\, dm \right)^{\bullet} + \int_{\mathscr{B}_t} \left(\overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot\, sym\, grad\, \mathbf{v} + \overset{\langle 3 \rangle}{\boldsymbol{T}} \cdots grad\, grad\, \mathbf{v} \right) dV$$

for the body.

This balance gives only a necessary, but not a sufficient condition for the admissibility of a motion.

Example[13]**: Third gradient continuum** ($K \equiv 3$)

In this case the virtual power can be brought with (1.100) into the form

$$\delta\Pi(\boldsymbol{\chi}\,,\,\delta\mathbf{v}) \;=\; \int\limits_{\mathscr{B}_t} (\overset{\langle 1\rangle}{\boldsymbol{T}} \cdot \delta\mathbf{v} + \overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot\, grad\,\delta\mathbf{v} + \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot\cdot\, grad^{\,2}\,\delta\mathbf{v})\; dV$$

$$= \int\limits_{\mathscr{B}_t} (\overset{\langle 1\rangle}{\boldsymbol{T}} - div\ \overset{\langle 2\rangle}{\boldsymbol{T}} + div^2\ \overset{\langle 3\rangle}{\boldsymbol{T}} - div^3 \overset{\langle 4\rangle}{\boldsymbol{T}}\,) \cdot \delta\mathbf{v}\; dV \tag{1.145}$$

$$+ \int\limits_{\partial\mathscr{B}_t} \{[(\overset{\langle 2\rangle}{\boldsymbol{T}} - div\ \overset{\langle 3\rangle}{\boldsymbol{T}}\ + div^2\ \overset{\langle 4\rangle}{\boldsymbol{T}}\,) \cdot \mathbf{n}] \cdot \delta\mathbf{v}$$

$$+ [(\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}\,) \cdot \mathbf{n}] \cdot\cdot\, grad\,\delta\mathbf{v}$$

$$+ (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot\cdot\, grad^{\,2}\,\delta\mathbf{v}\}\; dA\,.$$

This form can be further reformulated by applying the divergence theorem on the surface terms. Herein we follow the lines of MINDLIN (1965)[14]. We again split the gradient and the divergence operations in normal and tangential parts after (1.120) and (1.121).

In particular we have the term

$$\int\limits_{\partial\mathscr{B}_t} [(\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}\,) \cdot \mathbf{n}] \cdot\cdot\, grad\,\delta\mathbf{v}\; dA \tag{1.146}$$

by (1.125)

$$= \int\limits_{\partial\mathscr{B}_t} \Big\{[(\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}\,) \cdot \mathbf{n}] \cdot\cdot\, grad_n\,\delta\mathbf{v}$$

$$+ \big[(div_t\,\mathbf{n})\,(\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}\,) \cdot\cdot\, \mathbf{n}\otimes\mathbf{n} - div_t\,\{(\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}\,) \cdot \mathbf{n}\}\big] \cdot \delta\mathbf{v}\Big\}\; dA$$

$$= \int\limits_{\partial\mathscr{B}_t} \Big\{[(\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}\,) \cdot \mathbf{n}] \cdot\cdot\, grad_n\,\delta\mathbf{v}$$

$$+ \big[(div_t\,\mathbf{n})\,(\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}\,) \cdot\cdot\, \mathbf{n}\otimes\mathbf{n}$$

$$- (\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}\,) \cdot\cdot\, grad_t\,\mathbf{n} - \{div_t(\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}\,)\} \cdot \mathbf{n}\big] \cdot \delta\mathbf{v}\Big\}\; dA\,.$$

By using (1.123) and (1.125) we reformulate the term

---

[13] This part has been coauthored by CHRISTIAN REIHER.

[14] see also JAVILI/ DELL'ISOLA/ STEINMANN (2013) and CORDERO/ FOREST/ BUSSO.

(1.147)
$$\int_{\partial\mathscr{B}_t} (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad_t\, grad\, \delta\mathbf{v}\, dA$$

$$= \int_{\partial\mathscr{B}_t} \{(div_t\, \mathbf{n})\, [grad\ \delta\mathbf{v} \cdot\cdot (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n})] \cdot \mathbf{n} - div_t (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdot\cdot grad\ \delta\mathbf{v}\}\, dA$$

$$= \int_{\partial\mathscr{B}_t} \left[(div_t\, \mathbf{n})\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot \mathbf{n}\otimes\mathbf{n} - div_t (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n})\right] \cdot\cdot grad\ \delta\mathbf{v}\, dA\,.$$

We can use the product rule in the form

(1.148)
$$div_t (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) = \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot grad_t\, \mathbf{n} + (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}$$

to continue the previous equation with (1.120)

$$\int_{\partial\mathscr{B}_t} (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad_t\, grad\, \delta\mathbf{v}\, dA$$

(1.149)
$$= \int_{\partial\mathscr{B}_t} [(div_t\, \mathbf{n})\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot \mathbf{n}\otimes\mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot grad_t\, \mathbf{n} - (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}]$$

$$\cdot\cdot (grad_n\, \delta\mathbf{v} + grad_t\, \delta\mathbf{v})\, dA\,.$$

The last integrand is with (1.122) and (1.123)

$$\int_{\partial\mathscr{B}_t} [(div_t\, \mathbf{n})\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot \mathbf{n}\otimes\mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot grad_t\, \mathbf{n} - (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot\cdot grad_t\ \delta\mathbf{v}\, dA$$

(1.150)
$$= \int_{\partial\mathscr{B}_t} \big\{div_t\, \{[(div_t\, \mathbf{n})\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot \mathbf{n}\otimes\mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot grad_t\, \mathbf{n} - (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}]^T \cdot \delta\mathbf{v}\}$$

$$- \{div_t\, [(div_t\, \mathbf{n})\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot \mathbf{n}\otimes\mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot grad_t\, \mathbf{n} - (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}]\} \cdot \delta\mathbf{v}\big\}\, dA$$

$$= \int_{\partial\mathscr{B}_t} \big\{(div_t\, \mathbf{n})\, [(div_t\, \mathbf{n})\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n}\otimes grad_t\, \mathbf{n} - (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot \mathbf{n}\otimes\mathbf{n}] \cdot \delta\mathbf{v}$$

$$- \{div_t\, [(div_t\, \mathbf{n})\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot \mathbf{n}\otimes\mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot grad_t\, \mathbf{n} - (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}]\} \cdot \delta\mathbf{v}\big\}\, dA$$

$$= \int_{\partial\mathscr{B}_t} \{(div_t\, \mathbf{n})^2\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n} - (div_t\, \mathbf{n})\, \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n}\otimes grad_t\, \mathbf{n} - (div_t\, \mathbf{n})\, (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot \mathbf{n}\otimes\mathbf{n}$$

$$- \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n}\otimes\mathbf{n}\otimes grad_t\, div_t\, \mathbf{n} - (div_t\, \mathbf{n})\, (div_t\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot \mathbf{n}\otimes\mathbf{n} - 2\, (div_t\, \mathbf{n})\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n}\otimes grad_t\, \mathbf{n}$$

$$+ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots grad_t^2\, \mathbf{n} + 2\,(div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\, grad_t\, \mathbf{n} + (div_t^2 \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}\} \cdot \delta\mathbf{v}\; dA\,.$$

This gives analogously for the first part of (1.149)

$$\text{(1.151)} \qquad \int_{\partial\mathscr{B}_t} (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad_t\, grad_n\, \delta\mathbf{v}\; dA$$

$$= \int_{\partial\mathscr{B}_t} [(div_t\, \mathbf{n})\, \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot\, \mathbf{n} \otimes \mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot\, grad_t\, \mathbf{n} - (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot\cdot\, grad_n\; \delta\mathbf{v}\; dA\,.$$

Because of the subsymmetries of $\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}$ leading to the rule

$$\text{(1.152)} \qquad \int_{\partial\mathscr{B}_t} (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad_n\, grad_t\, \delta\mathbf{v}\; dA = \int_{\partial\mathscr{B}_t} (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad_t\, grad_n\, \delta\mathbf{v}\; dA$$

we obtain

$$\int_{\partial\mathscr{B}_t} (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad\; grad\; \delta\mathbf{v}\; dA$$

$$\text{(1.153)} \qquad = \int_{\partial\mathscr{B}_t} \{(\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad_n\, grad_n\, \delta\mathbf{v} + 2\,(\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad_t\, grad_n\, \delta\mathbf{v}$$

$$+ (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad_t\, grad_t\, \delta\mathbf{v}\} \; dA$$

$$= \int_{\partial\mathscr{B}_t} \{(\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad_n\, grad_n\, \delta\mathbf{v}$$

$$+ 2\,[(div_t\, \mathbf{n})\, \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot\, \mathbf{n} \otimes \mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot\, grad_t\, \mathbf{n} - (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot\cdot\, grad_n\; \delta\mathbf{v}$$

$$+ [(div_t\, \mathbf{n})^2\, \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} - (div_t\, \mathbf{n}) \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes grad_t\, \mathbf{n}$$

$$- (div_t\, \mathbf{n})\, (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\, \mathbf{n} \otimes \mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes \mathbf{n} \otimes grad_t\, div_t\, \mathbf{n}$$

$$- (div_t\, \mathbf{n})\, (div_t\, \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\, \mathbf{n} \otimes \mathbf{n} - 2\,(div_t\, \mathbf{n})\, \mathbf{n} \otimes grad_t\, \mathbf{n} \cdots \overset{\langle 4\rangle}{\boldsymbol{T}}$$

$$+ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots grad_t^2\, \mathbf{n} + 2\,(div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\, grad_t\, \mathbf{n} + (div_t^2 \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot \delta\mathbf{v}\} \; dA\,.$$

We now substitute this into the above expression for the virtual power (1.145)

$$\delta\Pi(\boldsymbol{\chi}\,, \delta\mathbf{v}) = \int_{\mathscr{B}_t} (\overset{\langle 1\rangle}{\boldsymbol{T}} - div\; \overset{\langle 2\rangle}{\boldsymbol{T}} + div^2\; \overset{\langle 3\rangle}{\boldsymbol{T}} - div^3 \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \delta\mathbf{v}\; dV$$

$$
\begin{aligned}
&+ \int_{\partial\mathscr{B}_t} \Big([(\overset{\langle 2\rangle}{\boldsymbol{T}} - div\ \overset{\langle 3\rangle}{\boldsymbol{T}} + div^2\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot \delta\mathbf{v} \\
&+ [(\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot\cdot\ grad_n\, \delta\mathbf{v} \\
&+ [(div_t\, \mathbf{n})\, (\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\ \mathbf{n} \otimes \mathbf{n} \\
&- (\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\ grad_t\, \mathbf{n} - \{div_t (\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}})\} \cdot \mathbf{n}] \cdot \delta\mathbf{v} \\
&+ (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad_n\, grad_n\, \delta\mathbf{v} \\
&+ 2\, [(div_t\, \mathbf{n})\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot\ \mathbf{n} \otimes \mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot\ grad_t\, \mathbf{n} - (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot\cdot\ grad_n\ \delta\mathbf{v} \\
&+ [(div_t\, \mathbf{n})^2\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} - (div_t\, \mathbf{n}) \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes grad_t\, \mathbf{n} \\
&- (div_t\, \mathbf{n})\, (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\ \mathbf{n} \otimes \mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes \mathbf{n} \otimes grad_t\, div_t\, \mathbf{n} \\
&- (div_t\, \mathbf{n})\, (div_t\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\ \mathbf{n} \otimes \mathbf{n} - 2\, (div_t\, \mathbf{n})\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes grad_t\, \mathbf{n} \\
&+ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots grad_t^2\, \mathbf{n} + 2\, (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\ grad_t\, \mathbf{n} + (div_t^2 \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}] \cdot \delta\mathbf{v} \Big)\, dA\,.
\end{aligned}
\tag{1.154}
$$

Because of the subsymmetries of $\overset{\langle 4\rangle}{\boldsymbol{T}}$ we have $div_t\, div_n\ \overset{\langle 4\rangle}{\boldsymbol{T}} = div_n\, div_t\ \overset{\langle 4\rangle}{\boldsymbol{T}}$. The surface terms of the virtual power linear in $\delta\mathbf{v}$ are then

$$
\begin{aligned}
\mathbf{t}_3 \ :=\ &\{(\overset{\langle 2\rangle}{\boldsymbol{T}} - div\ \overset{\langle 3\rangle}{\boldsymbol{T}} + div^2\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n} \\
&+ (div_t\, \mathbf{n})\, (\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\ \mathbf{n} \otimes \mathbf{n} \\
&- (\overset{\langle 3\rangle}{\boldsymbol{T}} - div\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\ grad_t\, \mathbf{n} - \{div_t (\overset{\langle 3\rangle}{\boldsymbol{T}} - div_t\ \overset{\langle 4\rangle}{\boldsymbol{T}} - div_n\ \overset{\langle 4\rangle}{\boldsymbol{T}})\} \cdot \mathbf{n} \\
&+ (div_t\, \mathbf{n})^2\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} - (div_t\, \mathbf{n}) \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes grad_t\, \mathbf{n} \\
&- (div_t\, \mathbf{n})\, (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\ \mathbf{n} \otimes \mathbf{n} - \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes \mathbf{n} \otimes grad_t\, div_t\, \mathbf{n} \\
&- (div_t\, \mathbf{n})\, (div_t\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\ \mathbf{n} \otimes \mathbf{n} - 2\, (div_t\, \mathbf{n})\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes grad_t\, \mathbf{n} \\
&+ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots grad_t^2\, \mathbf{n} + 2\, (div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\ grad_t\, \mathbf{n} + (div_t^2 \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n}\} \cdot \delta\mathbf{v} \\
=\ &\{(\overset{\langle 2\rangle}{\boldsymbol{T}} - div_n\ \overset{\langle 3\rangle}{\boldsymbol{T}} - 2\, div_t\ \overset{\langle 3\rangle}{\boldsymbol{T}} + 3\, div_t^2\ \overset{\langle 4\rangle}{\boldsymbol{T}} + 3\, div_t\, div_n\ \overset{\langle 4\rangle}{\boldsymbol{T}} + 2\, div_n^2\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n} \\
&+ (div_t\, \mathbf{n})\, (\overset{\langle 3\rangle}{\boldsymbol{T}} - div_n\ \overset{\langle 4\rangle}{\boldsymbol{T}} - 3\, div_t\ \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\ \mathbf{n} \otimes \mathbf{n} + (div_t\, \mathbf{n})^2\ \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n}
\end{aligned}
\tag{1.155}
$$

$$+ (-\overset{\langle 3\rangle}{\boldsymbol{T}} + div_n \overset{\langle 4\rangle}{\boldsymbol{T}} + 3\, div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\, grad_t \mathbf{n} - 3\,(div_t \mathbf{n}) \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes grad_t \mathbf{n}$$

$$- \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes \mathbf{n} \otimes grad_t\, div_t \mathbf{n} + \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots grad_t^2 \mathbf{n} \big\} \cdot \delta\mathbf{v}$$

so that we finally obtain for the virtual power

$$\delta\Pi(\boldsymbol{\chi}, \delta\mathbf{v}) = \int_{\mathscr{B}_t} \mathbf{b}_{3\,\mathrm{gen}} \cdot \delta\mathbf{v}\, dm \tag{1.156}$$

$$+ \int_{\partial\mathscr{B}_t} [\mathbf{t}_3 \cdot \delta\mathbf{v} + \mathbf{S}_3 \cdot\cdot\, grad_n\, \delta\mathbf{v} + (\overset{\langle 4\rangle}{\boldsymbol{T}} \cdot \mathbf{n}) \cdots grad_n^2\, \delta\mathbf{v}]\, dA$$

with the **generalized body force**

$$\mathbf{b}_{3\,\mathrm{gen}} := (\overset{\langle 1\rangle}{\boldsymbol{T}} - div \overset{\langle 2\rangle}{\boldsymbol{T}} + div^2 \overset{\langle 3\rangle}{\boldsymbol{T}} - div^3 \overset{\langle 4\rangle}{\boldsymbol{T}}) / \rho \tag{1.157}$$

and the **surface tractions**

$$\mathbf{t}_3 := (\overset{\langle 2\rangle}{\boldsymbol{T}} - div_n \overset{\langle 3\rangle}{\boldsymbol{T}} - 2\, div_t \overset{\langle 3\rangle}{\boldsymbol{T}} + 3\, div_t^2 \overset{\langle 4\rangle}{\boldsymbol{T}} + 3\, div_t div_n \overset{\langle 4\rangle}{\boldsymbol{T}} + 2\, div_n^2 \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n} \tag{1.158}$$

$$+ (div_t \mathbf{n})(\overset{\langle 3\rangle}{\boldsymbol{T}} - div_n \overset{\langle 4\rangle}{\boldsymbol{T}} - 3\, div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\, \mathbf{n} \otimes \mathbf{n} + (div_t \mathbf{n})^2 \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n}$$

$$+ (-\overset{\langle 3\rangle}{\boldsymbol{T}} + div_n \overset{\langle 4\rangle}{\boldsymbol{T}} + 3\, div_t \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot\cdot\, grad_t \mathbf{n} - 3\,(div_t \mathbf{n}) \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes grad_t \mathbf{n}$$

$$- \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots \mathbf{n} \otimes \mathbf{n} \otimes grad_t\, div_t \mathbf{n} + \overset{\langle 4\rangle}{\boldsymbol{T}} \cdots grad_t^2 \mathbf{n}$$

and the **surface couple tractions**

$$\mathbf{S}_3 := (\overset{\langle 3\rangle}{\boldsymbol{T}} - 3\, div_t \overset{\langle 4\rangle}{\boldsymbol{T}} - div_n \overset{\langle 4\rangle}{\boldsymbol{T}}) \cdot \mathbf{n} + 2\,(div_t \mathbf{n}) \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot\, \mathbf{n} \otimes \mathbf{n} - 2 \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot\, grad_t \mathbf{n}\,. \tag{1.159}$$

For a constant $\delta\mathbf{v}$ this gives the force in the form

$$\mathbf{f} = \int_{\mathscr{B}_t} (\mathbf{b}_{3\,\mathrm{gen}} - \mathbf{x}^{\bullet\bullet})\, dm + \int_{\partial\mathscr{B}_t} \mathbf{t}_3\, dA\,. \tag{1.160}$$

For $\delta\mathbf{v} \equiv \delta\boldsymbol{\omega} \times \mathbf{x}$ we obtain the torque from (1.156) using (1.89)

$$\mathbf{m}_O = \int_{\mathscr{B}_t} \mathbf{x} \times (\mathbf{b}_{3\,\mathrm{gen}} - \mathbf{x}^{\bullet\bullet})\, dm + \int_{\partial\mathscr{B}_t} (\mathbf{x} \times \mathbf{t}_3 + 2\, axi\, \mathbf{S})\, dA\,. \tag{1.161}$$

The next theorem is an alternative form of EULER´s laws of motion (1.73), (1.74) using the above equations.

Theorem 1.24 (generalized EULER´s laws of motion)
A motion of the body is dynamically admissible if and only if the laws of motion

$$\int_{\mathscr{B}_t} \mathbf{b}_3\, dm + \int_{\partial\mathscr{B}_t} \mathbf{t}_3\, dA = \int_{\mathscr{B}_t} \mathbf{x}^{\bullet\bullet}\, dm \tag{1.162}$$

$$\int_{\mathscr{B}_t} \mathbf{x}\times\mathbf{b}_3\, dm + \int_{\partial\mathscr{B}_t} (\mathbf{x}\times\mathbf{t}_3 + 2\, axi\, \mathbf{S}_3)\, dA = \int_{\mathscr{B}_t} \mathbf{x}\times\mathbf{x}^{\bullet\bullet}\, dm \tag{1.163}$$

hold for the body for one observer (and hence for all).

By use of (1.92), (1.93), and (1.157) we obtain the field formulations.

Theorem 1.25 (extended CAUCHY´s laws)
A motion of the body is dynamically admissible if and only if

$$\overset{\langle 1\rangle}{\boldsymbol{T}} - div\, \overset{\langle 2\rangle}{\boldsymbol{T}} + div^2\, \overset{\langle 3\rangle}{\boldsymbol{T}} - div^3 \overset{\langle 4\rangle}{\boldsymbol{T}} + \rho\, \mathbf{b}_3 = \rho\, \mathbf{x}^{\bullet\bullet} \tag{1.164}$$

$$\overset{\langle 2\rangle}{\boldsymbol{T}} = \overset{\langle 2\rangle}{\boldsymbol{T}}{}^{T} \tag{1.165}$$

hold everywhere in the body.

We are now able to reformulate the principle of virtual power extending Theorem 1.19 by (1.156) using (1.145).

Theorem 1.26 (integral version of PVP)
A motion of the body is dynamically admissible if and only if the balance of virtual power holds in the form

$$\int_{\mathscr{B}_t} (\mathbf{b}_3 - \mathbf{x}^{\bullet\bullet})\cdot\delta\mathbf{v}\, dm + \int_{\partial\mathscr{B}_t} \left[\mathbf{t}_3\cdot\delta\mathbf{v} + \mathbf{S}_3 \cdot\cdot\, grad_n\, \delta\mathbf{v} + (\overset{\langle 4\rangle}{\boldsymbol{T}}\cdot\mathbf{n}) \cdots grad_n^2\, \delta\mathbf{v}\right] dA \tag{1.166}$$

$$= \int_{\mathscr{B}_t} (\overset{\langle 2\rangle}{\boldsymbol{T}} \cdot\cdot\, sym\, grad\, \delta\mathbf{v} + \overset{\langle 3\rangle}{\boldsymbol{T}} \cdots grad^2\, \delta\mathbf{v} + \overset{\langle 4\rangle}{\boldsymbol{T}} \cdot\cdot\cdot\cdot grad^3\, \delta\mathbf{v})\, dV$$

for all vector fields $\delta\mathbf{v} \in \delta\mathscr{V}$ *for one observer (and hence for all).*

Proof. The form (1.166) comes from (1.145). It is obvious that (1.166) gives for a constant field for $\delta\mathbf{v}$ the balance of linear momentum (1.162) and for a rotational field the balance of angular momentum (1.163), since the right-hand side of the equation is zero in both cases; *q. e. d.*

By identifying $\delta\mathbf{v}$ with the current velocity field $\mathbf{v} \in \delta\mathscr{V}$, we obtain the balance of power.

Theorem 1.27 (global work balance)
If a motion of the body is dynamically admissible then the balance of power states that the power of the external loads equals the change of the kinetic energy plus the stress power

$$\int_{\mathscr{B}_t} \mathbf{b}_3\cdot\mathbf{v}\, dm + \int_{\partial\mathscr{B}_t} \left[\mathbf{t}_3\cdot\mathbf{v} + \mathbf{S}_3 \cdot\cdot\, grad_n\, \mathbf{v} + (\overset{\langle 4\rangle}{\boldsymbol{T}}\cdot\mathbf{n}) \cdot\cdot\, grad_n^2\, \mathbf{v}\right] dA \tag{1.167}$$

$$= \left(\int\limits_{\mathscr{B}_t} \tfrac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\, dm \right)^{\bullet} + \int\limits_{\mathscr{B}_t} \left(\overset{\langle 2 \rangle}{\boldsymbol{T}} \cdot\cdot\, sym\, grad\, \mathbf{v} + \overset{\langle 3 \rangle}{\boldsymbol{T}} \cdot\cdot\cdot\, grad^2\, \mathbf{v} + \overset{\langle 4 \rangle}{\boldsymbol{T}} \cdot\cdot\cdot\cdot\, grad^3\, \mathbf{v} \right) dV$$

for the body.

This balance gives only a necessary, but not a sufficient condition for the admissibility of a motion.

From Theorem 1.26 we also see the dynamic or **NEUMANN boundary conditions** for the surface tractions that can be prescribed, namely the vector field of the tractions

(1.168) $$\mathbf{t}_{3\,\mathrm{presc}} = \mathbf{t}_3 ,$$

the dyadic tensor field of the double tractions in normal direction in the following form

(1.169) $$\mathbf{s}_{3\,\mathrm{presc}} \otimes \mathbf{n} = \mathbf{S}_3$$

since $grad_n\, \delta\mathbf{v}$ has only a normal component in its second entry, and the triadic field

(1.170) $$\mathbf{r}_{3\,\mathrm{presc}} \otimes \mathbf{n} \otimes \mathbf{n} = \overset{\langle 4 \rangle}{\boldsymbol{T}} \cdot \mathbf{n}$$

since $grad_n^2\, \delta\mathbf{v}$ has only normal components in its second and third entry, so that the three vector fields $\mathbf{t}_{3\,\mathrm{presc}}$, $\mathbf{s}_{3\,\mathrm{presc}}$, and $\mathbf{r}_{3\,\mathrm{presc}}$ are needed on the surface.

The **DIRICHLET boundary conditions** are then the description of the displacement field $\mathbf{u}$, its first normal gradient $grad_n\, \mathbf{u}$, and the second one $grad_n^2\, \mathbf{u}$ on the surface of the body or parts of it.

Of course, one can also apply mixed boundary conditions of the two types.

2. Material Theory of Second Gradient Materials

This chapter is mainly based on

> Bertram, A.: *Finite gradient elasticity and plasticity: a constitutive mechanical framework*. Continuum Mech. Thermodyn., **27**,6, 1039-1058 (2015)[15]

In the preceding chapter we described gradient materials and, in particular, second gradient materials ($K \equiv 2$). In what follows, we will consider the material theory for this class of materials, and particularize it for elasticity and plasticity.

While there is already a high number of contributions to gradient elasticity and plasticity within the linear format, publications on the same class of materials for large deformations are still limited, and among the few one will mainly find particular cases rather than a general theory.

Examples are TOUPIN (1964), GREEN/ RIVLIN (1964b), TRIANTAFYLLIDIS/ AIFANTIS (1986), AIFANTIS (1987), LEROY/ MOLINARI (1993), HWANG et al. (2002), GURTIN/ ANAND (2005), GURTIN (2010), NEFF (2008), NEFF/ CHELMINSKI/ ALBER (2009), EKH/ GRYMER/ RUNESSON/ SVEDBERG (2007), POLIZZOTTO (2009), SIEVERT (2011), SVENDSEN/ NEFF/ MENZEL (2009), BAMMANN (2001), LUSCHER/ MCDOWELL/ BRONKHORST (2010), CLEJA-TIGOIU (2002, 2010, 2012), ANAND/ ASLAN/ CHESTER (2012), MIEHE (2014), to mention just a few.

The starting point for introduction of the stresses here is the internal stress power. Once having defined appropriate material stress and strain variables, we consider gradient elasticity. After working out the effect of a change of the reference placement, the concept of elastic isomorphy allows us to define elastic symmetry transformations. With these concepts we are able to distinguish isotropic and anisotropic elasticity. As an example we give a linear form of the gradient elasticity.

These are the variables (fields) that are needed for such a theory:

- the specific body force $\mathbf{b}$ after (1.128)
- the second-order stresses $\overset{\langle 2 \rangle}{\boldsymbol{T}}$
- the third-order hyperstresses $\overset{\langle 3 \rangle}{\boldsymbol{T}}$

and in the case of kinematical or DIRICHLET boundary conditions

- the displacements on the boundary $\mathbf{u}$
- and the normal gradient of the displacements $grad_n\, \mathbf{u}$

or in the case of dynamic or NEUMANN boundary conditions

- the tractions prescribed on the boundary after (1.142) $\mathbf{t}_{\text{presc}}$

[15] see also BERTRAM (2013)

- the tensor field of the double tractions in normal direction after (1.143) $\mathbf{s}_{\text{presc}} \otimes \mathbf{n}$

or an appropriate mixture of the two types.

For the specific body force we expect a gravitational or a magnetic law. For the two stress tensors $\overset{\langle 2 \rangle}{\boldsymbol{T}}$ and $\overset{\langle 3 \rangle}{\boldsymbol{T}}$ material laws are needed. We will further on write $\mathbf{T} := \overset{\langle 2 \rangle}{\boldsymbol{T}}$ for the CAUCHY stresses and $\boldsymbol{G} := \overset{\langle 3 \rangle}{\boldsymbol{T}}$ for the hyperstresses to facilitate the notations.

To find appropriate forms of such material laws is the subject of material theory within continuum mechanics. They are subject of certain axioms called *principles of material theory* such as the

- *Principle of Determinism*
- *Principle of Local Action*
- *Principle of EUCLIDean Invariance* (or *Objectivity*)
- *Principle of Invariance under Rigid Body Modifications*
- *Principle of Thermodynamical Consistency*

and the like[16]. These principles can be straightforward extended from simple materials to gradient materials, as we will show later.

The *Principle of EUCLIDean Invariance* is already fulfilled by the statement of Theorem 1.17 that the higher-order stress tensors are objective. The other principles, however, lead to some concretization of the constitutive format.

[16] see TRUESDELL/NOLL (1965), BERTRAM (2005), and many other books.

Second-Order Kinematics

A non-trivial problem in finite deformation theory is the choice of appropriate and practical third-order variables. In contrast to POLIZZOTTO (2009), CIARLETTA/ MAUGIN (2011), and many others, we use material variables and reduced forms. This saves us from introducing objective time derivatives for the stresses and strains.

Some authors like TOUPIN (1962), SVENDSEN/ NEFF/ MENZEL (2009), and HWANG et al. (2002) use the gradient of GREEN´s strain tensor (0.38) as a higher-order material variable. Although principally equivalent to our procedure, it turns out that this choice leads to rather complicated expressions in elasto-plasticity even if applied only for the isotropic case.

In WANG (1973), TESTA/ VIANELLO (2005), and PODIO-GUIDUGLI/ VIANELLO (2013) the symmetry properties of gradient materials are also investigated (in the context of elastic fluids). These authors, however, apply them to variables which are neither spatial nor material and, thus, inappropriate for material modeling of solids.

We will next provide the necessary kinematical variables for a second gradient theory.

With respect to the natural bases of a spatial coordinate system $\{\varphi^i\}$ and a material one $\{\Psi^i\}$ the deformation gradient can be calculated as

$$\mathbf{F} = \frac{\partial \varphi^k}{\partial \Psi^i}\, \mathbf{r}_{\varphi k} \otimes \mathbf{r}_{\Psi}{}^{i}. \tag{2.1}$$

With the transformation of the nablas are after (0.29) we get for the gradient of any tensor field ϕ the transformations

$$Grad\, \phi = (grad\, \phi)\, \mathbf{F} \qquad \text{and} \qquad grad\, \phi = (Grad\, \phi)\, \mathbf{F}^{-1} \tag{2.2}$$

where it is understood that ϕ is in the LAGRANGEan description if *Grad* is applied, and in the EULERean one if *grad* is.

In particular, we have the identities

$$grad\, \mathbf{x} = \mathbf{x} \otimes \nabla = \mathbf{I} \quad \Rightarrow \quad grad\, grad\, \mathbf{x} = \boldsymbol{0} \tag{2.3}$$

$$Grad\, \mathbf{x}_0 = \mathbf{x}_0 \otimes \nabla_0 = \mathbf{I} \quad \Rightarrow \quad Grad\, Grad\, \mathbf{x}_0 = \boldsymbol{0} \tag{2.4}$$

with $\mathbf{x}$ being the position vector in the current placement and $\mathbf{x}_0$ in the reference placement.

The second gradient of the motions is the gradient of the deformation gradient

$$Grad\, Grad\, \boldsymbol{\chi} = Grad\, \mathbf{F} = \boldsymbol{\chi} \otimes \nabla_0 \otimes \nabla_0 \tag{2.5}$$

which is a triadic (field) with the right subsymmetry by definition.

We will later-on need the differential of the inverse of the deformation gradient. First we note that the product rule gives

$$d(\mathbf{F}^{-1} \cdot \mathbf{F}) = \mathbf{0} = d\mathbf{F}^{-1} \cdot \mathbf{F} + \mathbf{F}^{-1} \cdot d\mathbf{F}$$

so that

(2.6) $$d\mathbf{F}^{-1} = -\mathbf{F}^{-1}\cdot d\mathbf{F}\cdot\mathbf{F}^{-1}.$$

With this representation we obtain

$$\begin{aligned} d(\mathbf{F}^{-1}) &= Grad\,\mathbf{F}^{-1}\cdot d\mathbf{x}_0 \\ &= -\mathbf{F}^{-1}\cdot d\mathbf{F}\cdot\mathbf{F}^{-1} \\ &= -\mathbf{F}^{-1}\cdot(Grad\,\mathbf{F}\cdot d\mathbf{x}_0)\cdot\mathbf{F}^{-1} \\ &= -\mathbf{F}^{-1}\cdot[(Grad\,\mathbf{F})^t\cdot\mathbf{F}^{-1}]^t\cdot d\mathbf{x}_0 \\ &= -\mathbf{F}^{-1}\cdot[(Grad\,\mathbf{F})\cdot\mathbf{F}^{-1}]^t\cdot d\mathbf{x}_0\,. \end{aligned}$$

Thus

(2.7) $$Grad\,\mathbf{F}^{-1} = -\mathbf{F}^{-1}\cdot[(Grad\,\mathbf{F})\cdot\mathbf{F}^{-1}]^t.$$

Later, we will use the following expression for the gradient of the product of two second-order tensor fields $\mathbf{T}$ and $\mathbf{S}$

(2.8) $$Grad\,(\mathbf{S}\cdot\mathbf{T}) = \mathbf{S}\cdot Grad\,\mathbf{T} + [(Grad\,\mathbf{S})^t\cdot\mathbf{T}]^t.$$

This can be verified by the following calculation

$$Grad\,(\mathbf{S}\,\mathbf{T}) = (\mathbf{S}\,\mathbf{T})\otimes\nabla_L = \mathbf{S}\,Grad\,\mathbf{T} + \overset{\downarrow}{\mathbf{S}}\,\mathbf{T}\otimes\nabla_L$$

where the arrows indicate the term to which nabla has to be applied. The last term is with respect to an orthonormal basis

$$\overset{\downarrow}{\mathbf{S}}\,\mathbf{T}\otimes\nabla_L = S_{ij,k}\,T_{jm}\,\mathbf{e}_i\otimes\mathbf{e}_m\otimes\mathbf{e}_k$$

while

$$\begin{aligned} [(Grad\,\mathbf{S})^t\,\mathbf{T}]^t &= [(S_{ij,k}\,\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k)^t\,\mathbf{T}]^t = [S_{ij,k}\,\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_j\,\mathbf{T}]^t \\ &= [S_{ij,k}\,T_{jm}\,\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_m]^t = S_{ij,k}\,T_{jm}\,\mathbf{e}_i\otimes\mathbf{e}_m\otimes\mathbf{e}_k \end{aligned}$$

gives the same. An analogous result holds also for the gradient in the EULERean description.

The second velocity gradient can be related to the material time derivative of a material deformation tensor

(2.9) $$grad\,grad\,\mathbf{v} = grad\,\mathbf{L}$$

with (0.35) $$= grad\,(\mathbf{F}^{\bullet}\cdot\mathbf{F}^{-1})$$

with (2.2) $$= Grad\,(\mathbf{F}^{\bullet}\cdot\mathbf{F}^{-1})\cdot\mathbf{F}^{-1}$$

with (2.8) $$= \mathbf{F}^{\bullet}\cdot(Grad\,\mathbf{F}^{-1})\cdot\mathbf{F}^{-1} + [(Grad\,\mathbf{F}^{\bullet})^t\cdot\mathbf{F}^{-1}]^t\cdot\mathbf{F}^{-1}$$

with (2.7) $$= -\mathbf{F}^{\bullet}\cdot\mathbf{F}^{-1}\cdot[(Grad\,\mathbf{F})\cdot\mathbf{F}^{-1}]^t\cdot\mathbf{F}^{-1} + [(Grad\,\mathbf{F}^{\bullet})\cdot\mathbf{F}^{-1}]^t\cdot\mathbf{F}^{-1}$$

with (0.15) $$= \mathbf{F}^{-T} * [-\mathbf{F}^T\cdot\mathbf{F}^{\bullet}\cdot\mathbf{F}^{-1}\cdot Grad\,\mathbf{F} + \mathbf{F}^T\cdot Grad\,\mathbf{F}^{\bullet}]$$

$$= \mathbf{F}^{-T} * [-\mathbf{F}^T\cdot\mathbf{F}\cdot\mathbf{F}^{-1}\cdot\mathbf{F}^{\bullet}\cdot\mathbf{F}^{-1}\cdot Grad\,\mathbf{F} + \mathbf{F}^T\cdot\mathbf{F}\cdot\mathbf{F}^{-1}\cdot Grad\,\mathbf{F}^{\bullet}]$$

with (0.16) $$= \mathbf{F}^{-T}\circ[-\mathbf{F}^{-1}\cdot\mathbf{F}^{\bullet}\cdot\mathbf{F}^{-1}\cdot Grad\,\mathbf{F} + \mathbf{F}^{-1}\cdot Grad\,\mathbf{F}^{\bullet}]$$

with (2.7) $= \mathbf{F}^{-T} \circ [\mathbf{F}^{-1\bullet} \cdot Grad\, \mathbf{F} + \mathbf{F}^{-1} \cdot Grad\, \mathbf{F}^{\bullet}]$

$= \mathbf{F}^{-T} \circ \boldsymbol{K}^{\bullet}$

with

(2.10) $\boldsymbol{K} := \mathbf{F}^{-1} \cdot Grad\, \mathbf{F}$

which has the right subsymmetry by definition.

This triadic has been used by CHAMBON/ CAILLERIE/ TAMAGNINI (2001), FOREST/ SIEVERT (2003), CLEJA-TIGOIU (2013), STEINMANN (2015), and other authors[17]. It is sometimes called the *connection* or *curvature*, although this might lead to confusion with nabla or the well-known RIEMANNean curvature tensor. Therefore, we prefer to call $\boldsymbol{K}$ **configuration tensor**.

The product $\circ$ in (2.9) can be interpreted as the push-forward from the reference placement to the current placement taking into account the different transformation behavior of tangent and cotangent vectors.

It can be shown[18] that the configuration tensor $\boldsymbol{K}$ can be determined by the right CAUCHY-GREEN tensor $\mathbf{C}$ and its gradient according to

(2.11) $\boldsymbol{K} = \mathbf{C}^{-1} \cdot Sym\, Grad\, \mathbf{C}$

with the following symmetrization of a triadic

(2.12) $Sym\, T^{ijk} := \frac{1}{2} (T^{ijk} + T^{ikj} - T^{kji})$.

These fields can be calculated with respect to the natural bases of the coordinate systems $\{\varphi^i\}$ in the current placement and $\{\Psi^i\}$ in the reference placement

(2.13) $\mathbf{F} = \frac{\partial \varphi^k}{\partial \Psi^i}\, \mathbf{r}_{\varphi k} \otimes \mathbf{r}_{\Psi}{}^{i}$

(2.14) $\mathbf{F}^{-1} = \frac{\partial \Psi^l}{\partial \varphi^k}\, \mathbf{r}_{\Psi l} \otimes \mathbf{r}_{\varphi}{}^{k}$

(2.15) $Grad\, \mathbf{F} = (\frac{\partial \varphi^k}{\partial \Psi^i}\, \mathbf{r}_{\varphi k} \otimes \mathbf{r}_{\Psi}{}^{i}) \otimes \frac{\partial}{\partial \Psi^j}\, \mathbf{r}_{\Psi}{}^{j}$

$= \frac{\partial^2 \varphi^k}{\partial \Psi^i \partial \Psi^j}\, \mathbf{r}_{\varphi k} \otimes \mathbf{r}_{\Psi}{}^{i} \otimes \mathbf{r}_{\Psi}{}^{j} + \frac{\partial \varphi^k}{\partial \Psi^i}\, (\frac{\partial}{\partial \Psi^j}\, \mathbf{r}_{\varphi k}) \otimes \mathbf{r}_{\Psi}{}^{i} \otimes \mathbf{r}_{\Psi}{}^{j}$

$+ \frac{\partial \varphi^k}{\partial \Psi^i}\, \mathbf{r}_{\varphi k} \otimes (\frac{\partial}{\partial \Psi^j}\, \mathbf{r}_{\Psi}{}^{i}) \otimes \mathbf{r}_{\Psi}{}^{j}$

$= \frac{\partial^2 \varphi^k}{\partial \Psi^i \partial \Psi^j}\, \mathbf{r}_{\varphi k} \otimes \mathbf{r}_{\Psi}{}^{i} \otimes \mathbf{r}_{\Psi}{}^{j} + \frac{\partial \varphi^k}{\partial \Psi^i}\, (\frac{\partial \mathbf{r}_{\varphi k}}{\partial \varphi^m}\, \frac{\partial \varphi_m}{\partial \Psi^j}) \otimes \mathbf{r}_{\Psi}{}^{i} \otimes \mathbf{r}_{\Psi}{}^{j}$

[17] see also NOLL (1967)

[18] KRAWIETZ (1993), see also HWANG et al. (2002)

$$+\frac{\partial \varphi^k}{\partial \Psi^i}\ \mathbf{r}_{\varphi k} \otimes \frac{\partial \mathbf{r}_\Psi{}^i}{\partial \Psi^j} \otimes \mathbf{r}_\Psi{}^j$$

$$= \frac{\partial^2 \varphi^k}{\partial \Psi^i \partial \Psi^j}\ \mathbf{r}_{\varphi k} \otimes \mathbf{r}_\Psi{}^i \otimes \mathbf{r}_\Psi{}^j + \frac{\partial \varphi^l}{\partial \Psi^i}\frac{\partial \varphi^m}{\partial \Psi^j}\ (\frac{\partial \mathbf{r}_{\varphi l}}{\partial \varphi^m} \cdot \mathbf{r}_\varphi{}^k)\ \mathbf{r}_{\varphi k} \otimes \mathbf{r}_\Psi{}^i \otimes \mathbf{r}_\Psi{}^j$$

$$+\frac{\partial \varphi^k}{\partial \Psi^l}\ (\frac{\partial \mathbf{r}_\Psi{}^l}{\partial \Psi^j} \cdot \mathbf{r}_{\Psi i})\ \mathbf{r}_{\varphi k} \otimes \mathbf{r}_\Psi{}^i \otimes \mathbf{r}_\Psi{}^j$$

$$= [\frac{\partial^2 \varphi^k}{\partial \Psi^i \partial \Psi^j} + \frac{\partial \varphi^l}{\partial \Psi^i}\frac{\partial \varphi^m}{\partial \Psi^j}\ (\frac{\partial \mathbf{r}_{\varphi l}}{\partial \varphi^m} \cdot \mathbf{r}_\varphi{}^k) + \frac{\partial \varphi^k}{\partial \Psi^l}(\frac{\partial \mathbf{r}_\Psi{}^l}{\partial \Psi^j} \cdot \mathbf{r}_{\Psi i})]\ \mathbf{r}_{\varphi k} \otimes \mathbf{r}_\Psi{}^i \otimes \mathbf{r}_\Psi{}^j$$

and

$$\boldsymbol{K} = \frac{\partial \Psi^p}{\partial \varphi^q}\ \mathbf{r}_{\Psi p} \otimes \mathbf{r}_\varphi{}^q\ [\frac{\partial^2 \varphi^k}{\partial \Psi^i \partial \Psi^j} + \frac{\partial \varphi^l}{\partial \Psi^i}\frac{\partial \varphi^m}{\partial \Psi^j}\ (\frac{\partial \mathbf{r}_{\varphi l}}{\partial \varphi^m} \cdot \mathbf{r}_\varphi{}^k)$$

$$+\frac{\partial \varphi^k}{\partial \Psi^l}\ (\frac{\partial \mathbf{r}_\Psi{}^l}{\partial \Psi^j} \cdot \mathbf{r}_{\Psi i})]\ \mathbf{r}_{\varphi k} \otimes \mathbf{r}_\Psi{}^i \otimes \mathbf{r}_\Psi{}^j$$

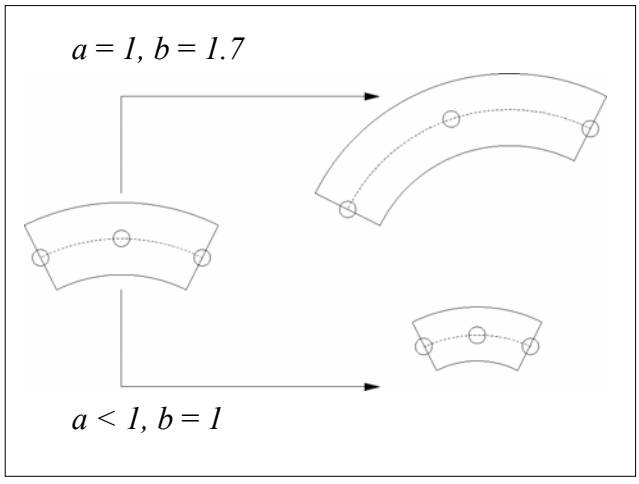
$$= \frac{\partial \Psi^p}{\partial \varphi^k}\ [\frac{\partial^2 \varphi^k}{\partial \Psi^i \partial \Psi^j} + \frac{\partial \varphi^l}{\partial \Psi^i}\frac{\partial \varphi^m}{\partial \Psi^j}\ (\frac{\partial \mathbf{r}_{\varphi l}}{\partial \varphi^m} \cdot \mathbf{r}_\varphi{}^k)$$

$$+ \frac{\partial \varphi^k}{\partial \Psi^l}(\frac{\partial \mathbf{r}_\Psi{}^l}{\partial \Psi^j} \cdot \mathbf{r}_{\Psi i})]\ \mathbf{r}_{\Psi p} \otimes \mathbf{r}_\Psi{}^i \otimes \mathbf{r}_\Psi{}^j. \tag{2.16}$$

Example We consider the bending and tension of a ring segment. The motion is given by the coordinate transformation

$$\begin{aligned} r &= \chi^1(R, \Theta, Z) = a\,R \\ \vartheta &= \chi^2(R, \Theta, Z) = b\,\Theta \\ z &= \chi^3(R, \Theta, Z) = Z \end{aligned} \tag{2.17}$$

with respect to a cylindrical COOS $\{R, \Theta, Z\}$ for the initial or reference placement, and another cylindrical COOS $\{r, \vartheta, z\}$ for the current placement.



The tangent bases to these coordinates are

$$\mathbf{r}_{\varphi 1} = \mathbf{e}_r(\vartheta) \qquad \mathbf{r}_{\varphi 2} = r\,\mathbf{e}_\vartheta(\vartheta) \qquad \mathbf{r}_{\varphi 3} = \mathbf{e}_z$$

$$\mathbf{r}_{\Psi 1} = \mathbf{e}_r(\Theta) \qquad \mathbf{r}_{\Psi 2} = R\,\mathbf{e}_\theta(\Theta) \qquad \mathbf{r}_{\Psi 3} = \mathbf{e}_z$$

and the gradient bases

$$\mathbf{r}_\varphi{}^1 = \mathbf{e}_r(\vartheta) \qquad \mathbf{r}_\varphi{}^2 = \frac{1}{r}\,\mathbf{e}_\vartheta(\vartheta) \qquad \mathbf{r}_\varphi{}^3 = \mathbf{e}_z$$

$$\mathbf{r}_\Psi{}^1 = \mathbf{e}_R(\Theta) \qquad \mathbf{r}_\Psi{}^2 = \frac{1}{R}\,\mathbf{e}_\theta(\Theta) \qquad \mathbf{r}_\Psi{}^3 = \mathbf{e}_z\,.$$

The deformation gradient with respect to the natural bases of these coordinates and the normed ones is

$$\mathbf{F} = \frac{\partial \varphi^k}{\partial \Psi^i}\,\mathbf{r}_{\varphi k} \otimes \mathbf{r}_\Psi{}^i = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{r}_{\varphi k} \otimes \mathbf{r}_\Psi{}^i = \begin{bmatrix} a & 0 & 0 \\ 0 & \dfrac{br}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_{\varphi k} \otimes \mathbf{e}_{\Psi i} \tag{2.18}$$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & ab & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_{\varphi k} \otimes \mathbf{e}_{\Psi i}$$

so that

$$\mathbf{F}^{-1} = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & (ab)^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_{\Psi i} \otimes \mathbf{e}_{\varphi k}$$

which describes a state of plane strain. The right CAUCHY-GREEN tensor is

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$$

$$= (a\,\mathbf{r}_\Psi{}^1 \otimes \mathbf{r}_{\varphi 1} + b\,\mathbf{r}_\Psi{}^2 \otimes \mathbf{r}_{\varphi 2} + \mathbf{r}_\Psi{}^3 \otimes \mathbf{r}_{\varphi 3}) \cdot (a\,\mathbf{r}_{\varphi 1} \otimes \mathbf{r}_\Psi{}^1 + b\,\mathbf{r}_{\varphi 2} \otimes \mathbf{r}_\Psi{}^2 + \mathbf{r}_{\varphi 3} \otimes \mathbf{r}_\Psi{}^3)$$

$$= \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{r}_\Psi{}^i \otimes \mathbf{r}_\Psi{}^j = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & \dfrac{b^2 r^2}{R^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_{\Psi i} \otimes \mathbf{e}_{\Psi j}$$

$$= \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 b^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_{\Psi i} \otimes \mathbf{e}_{\Psi j} \,. \tag{2.19}$$

The determinant of $\mathbf{F}$ is $J = a^2\, b$. Incompressibility would be characterized by $a^2 = 1/b$. In this case we would obtain

$$\mathbf{F}_{inc} = \begin{bmatrix} a & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_{\varphi k} \otimes \mathbf{e}_{\Psi i} \,. \tag{2.20}$$

The LAGRANGEan and the EULERean nabla operators are

$$\boldsymbol{\nabla}_0 := \frac{\partial}{\partial \Psi^i}\, \mathbf{r}_\Psi{}^i \qquad\qquad \boldsymbol{\nabla} := \frac{\partial}{\partial \varphi^i}\, \mathbf{r}_\varphi{}^i .$$

Further

$$Grad\, \mathbf{F} = \begin{bmatrix} a & 0 & 0 \\ 0 & ab & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_{\varphi k} \otimes \mathbf{e}_{\Psi j} \otimes \frac{\partial}{\partial \Psi^i}\, \mathbf{r}_\Psi{}^i \tag{2.21}$$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & ab & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_{\varphi k} \otimes \mathbf{e}_{\Psi j} \otimes \frac{\partial}{\partial \Theta}\, \mathbf{r}_\Psi{}^2$$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & ab & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\frac{\partial}{\partial \vartheta}\, \mathbf{e}_{\varphi k}\, \frac{\partial \vartheta}{\partial \Theta} \otimes \mathbf{e}_{\Psi j} + \mathbf{e}_{\varphi k} \otimes \frac{\partial}{\partial \Theta}\, \mathbf{e}_{\Psi j} \right) \otimes \frac{1}{R}\, \mathbf{e}_{\Psi 2}$$

$$= \left(a \frac{\partial}{\partial \vartheta}\, \mathbf{e}_{\varphi 1}\, b \otimes \mathbf{e}_{\Psi 1} + ab \frac{\partial}{\partial \vartheta}\, \mathbf{e}_{\varphi 2}\, b \otimes \mathbf{e}_{\Psi 2} + a\, \mathbf{e}_{\varphi 1} \otimes \frac{\partial}{\partial \Theta}\, \mathbf{e}_{\Psi 1} \right.$$

$$\left. + ab\, \mathbf{e}_{\varphi 2} \otimes \frac{\partial}{\partial \Theta}\, \mathbf{e}_{\Psi 2} \right) \otimes \frac{1}{R}\, \mathbf{e}_{\Psi 2}$$

$$= \frac{1}{R} \left(ab\, \mathbf{e}_{\varphi 2} \otimes \mathbf{e}_{\Psi 1} - ab^2\, \mathbf{e}_{\varphi 1} \otimes \mathbf{e}_{\Psi 2} + a\, \mathbf{e}_{\varphi 1} \otimes \mathbf{e}_{\Psi 2} - ab\, \mathbf{e}_{\varphi 2} \otimes \mathbf{e}_{\Psi 1} \right) \otimes \mathbf{e}_{\Psi 2}$$

$$= R\, a\, (1 - b^2)\, \mathbf{r}_{\varphi 1} \otimes \mathbf{r}_\Psi{}^2 \otimes \mathbf{r}_\Psi{}^2 = \frac{1}{R}\, (a - ab^2)\, \mathbf{e}_{\varphi 1} \otimes \mathbf{e}_{\Psi 2} \otimes \mathbf{e}_{\Psi 2}$$

and

$$\boldsymbol{K} = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & (ab)^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_{\Psi i} \otimes \mathbf{e}_{\varphi k} \, \frac{1}{R} \, (a - ab^2) \, \mathbf{e}_{\varphi 1} \otimes \mathbf{e}_{\Psi 2} \otimes \mathbf{e}_{\Psi 2} \tag{2.22}$$

$$= R \, (1 - b^2) \, \mathbf{r}_{\Psi 1} \otimes \mathbf{r}_{\Psi}{}^{2} \otimes \mathbf{r}_{\Psi}{}^{2} = \frac{1 - b^2}{R} \, \mathbf{e}_{\Psi 1} \otimes \mathbf{e}_{\Psi 2} \otimes \mathbf{e}_{\Psi 2}$$

which is independent of a, and for $b \equiv 1$ it vanishes completely.

Stress Power

As a starting point for our gradient theory we have chosen the stress power after (1.144)

$$\Pi_i := \int_{\mathscr{B}_t} 1/\rho\,(\mathbf{T} \cdot\cdot \mathit{grad}\,\mathbf{v} + \boldsymbol{G} \cdots \mathit{grad}\,\mathit{grad}\,\mathbf{v})\,dm\,. \tag{2.23}$$

$\mathbf{T}$ is symmetric because of the balance of angular momentum (1.134) , and the first term can be substituted by $\mathbf{T} \cdot\cdot \mathbf{D}$.

grad grad $\mathbf{v}$ has the right subsymmetry by definition. So the same symmetry can be imposed on $\boldsymbol{G}$ without loss of generality within the present format. The balance of angular momentum does not impose any restriction on $\boldsymbol{G}$.

Since $\mathbf{D}$ and *grad grad* $\mathbf{v}$ are objective fields after (1.39), we can again see that the stress power is objective if $\mathbf{T}$ and $\boldsymbol{G}$ are objective.

For material modeling, however, it is more practical to use invariant variables instead of objective ones. Therefore we will next bring the stress power into a material form which is invariant under EUCLIDean transformations. For this purpose we use (0.13), (0.17), (0.38), (2.9)

$$\Pi_i = \int_{\mathscr{B}_0} \pi_i\,dm$$

with the specific stress power

$$\begin{aligned} \pi_i &:= 1/\rho_0\,J\,[\mathbf{T} \cdot\cdot (\mathbf{F}^{-T} * \tfrac{1}{2}\,\mathbf{C}^{\bullet}) + \boldsymbol{G} \cdots (\mathbf{F}^{-T} \circ \boldsymbol{K}^{\bullet})] \\ &= 1/\rho_0\,[\tfrac{1}{2}\,J\,(\mathbf{F}^{-1} * \mathbf{T}) \cdot\cdot \mathbf{C}^{\bullet} + (\mathbf{F}^{-1} \circ J\,\boldsymbol{G}) \cdots \boldsymbol{K}^{\bullet}] \\ &= 1/\rho_0\,[\tfrac{1}{2}\,\mathbf{S} \cdot\cdot \mathbf{C}^{\bullet} + \boldsymbol{H} \cdots \boldsymbol{K}^{\bullet}] \end{aligned} \tag{2.24}$$

with the two material stress tensors, namely the **second PIOLA-KIRCHHOFF tensor**

$$\mathbf{S} := \mathbf{F}^{-1} * J\,\mathbf{T} \tag{2.25}$$

and the third-order **material hyperstress tensor** defined as

$$\boldsymbol{H} := \mathbf{F}^{-1} \circ J\,\boldsymbol{G}\,. \tag{2.26}$$

The product $\circ$ in (2.26) is the pull-back of $\boldsymbol{G}$ from the current placement to the reference placement, the same as the RAYLEIGH product in (2.25).

The following tensors are invariant: $\mathbf{C}$, $\boldsymbol{K}$ and their duals $\mathbf{S}$, $\boldsymbol{H}$, which makes them good candidates for material modeling.

2.1 Finite Gradient Elasticity

We will now extend the definition of a simple elastic material to a second-order one by enlarging the set of independent variables by the second deformation gradient.

Definition 2.1. We call a material **second-order elastic** if the stress tensors are functions of the motion, the deformation gradient, and the gradient of the deformation gradient

(2.27) $$\mathbf{T} = f(\boldsymbol{\chi}, \mathit{Grad}\,\boldsymbol{\chi}, \mathit{Grad\,Grad}\,\boldsymbol{\chi}) = f(\boldsymbol{\chi}, \mathbf{F}, \mathit{Grad}\,\mathbf{F})$$

(2.28) $$\boldsymbol{G} = F(\boldsymbol{\chi}, \mathit{Grad}\,\boldsymbol{\chi}, \mathit{Grad\,Grad}\,\boldsymbol{\chi}) = F(\boldsymbol{\chi}, \mathbf{F}, \mathit{Grad}\,\mathbf{F})$$

where all variables are taken at the same material point at the same instant of time.

We already know that both stress tensors $\mathbf{T}$ and $\boldsymbol{G}$ behave like objective tensors under EUCLIDean transformations (Theorem 1.17).

These constitutive equations can be further reduced by means of the EUCLIDean invariance principle[19], which we assume in the following form.

Axiom 2.1 Principle of Invariance under Rigid Body Modifications
The stress power at the end of a motion $\boldsymbol{\chi}(\mathbf{x}_0, \tau)\big|_{\tau=0}^{t}$ *in some time interval* $[0,t]$ *equals the stress power after superimposing a rigid body motion upon the original motion*

(2.29) $$\{\mathbf{Q}(\tau) \cdot \boldsymbol{\chi}(\mathbf{x}_0, \tau) + \mathbf{c}(\tau)\big|_{\tau=0}^{t}\}$$

with arbitrary differentiable time functions $\mathbf{Q}(\tau) \in \mathcal{O}rth$ *and* $\mathbf{c}(\tau) \in \mathcal{V}^3$.

The transformation (2.29) has nothing to do with changes of observers (1.33). The invariance under observer changes does not lead to reduced forms (see BERTRAM/ SVENDSEN 2001), in contrast to the *Principle of Invariance under Rigid Body Modifications* above, as we will next show.

We will further on denote the binary set of elements like $\{\mathbf{T}, \boldsymbol{G}\}$ consisting of all dyadics $\mathbf{T}$ and triadics $\boldsymbol{G}$ with right subsymmetry by $\mathcal{L}in\mathcal{C}omb$. This space has the dimension 9 + 18 = 27. A subset of this space is formed by all positive-definite and symmetric second-order tensors $\mathbf{C}$ and all triadics with right subsymmetry $\boldsymbol{K}$, which we call the **space of configurations** $\mathcal{C}onf$. This set is imbedded in a space with dimension 6 + 18 = 24. Another subset of $\mathcal{L}in\mathcal{C}omb$ is formed by all invertible dyadics and all triadics with right subsymmetry, which we denote by $\mathcal{I}nv\mathcal{C}omb$. We can further restrict this subset to those dyadics which are unimodular (determinant equal ± 1) denoted by $\mathcal{U}nim\mathcal{C}omb$.

Reduced forms of the elastic laws (2.27) and (2.28) are then

(2.30) $$\mathbf{S} = k(\mathbf{C}, \boldsymbol{K})$$

(2.31) $$\boldsymbol{H} = K(\mathbf{C}, \boldsymbol{K})$$

by two elastic laws which are defined on the space of configurations

[19] see BERTRAM (2005), therein called PISM

$$k : \mathscr{Conf} \to \mathscr{Sym}$$

$$K : \mathscr{Conf} \to \mathscr{Triad}$$

since all involved variables are material and thus invariant under both changes of observers and superimposed rigid body motions. The proof of this statement is a straightforward extension of the rationale given in, *e.g.*, TRUESDE/ NOLL (1965). This means that the elastic laws (2.27) and (2.28) of every second-order elastic material law that obeys the *Principle of Invariance under Rigid Body Modifications* can be brought into these forms.

There are strong physical arguments that all elastic materials should also be hyperelastic.

Definition 2.2. We will call a material a **second-order hyperelastic material** if there exists a specific **elastic energy**

$$w : \mathscr{Conf} \to \mathscr{R}$$

such that the specific stress power after (2.24) equals

(2.32) $$\pi_i = w(\mathbf{C}, \boldsymbol{K})^{\bullet}$$

Note that the elastic energy defined on the configurations is already in a reduced form. By the chain rule and (2.24), this gives

(2.33) $$1/\rho_0 \, (\tfrac{1}{2} \, \mathbf{S} \cdot\cdot \mathbf{C}^{\bullet} + \boldsymbol{H} \cdots \boldsymbol{K}^{\bullet}) = \partial_{\mathbf{C}} w(\mathbf{C}, \boldsymbol{K}) \cdot\cdot \mathbf{C}^{\bullet} + \partial_{\boldsymbol{K}} w(\mathbf{C}, \boldsymbol{K}) \cdots \boldsymbol{K}^{\bullet}$$

and by comparison we obtain the potential relations for (2.30) and (2.31)

(2.34) $$k(\mathbf{C}, \boldsymbol{K}) = 2\rho_0 \, \partial_{\mathbf{C}} w(\mathbf{C}, \boldsymbol{K})$$

(2.35) $$K(\mathbf{C}, \boldsymbol{K}) = \rho_0 \, \partial_{\boldsymbol{K}} w(\mathbf{C}, \boldsymbol{K}) \, .$$

So there is a clear mathematical distinction between elastic and the hyperelastic materials, and the latter form a proper subset of the first. However, elastic materials that are not hyperelastic, have hardly any physical relevance. In what follows, we will only consider elastic materials that are also hyperelastic, so that both terms are used in a synonymous sense.

Change of Reference Placement

The reduced forms (2.30) and (2.31) and the hyperelastic energy depend on the choice of the reference placement κ_0 . If we want to indicate this dependence, we will write for example $k(\kappa_0, \bullet)$ and $K(\kappa_0, \bullet)$ for the elastic laws. Their transformation behavior under change of the reference placement plays an important role for isomorphisms and symmetry transformations, as we will show later.

While the spatial quantities $grad\,\mathbf{v}$, $grad\,grad\,\mathbf{v}$, $\mathbf{T}$, $\boldsymbol{G}$ do not depend on the reference placement, the material ones like $\mathbf{C}$, $\boldsymbol{K}$, $\mathbf{S}$, $\boldsymbol{H}$ do so. We will next investigate their transformation behavior under change of the reference placement. We will therefore consider a second reference placement $\underline{\kappa_0}$ indicated by underlining, like all variables with respect to $\underline{\kappa_0}$.

For an arbitrary differentiable field ϕ given in the LAGRANGEan representation with respect to both reference placements, we obtain by the chain rule

(2.36) $$\underline{Grad}\,\phi = (Grad\,\phi)\,\mathbf{A}$$

where $\mathbf{A} := \underline{Grad}\,(\kappa_0\,\underline{\kappa_0}^{-1}) \in \mathcal{I}nv^+$ is the gradient of the change of reference placement. It is understood that the field ϕ is defined as a field on the corresponding reference placements, which has identical values for identical material points.

In particular, we find

(2.37) $$\underline{\mathbf{F}} = \underline{Grad}\,\chi = \mathbf{F}\cdot\mathbf{A}$$

so that

$$\underline{J} = \rho / \underline{\rho_0} = J\,det(\mathbf{A}) = \rho / \rho_0\; J_A \quad \text{or} \quad \rho_0 / \underline{\rho_0} = J_A \qquad \text{with } J_A := det(\mathbf{A})$$

and

$$\underline{\mathbf{C}} = \mathbf{A}^T\cdot\mathbf{C}\cdot\mathbf{A} = \mathbf{A}^T * \mathbf{C}\,.$$

For the second gradient the transformation is

(2.38) $$\underline{Grad}\;\underline{\mathbf{F}} = \underline{Grad}\,(\mathbf{F}\,\mathbf{A})$$

with (2.8) $$= \mathbf{F}\cdot\underline{Grad}\,\mathbf{A} + [(\underline{Grad}\;\mathbf{F})^t\cdot\mathbf{A}]^t$$

with (2.36) $$= \mathbf{F}\cdot\underline{Grad}\,\mathbf{A} + [\{(Grad\;\mathbf{F})\cdot\mathbf{A}\}^t\cdot\mathbf{A}]^t$$

with (0.15) $$= \mathbf{F}\cdot\underline{Grad}\,\mathbf{A} + \mathbf{A}^T * (\mathbf{A}^{-T}\cdot Grad\;\mathbf{F})\,.$$

Thus

(2.39) $$\underline{\boldsymbol{K}} = \underline{\mathbf{F}}^{-1}\cdot\underline{Grad}\;\underline{\mathbf{F}}$$

$$= \mathbf{A}^{-1}\cdot\mathbf{F}^{-1}\cdot\{\mathbf{F}\cdot\underline{Grad}\,\mathbf{A} + \mathbf{A}^T * (\mathbf{A}^{-T}\cdot Grad\;\mathbf{F})\}$$

$$= \mathbf{A}^{-1}\cdot\underline{Grad}\,\mathbf{A} + \mathbf{A}^{-1}\cdot\mathbf{F}^{-1}\cdot[\mathbf{A}^T * (\mathbf{A}^{-T}\cdot Grad\;\mathbf{F})]$$

with (0.11) $$= \mathbf{A}^{-1}\cdot\underline{Grad}\,\mathbf{A} + \mathbf{A}^{-1}\cdot\mathbf{F}^{-1}\cdot[\mathbf{A}^{-T}\cdot(\mathbf{A}^T * Grad\;\mathbf{F})]$$

with (0.12) $$= \mathbf{A}^{-1}\cdot\underline{Grad}\,\mathbf{A} + \mathbf{A}^T * (\mathbf{A}^{-T}\cdot\mathbf{A}^{-1}\cdot\mathbf{F}^{-1}\,Grad\;\mathbf{F})$$

with (0.16) $= \boldsymbol{K}_{\mathbf{A}} + \mathbf{A}^{T} \circ \boldsymbol{K}$

with the configuration of κ_0 with respect to $\underline{\kappa}_0$

(2.40) $\boldsymbol{K}_{\mathbf{A}} := \mathbf{A}^{-1} \cdot \underline{Grad}\, \mathbf{A}$.

For the material stresses we obtain

(2.41) $\underline{\mathbf{S}} = \underline{J}\, \underline{\mathbf{F}}^{-1} \cdot \mathbf{T} \cdot \underline{\mathbf{F}}^{-T}$

with (2.37) $= (det\, \mathbf{A})\, J\, \mathbf{A}^{-1} \cdot \mathbf{F}^{-1} \cdot \mathbf{T} \cdot \mathbf{F}^{-T} \cdot \mathbf{A}^{-T}$

$= (det\, \mathbf{A})\, \mathbf{A}^{-1} \cdot \mathbf{S} \cdot \mathbf{A}^{-T}$

$= \mathbf{A}^{-1} * J_A\, \mathbf{S}$

and with (2.26) for the hyperstresses

(2.42) $\underline{\boldsymbol{H}} = \underline{\mathbf{F}}^{-1} \circ \underline{J}\, \boldsymbol{G}$

with (2.37) $= (\mathbf{A}^{-1} \cdot \mathbf{F}^{-1}) \circ (det(\mathbf{A})\, J\, \boldsymbol{G})$

with (0.20) $= \mathbf{A}^{-1} \circ [\, \mathbf{F}^{-1} \circ (J_A\, J\, \boldsymbol{G})]$

with (2.26) $= \mathbf{A}^{-1} \circ J_A\, \boldsymbol{H}$

or inversely

(2.43) $\boldsymbol{H} = \mathbf{A} \circ J_A^{-1}\, \underline{\boldsymbol{H}}$.

The above formulae hold for arbitrary changes of reference placements. If we particularize these results to rigid rotations of the reference placements $\mathbf{A} \in \mathcal{O}rth^{+}$ we have

(2.44) $J_A = 1 \qquad Grad\, \mathbf{A} \equiv \boldsymbol{0}$

$\underline{\boldsymbol{K}} = \mathbf{A}^{T} * \boldsymbol{K} \qquad \underline{\boldsymbol{H}} = \mathbf{A}^{T} * \boldsymbol{H}$

Coming back to the general case, the elastic laws (2.30) and (2.31) with respect to two different reference placements are transformed as

(2.45) $k(\kappa_0, \mathbf{C}, \boldsymbol{K}) = \mathbf{A} * [J_A^{-1}\, k(\underline{\kappa}_0, \mathbf{A}^{T} * \mathbf{C}, \boldsymbol{K}_{\mathbf{A}} + \mathbf{A}^{T} \circ \boldsymbol{K})]$

(2.46) $K(\kappa_0, \mathbf{C}, \boldsymbol{K}) = \mathbf{A} \circ [J_A^{-1}\, K(\underline{\kappa}_0, \mathbf{A}^{T} * \mathbf{C}, \boldsymbol{K}_{\mathbf{A}} + \mathbf{A}^{T} \circ \boldsymbol{K})]$

for all $(\mathbf{C}, \boldsymbol{K}) \in \mathcal{C}onf$ using (2.37), (2.40), (2.41), and (2.43).

For the hyperelastic energy introduced in (2.32) we obtain for the change of the reference placement

(2.47) $w(\kappa_0, \mathbf{C}, \boldsymbol{K}) = w(\underline{\kappa}_0, \underline{\mathbf{C}}, \underline{\boldsymbol{K}}) = w(\underline{\kappa}_0, \mathbf{A}^{T} * \mathbf{C}, \boldsymbol{K}_{\mathbf{A}} + \mathbf{A}^{T} \circ \boldsymbol{K}) \qquad \forall\, (\mathbf{C}, \boldsymbol{K}) \in \mathcal{C}onf$

which is in accordance with (2.34), (2.35), (2.37), (2.39) and would again give (2.45) and (2.46).

The change of the reference placement enters into these transformations only through the couple $(\mathbf{A}, \boldsymbol{K}_{\mathbf{A}}) \in \mathcal{I}nv\mathcal{C}omb$. *Vice versa*, for any such couple one can surely find a corresponding reference placement obeying (2.37) and (2.39). In fact, there are always infinitely many reference placements that do so.

Elastic Isomorphy

This concept plays an important role for the formulation of elasticity and elastoplasticity[20]. It is used to precisely define the notion that two elastic points show the *same elastic behavior*.

We have already seen that our elastic laws depend on the choice of the reference placement. If we want to compare two different elastic points, we have to take these dependencies of the respective elastic laws into account.

Definition 2.3. Two elastic material points X and Y are called **elastically isomorphic** if we can find reference placements κ_X for X and κ_Y for Y such that the following two conditions hold.

- In κ_X and κ_Y the mass densities are equal

(2.48) $$\rho_{0X} = \rho_{0Y}.$$

- With respect to κ_X and κ_Y the elastic laws are identical

(2.49) $$k_X(\kappa_X, \bullet) = k_Y(\kappa_Y, \bullet)$$

(2.50) $$K_X(\kappa_X, \bullet) = K_Y(\kappa_Y, \bullet)$$

In the hyperelastic case one can equivalently demand that the energy functions coincide

(2.51) $$w_Y(\kappa_Y, \bullet) = w_X(\kappa_X, \bullet) + w_c$$

up to some constant $w_c \in \mathscr{R}$ instead of (2.49) and (2.50).

TESTA/ VIANELLO (2005) demand in addition to (2.48) that also the gradient of the density is equal in the two points, an assumption which makes sense in the context of elastic gradient fluids. In the present context, however, we do not see any reason for such a restriction.

If two isomorphic elastic laws are given with respect to arbitrary reference placements $\underline{\kappa}_X$ and $\underline{\kappa}_Y$, then we must probably first transform them to appropriate κ_X and κ_Y using (2.45) and (2.46)

(2.52) $$k_X(\kappa_X, \mathbf{C}_X, \boldsymbol{K}_X) = \mathbf{A}_X * [J_X^{-1} k_X(\underline{\kappa}_X, \mathbf{A}_X^T * \mathbf{C}_X, \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^T \circ \boldsymbol{K}_X)]$$

(2.53) $$K_X(\kappa_X, \mathbf{C}_X, \boldsymbol{K}_X) = \mathbf{A}_X \circ [J_X^{-1} K_X(\underline{\kappa}_X, \mathbf{A}_X^T * \mathbf{C}_X, \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^T \circ \boldsymbol{K}_X)]$$

and

(2.54) $$k_Y(\kappa_Y, \mathbf{C}_Y, \boldsymbol{K}_Y) = \mathbf{A}_Y * [J_Y^{-1} k_Y(\underline{\kappa}_Y, \mathbf{A}_Y^T * \mathbf{C}_Y, \boldsymbol{K}_{\mathbf{A}Y} + \mathbf{A}_Y^T \circ \boldsymbol{K}_Y)]$$

(2.55) $$K_Y(\kappa_Y, \mathbf{C}_Y, \boldsymbol{K}_Y) = \mathbf{A}_Y \circ [J_Y^{-1} K_Y(\underline{\kappa}_Y, \mathbf{A}_Y^T * \mathbf{C}_Y, \boldsymbol{K}_{\mathbf{A}Y} + \mathbf{A}_Y^T \circ \boldsymbol{K}_Y)]$$

with $\mathbf{A}_{X,Y} := Grad(\kappa_{X,Y}\ \underline{\kappa}_{X,Y}^{-1})$, $J_{X,Y} := det(\mathbf{A}_{X,Y})$, and $\boldsymbol{K}_{\mathbf{A}X,Y} := \mathbf{A}_{X,Y}^{-1}\ \underline{Grad}\ \mathbf{A}_{X,Y}$

as well as the mass densities in these reference placements

[20] see BERTRAM (1998), (2005)

(2.56) $\underline{\rho}_{0X} / \rho_{0X} = J_X$ and $\underline{\rho}_{0Y} / \rho_{0Y} = J_Y$.

So the above isomorphy conditions hold if and only if

$$\underline{\rho}_{0X}\, J_X^{-1} = \underline{\rho}_{0Y}\, J_Y^{-1}$$

(2.57) $$k_X(\kappa_X, \mathbf{C}, \boldsymbol{K}) = k_Y(\kappa_Y, \mathbf{C}, \boldsymbol{K})$$

$$K_X(\kappa_X, \mathbf{C}, \boldsymbol{K}) = K_Y(\kappa_Y, \mathbf{C}, \boldsymbol{K}) \qquad \forall\, (\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}.$$

Then they certainly also hold for the inverse transformation

(2.58) $$(\mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K}) \in \mathscr{Conf}$$

so that

(2.59) $$k_X(\kappa_X, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K})$$
$$= k_Y(\kappa_Y, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K})$$

(2.60) $$K_X(\kappa_X, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K})$$
$$= K_Y(\kappa_Y, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K}) \qquad \forall\, (\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}.$$

If we multiply the first equation by $\mathbf{A}_X^{-1} * J_X$ and the second one by $\mathbf{A}_X^{-1} \circ J_X$, then we see that the left-hand sides give the elastic laws in the reference placement $\underline{\kappa}_X$, so that we achieve

(2.61) $$k_X(\underline{\kappa}_X, \mathbf{C}, \boldsymbol{K}) = \mathbf{A}_X^{-1} * [J_X\, k_Y(\kappa_Y, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K})]$$

(2.62) $$K_X(\underline{\kappa}_X, \mathbf{C}, \boldsymbol{K}) = \mathbf{A}_X^{-1} \circ [J_X\, K_Y(\kappa_Y, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K})].$$

By interpreting the right-hand side as a change of the reference placement for Y, we see that the isomorphy conditions hold for this choice of reference placements as well. By the notations $\mathbf{A}_X =: \mathbf{P}$ and $\boldsymbol{K}_{\mathbf{A}} =: \boldsymbol{P}$, the following equivalent, but simpler isomorphy conditions result.[21]

The dyadic $\mathbf{P}$ can be interpreted as the gradient of the change of the reference placement and the triadic $\boldsymbol{P}$ as its second gradient. However, in a local theory, these two tensors can be considered as being independent of each other.

Theorem 2.1. *Two elastic material points X and Y with elastic laws k_X, K_X and k_Y, K_Y with respect to arbitrary reference placements are elastically isomorphic if and only if there exist two tensors $(\mathbf{P}, \boldsymbol{P}) \in \mathscr{InvComb}$ such that*

(2.63) $$\rho_{0X} = \rho_{0Y}\, det(\mathbf{P})$$

(2.64) $$k_Y(\mathbf{C}, \boldsymbol{K}) = det^{-1}(\mathbf{P})\, [\mathbf{P} * k_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P})]$$

(2.65) $$K_Y(\mathbf{C}, \boldsymbol{K}) = det^{-1}(\mathbf{P})\, [\mathbf{P} \circ K_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P})]$$

hold for all $(\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}$ with ρ_{0X} and ρ_{0Y} being the mass densities in the reference placements of X and Y, respectively.

This gives for the elastic energy in the case of isomorphy

[21] This theorem has been used for simple elastic materials in (BERTRAM 2005). In a more general format is has been shown already in BERTRAM (1982, page 111) and BERTRAM (1989, page 206).

(2.66) $$w_Y(\mathbf{C}, \boldsymbol{K}) = w_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) + w_c \qquad \forall\, (\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}$$

equivalent to (2.64) and (2.65) with some constant $w_c \in \mathscr{R}$.

Material Symmetry

If we particularize the concept of isomorphy to identical points $X \equiv Y$, it defines *automorphy* or **symmetry**. In this case we consider only one point so that we can drop the point index, and denote the automorphism by $(\mathbf{A}, \boldsymbol{A}) \in \mathscr{I}nv\mathscr{C}omb$ to distinguish from the isomorphisms of the previous section. Because of the first isomorphy condition, any automorphism must be proper unimodular in its first entry: $(\mathbf{A}, \boldsymbol{A}) \in \mathscr{U}nim\mathscr{C}omb$. This leads us to the following definition using (2.66).

Definition 2.4. For a gradient hyperelastic material with elastic energy w a **symmetry transformation** is a pair $(\mathbf{A}, \boldsymbol{A}) \in \mathscr{U}nim\mathscr{C}omb$ such that

$$w(\mathbf{C}, \boldsymbol{K}) = w(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A}) \qquad \forall\, (\mathbf{C}, \boldsymbol{K}) \in \mathscr{C}onf. \tag{2.67}$$

This gives with (2.34) and (2.35) the transformations for the elastic laws

$$k(\mathbf{C}, \boldsymbol{K}) = \mathbf{A} * k(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A}) \tag{2.68}$$

$$K(\mathbf{C}, \boldsymbol{K}) = \mathbf{A} \circ K(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A})\,. \tag{2.69}$$

The set of all such symmetry transformations represented by such a couple $(\mathbf{A}, \boldsymbol{A}) \in \mathscr{U}nim\mathscr{C}omb$ forms the **symmetry group** of the material. In fact, the transformation is a group under composition in the algebraic sense. Its identity is $(\mathbf{I}, \boldsymbol{0}) \in \mathscr{U}nim\mathscr{C}omb$. The composition of two elements $(\mathbf{A}, \boldsymbol{A})$ and $(\mathbf{B}, \boldsymbol{B}) \in \mathscr{U}nim\mathscr{C}omb$ is

$$(\mathbf{A}, \boldsymbol{A})\,(\mathbf{B}, \boldsymbol{B}) := (\mathbf{A} \cdot \mathbf{B}, \mathbf{B}^T \circ \boldsymbol{A} + \boldsymbol{B}) \in \mathscr{U}nim\mathscr{C}omb$$

which does not commute. The inverse of some $(\mathbf{A}, \boldsymbol{A}) \in \mathscr{U}nim\mathscr{C}omb$ is

$$(\mathbf{A}^{-1}, -\mathbf{A}^{-T} \circ \boldsymbol{A}) \in \mathscr{U}nim\mathscr{C}omb\,.$$

This group is used to define isotropy or anisotropy. If the symmetry group is a subgroup of $\mathscr{O}rth^+$ in the first entry and the zero in the second, $(\mathbf{Q}, \boldsymbol{O})$, these transformations can be interpreted as rigid rotations, and we call the respective reference placement an **undistorted state**. If a material allows for such undistorted states, it is a **solid**. If it contains all orthogonal dyadics in the first entry, then the material is called **isotropic**. These definitions apply not only to gradient elasticity and hyperelasticity, but also to any inelastic gradient material in an analogous way.

Here one could also allow for improper symmetry transformations like $(-\mathbf{Q}, \boldsymbol{O})$ with $\mathbf{Q} \in \mathscr{O}rth^+$. Since we are dealing with even *and* odd-order tensors, this minus sign does not cancel out in (2.67) - (2.69). Thus, for gradient materials improper symmetry transformations do play a non-trivial role, in contrast to simple materials.

If a material contains with all proper symmetry transformations also the corresponding improper ones, it is called **centro-symmetric**. For isotropic materials this centro-symmetry is often tacitely included.

In all of these cases, we obtain after (2.44) with respect to undistorted states

(2.70) $\mathbf{A} * k(\mathbf{C}, \boldsymbol{K}) = k(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K})$

(2.71) $\mathbf{A} * K(\mathbf{C}, \boldsymbol{K}) = K(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K})$

(2.72) $w(\mathbf{C}, \boldsymbol{K}) = w(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K})$

$\forall\ (\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}$. Thus, for an isotropic material the elastic laws are isotropic tensor functions.

In MURDOCH (1979)[22] one finds interesting considerations about the symmetry of second gradient materials. MURDOCH uses other configuration variables than we do, namely $\mathbf{F}$ and $\mathbf{F}^T \cdot \mathit{Grad}\,\mathbf{F}$, the latter being a material quantity, in contrast to the first one.

[22] see also CROSS (1973), ELZANOWSKI/ EPSTEIN (1992) and LEON/ EPSTEIN (1996) for such considerations related to the symmetry group.

Finite Linear Elasticity

For many applications the elastic deformations are rather small which justifies the linearization of the hyperelastic laws. In-order to avoid the introduction of new notations like a generalized VOIGT notation, we use a tensor notation.

In the physically linear elasticity theory, the elastic energy is assumed to be a (symmetric) square form of the configuration $(\mathbf{C}, \boldsymbol{K})$. Such a form on $\mathscr{Conf}$ would have $24^2/2 + 24/2 = 300$ parameters. In tensor notations it can be represented by

$$w(\mathbf{C}, \boldsymbol{K}) = \frac{1}{2\rho_0}\, \Delta\mathbf{C} \cdot\cdot \overset{\langle 4\rangle}{\boldsymbol{E}} \cdot\cdot \Delta\mathbf{C} + \frac{1}{\rho_0}\, \Delta\mathbf{C} \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{E}} \cdots \Delta\boldsymbol{K} + \frac{1}{2\rho_0}\, \Delta\boldsymbol{K} \cdots \overset{\langle 6\rangle}{\boldsymbol{E}} \cdots \Delta\boldsymbol{K} \tag{2.73}$$

with the abbreviations

$$\Delta\mathbf{C} := \tfrac{1}{2}(\mathbf{C} - \mathbf{C}_u) \quad \text{and} \quad \Delta\boldsymbol{K} := \boldsymbol{K} - \boldsymbol{K}_u$$

with respect to some unloaded configuration $(\mathbf{C}_u, \boldsymbol{K}_u) \in \mathscr{Conf}$ and higher-order elasticity tensors $\overset{\langle 4\rangle}{\boldsymbol{E}}, \overset{\langle 5\rangle}{\boldsymbol{E}}, \overset{\langle 6\rangle}{\boldsymbol{E}}$.

These elasticities can be submitted to the following symmetry conditions:

$\overset{\langle 4\rangle}{\boldsymbol{E}}$:

- left subsymmetry $\{ijkl\} = \{jikl\}$
- right subsymmetry $\{ijkl\} = \{ijlk\}$
- and the major symmetry $\{ijkl\} = \{lkji\}$

with 21 independent constants as customary from classical elasticity

$\overset{\langle 5\rangle}{\boldsymbol{E}}$:

- left subsymmetry $\{ijklm\} = \{jiklm\}$
- right subsymmetry $\{ijklm\} = \{ijkml\}$

with 18 x 6 = 108 independent parameters

$\overset{\langle 6\rangle}{\boldsymbol{E}}$:

- subsymmetry in the 2nd and 3rd indices $\{ijklmn\} = \{ikjlmn\}$
- right subsymmetry $\{ijklmn\} = \{ijklnm\}$
- and major symmetry $\{ijklmn\} = \{lmnijk\}$

with $18^2/2 + 18/2 = 171$ independent parameters

This gives in total again 300 constants, which can eventually be reduced by the exploitation of symmetry properties.

The **isotropic** versions of the elastic energy can already be found in MINDLIN/ ESHEL (1968) with only 7 independent parameters including the two LAMÉ constants from classical elasticity[23]. Here we obtain for the elastic energy

$$w = a_1 (\Delta\mathbf{C} \cdot\cdot \mathbf{I})^2 + a_2 \Delta\mathbf{C} \cdot\cdot \Delta\mathbf{C} + a_3 \mathbf{I} \cdot\cdot \Delta\boldsymbol{K} \cdot \Delta\mathbf{C} \cdot\cdot \mathbf{I} + a_4 (\Delta\boldsymbol{K} \cdot\cdot \mathbf{I}) \cdot (\Delta\mathbf{C} \cdot\cdot \mathbf{I}) + a_5 (\mathbf{I} \cdot\cdot \Delta\boldsymbol{K}) \cdot (\mathbf{I} \cdot\cdot \Delta\boldsymbol{K}) + a_6 \Delta\boldsymbol{K} \cdots \Delta\boldsymbol{K} + a_7 \Delta\boldsymbol{K} \cdots \Delta\boldsymbol{K}^T \tag{2.74}$$

where the following particular transposition for a third-order tensor $[\Delta\boldsymbol{K}^T]_{ijk} := [\Delta\boldsymbol{K}]_{kji}$ is used. a_i are the seven scalar material constants.

This gives the following stresses after (2.34) and (2.35)

$$\mathbf{S} = 2a_1 (\Delta\mathbf{C} \cdot\cdot \mathbf{I})\, \mathbf{I} + 2a_2 \Delta\mathbf{C} \tag{2.75}$$

$$\boldsymbol{H} = 2a_3 \mathbf{I} \otimes \Delta\boldsymbol{K} \cdot\cdot \mathbf{I} + 2a_4 (\Delta\boldsymbol{K} \cdot\cdot \mathbf{I}) \otimes \mathbf{I} + 2a_5 \mathbf{I} \otimes \mathbf{I} \cdot\cdot \Delta\mathbf{K} + 2a_6 \Delta\boldsymbol{K} + 2a_7 \Delta\boldsymbol{K}^T. \tag{2.76}$$

In the general (anisotropic) case, the elastic energy (2.73) acts as a potential for the stresses with (2.34) and (2.35)

$$k(\mathbf{C}, \boldsymbol{K}) = \overset{\langle 4 \rangle}{\boldsymbol{E}} \cdot\cdot \Delta\mathbf{C} + \overset{\langle 5 \rangle}{\boldsymbol{E}} \cdots \Delta\boldsymbol{K} \tag{2.77}$$

$$K(\mathbf{C}, \boldsymbol{K}) = \Delta\mathbf{C} \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{E}} + \overset{\langle 6 \rangle}{\boldsymbol{E}} \cdots \Delta\boldsymbol{K} \qquad \forall\, (\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}. \tag{2.78}$$

These laws are straightforward extensions of the ST.-VENANT-KIRCHHOFF law to gradient elasticity. They are physically linear, but geometrically nonlinear, and they fulfill the EUCLIDean invariance requirement. Note that the linear theory depends on the choice of the stress and configuration variables, in contrast to the preceding non-linear theory. However, for small deformations, the differences remain negligible.

In the linear case the isomorphy conditions (2.49) and (2.50) become with $(\mathbf{P}, \boldsymbol{P}) \in \mathscr{Inv}$

$$\begin{aligned} &\overset{\langle 4 \rangle}{\boldsymbol{E}}_Y \cdot\cdot \tfrac{1}{2}(\mathbf{C} - \mathbf{C}_{uY}) + \overset{\langle 5 \rangle}{\boldsymbol{E}}_Y \cdots (\boldsymbol{K} - \boldsymbol{K}_{uY}) \\ &= \mathbf{P} * det^{-1}(\mathbf{P}) \{ \overset{\langle 4 \rangle}{\boldsymbol{E}}_X \cdot\cdot \tfrac{1}{2}(\mathbf{P}^T * \mathbf{C} - \mathbf{C}_{uX}) + \overset{\langle 5 \rangle}{\boldsymbol{E}}_X \cdots (\mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P} - \boldsymbol{K}_{uX}) \} \end{aligned} \tag{2.79}$$

and

$$\begin{aligned} &\tfrac{1}{2}(\mathbf{C} - \mathbf{C}_{uY}) \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{E}}_Y + \overset{\langle 6 \rangle}{\boldsymbol{E}}_Y \cdots (\boldsymbol{K} - \boldsymbol{K}_{uY}) \\ &= \mathbf{P} \circ (det^{-1}\, \mathbf{P}) \{ \tfrac{1}{2}(\mathbf{P}^T * \mathbf{C} - \mathbf{C}_{uX}) \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{E}}_X + \overset{\langle 6 \rangle}{\boldsymbol{E}}_X \cdots (\mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P} - \boldsymbol{K}_{uX}) \}). \end{aligned} \tag{2.80}$$

By a comparison in the independent variables $(\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}$ one can determine the transformations of the elasticities and the unloaded configuration like

$$\overset{\langle 4 \rangle}{\boldsymbol{E}}_Y = \mathbf{P} * (det^{-1}\, \mathbf{P})\, \overset{\langle 4 \rangle}{\boldsymbol{E}}_X \qquad \text{and} \qquad \mathbf{C}_{uY} = \mathbf{P}^{-T} * \mathbf{C}_{uX}. \tag{2.81}$$

For the other elastic tensors such relations can also be obtained but become more complicated.

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[23] see SUIKER/ CHANG (2000), DELL´ISOLA/ SCIARRA/ VIDOLI (2009), AUFFRAY/ LE QUANG/ HE (2013), and BERTRAM/ FOREST (2014)

The linearity of the elastic laws will not be assumed in what follows, in order to preserve full generality.

2.2 Finite Gradient Elastoplasticity

There are not many papers in the literature dealing with finite plastic deformations of gradient materials. In most of them, the internal length scale enters through the gradient of some internal variable, like hardening variables or the plastic deformation. Examples for such approaches are given by GURTIN/ ANAND (2005), GURTIN/ FRIED/ ANAND (2009), GURTIN (2010), LUSCHER et al. (2010), and CLEJA-ŢIGOIU (2013).

Our intention here, however, is to not limit the theory to such special cases, but instead to allow for a second and a third-order plastic variable as being substantially independent of each other (*unconstrained gradient plasticity*). This is the more general case, and there is no rationale known which would exclude this choice. Particular cases, in which this independence is not given, should be nevertheless contained in this general setting.

By elastoplasticity we understand rate-independent materials with elastic ranges. For a gradient theory of elastoplasticity, we consider materials for which both the elastic and the plastic behavior are assumed to be of gradient type.

One assumes that after some deformation process the material is within some elastic range for which elastic laws for the stresses exist. Thus, the stresses can be determined by these current elastic laws. And this holds also for any continuation of the deformation process as long as it does not leave the current elastic range. If this happens, the material continuously passes through different elastic ranges, a process which characterizes *yielding*.

We want to make these concepts more precise.

Definition 2.5. A (hyper)**elastic range** is a pair $\{\mathscr{E}_p, w_p\}$ consisting of

1.) a path-connected submanifold with boundary $\mathscr{E}_p \subset \mathscr{C}onf$

2.) and the elastic energy

(2.82) $w_p(\mathbf{C}, \boldsymbol{K})$

such that after any continuation process $\{\mathbf{C}(\tau), \boldsymbol{K}(\tau)\} \,\big|\, {}^{t}_{t_0}$, which remains entirely in $\mathscr{E}_p$

$$\{\mathbf{C}(\tau), \boldsymbol{K}(\tau)\} \in \mathscr{E}_p \qquad \forall\ \tau \in [t_0, t]$$

the stresses are determined after (2.34) and (2.35) by the final values of the process by

(2.83) $$\mathbf{S}(t) = 2\rho_0\, \partial_{\mathbf{C}}\, w_p(\mathbf{C}, \boldsymbol{K}) := k_p(\mathbf{C}, \boldsymbol{K})$$

(2.84) $$\boldsymbol{H}(t) = \rho_0\, \partial_{\boldsymbol{K}}\, w_p(\mathbf{C}, \boldsymbol{K}) := K_p(\mathbf{C}, \boldsymbol{K}).$$

The elastic laws are physically determined only for configurations within the specific elastic range $\mathscr{E}_p$. However, in what follows we will extend them to the entire space $\mathscr{C}onf$ for simplicity.

In contrast to many other authors, we introduce the elastic ranges in the configuration space. By the elastic laws (2.83) and (2.84), one can easily transform them into the space of stresses and hyperstresses if this is preferred.

Assumption 2.1. *At each instant the elastoplastic material point is associated with an elastic range so that the stresses are given by* (2.83) *and* (2.84) .

Isomorphy of the Elastic Ranges

During yielding two effects have to be considered. Firstly, the elastic range $\mathscr{E}_p$ as a subset of $\mathscr{C}onf$ evolves reflecting the hardening or softening behavior of the material. And secondly, the elastic energy function associated with these elastic ranges can also evolve. We will first address this second effect.

For many materials it is a microphysically and experimentally well-substantiated fact that during yielding the elastic behavior hardly alters even under very large deformations. This substantiates the assumption, that the elastic behavior remains identical. Such an assumption reduces the effort for the identification tremendously, since otherwise one would have to identify the elastic constants at each step of the deformation anew[24].

We now give this assumption a precise form.

Assumption 2.2. *The elastic laws of all elastic ranges are isomorphic.*

Note that in Assumption 2.2 nothing is said about the form or size of the elastic ranges $\mathscr{E}_p$ in the configuration space. So the hardening behavior is not at all addressed by it.

As a consequence, if $\{\mathscr{E}_1, w_1\}$ and $\{\mathscr{E}_2, w_2\}$ are two elastic ranges, then according Theorem 2.1 there exist two tensors $(\mathbf{P}_{12}, \boldsymbol{P}_{12}) \in \mathscr{I}nv\mathscr{C}omb$ and a scalar w_c such that

- for the mass densities in the reference placements ρ_{01} and ρ_{02} holds

(2.85) $$\rho_{01} = \rho_{02} \det \mathbf{P}_{12}$$

- and for the elastic energies we have the equality after (2.66)

(2.86) $$w_2(\mathbf{C}, \boldsymbol{K}) = w_1(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) + w_c$$

such that the elastic stress laws are after (2.34) and (2.35)

(2.87) $$k_2(\mathbf{C}, \boldsymbol{K}) = (det^{-1} \mathbf{P}_{12}) [\mathbf{P}_{12} * k_1(\mathbf{P}_{12}{}^T * \mathbf{C}, \mathbf{P}_{12}{}^T \circ \boldsymbol{K} + \boldsymbol{P}_{12})]$$

(2.88) $$K_2(\mathbf{C}, \boldsymbol{K}) = (det^{-1} \mathbf{P}_{12}) [\mathbf{P}_{12} \circ K_1(\mathbf{P}_{12}{}^T * \mathbf{C}, \mathbf{P}_{12}{}^T \circ \boldsymbol{K} + \boldsymbol{P}_{12})]$$

$\forall$ $(\mathbf{C}, \boldsymbol{K}) \in \mathscr{C}onf$. As we have chosen a joint reference placement for all elastic laws of one particular material point (this is, however, not compulsory), we already have $\rho_{01} \equiv \rho_{02}$, and therefore $\mathbf{P}_{12}$ must be proper unimodular, so that the first isomorphy condition (2.85) is always fulfilled.

If all elastic energies belonging to different elastic ranges are mutually isomorphic, then because of the group property of isomorphy transformations, they all are isomorphic to some freely chosen **elastic reference energy** w_0. While the current elastic energy function w_p varies with time during yielding, the reference energy function can always be chosen as constant in time. We thus have the isomorphy condition in the following form.

[24] One of the few examples where this assumption has not been made is BÖHLKE/ BERTRAM (2001).

Theorem 2.2. *Let* w_0 *be the elastic reference energy for an elasto-plastic material. Then for each elastic range* $\{\mathscr{E}_p, w_p\}$ *there are two tensors* $(\mathbf{P}, \boldsymbol{P}) \in \mathscr{U}nim\mathscr{C}omb$ and a scalar w_{c0} *such that*

$$w_p(\mathbf{C}, \boldsymbol{K}) = w_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) + w_{c0} \qquad \forall\, (\mathbf{C}, \boldsymbol{K}) \in \mathscr{C}onf. \tag{2.89}$$

The elastic laws are then given by

$$\begin{aligned} \mathbf{S} &= k_p(\mathbf{C}, \boldsymbol{K}) = 2\rho_0\, \partial_{\mathbf{C}}\, w_p(\mathbf{C}, \boldsymbol{K}) \\ &= 2\rho_0\, \partial_{\mathbf{C}}\, w_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) \\ &= \mathbf{P} * k_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) \end{aligned} \tag{2.90}$$

and

$$\begin{aligned} \boldsymbol{H} &= K_p(\mathbf{C}, \boldsymbol{K}) = \rho_0\, \partial_{\boldsymbol{K}}\, w_p(\mathbf{C}, \boldsymbol{K}) \\ &= \rho_0\, \partial_{\boldsymbol{K}}\, w_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) \\ &= \mathbf{P} \circ K_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) \qquad \forall\, (\mathbf{C}, \boldsymbol{K}) \in \mathscr{C}onf. \end{aligned} \tag{2.91}$$

In the present theory, the two variables $(\mathbf{P}, \boldsymbol{P}) \in \mathscr{U}nim\mathscr{C}omb$ are chosen as the plastic internal variables. One might be tempted to interpret the argument of (2.90) and (2.91) as both an additive *and* a multiplicative decomposition of the kinematical variables[25]. They are, however, not introduced as deformations but rather as a transformation of the current elastic energy (not of a placement) to a time-independent reference energy function, which results in a natural way from the isomorphy condition. We avoid the introduction of an intermediate configuration or a split of some deformation into elastic and plastic parts since it is misleading in a finite deformation theory[26].

If one linearizes these elastic laws, then we obtain after (2.79) and (2.80)

$$\begin{aligned} &\overset{\langle 4\rangle}{\boldsymbol{E}}_p \cdot\cdot\, \tfrac{1}{2}(\mathbf{C} - \mathbf{C}_{up}) + \overset{\langle 5\rangle}{\boldsymbol{E}}_p \cdots (\boldsymbol{K} - \boldsymbol{K}_{up}) \\ &= \mathbf{P} * \{\overset{\langle 4\rangle}{\boldsymbol{E}}_0 \cdot\cdot\, \tfrac{1}{2}(\mathbf{P}^T * \mathbf{C} - \mathbf{C}_{u0}) + \overset{\langle 5\rangle}{\boldsymbol{E}}_0 \cdots (\mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P} - \boldsymbol{K}_{u0})\} \end{aligned} \tag{2.92}$$

$$\begin{aligned} &\tfrac{1}{2}(\mathbf{C} - \mathbf{C}_{up}) \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{E}}_p + \overset{\langle 6\rangle}{\boldsymbol{E}}_p \cdots (\boldsymbol{K} - \boldsymbol{K}_{up}) \\ &= \mathbf{P} \circ \{\tfrac{1}{2}(\mathbf{P}^T * \mathbf{C} - \mathbf{C}_{u0}) \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{E}}_0 + \overset{\langle 6\rangle}{\boldsymbol{E}}_0 \cdots (\mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P} - \boldsymbol{K}_{u0})\}) \end{aligned} \tag{2.93}$$

where the suffix p indicates the (time-dependent) quantities related to the linear forms of k_p and K_p, and the suffix 0 to the (time-independent) ones of the linear forms of the elastic reference laws k_0 and K_0. Again one can determine the transformations of the elasticities and the unloaded configuration as in (2.81)

$$\overset{\langle 4\rangle}{\boldsymbol{E}}_p = \mathbf{P} * \overset{\langle 4\rangle}{\boldsymbol{E}}_0 \qquad \text{and} \qquad \mathbf{C}_{up} = \mathbf{P}^{-T} * \mathbf{C}_{u0} \tag{2.94}$$

etc.

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[25] In CHAMBON/ CAILLERIE/ TAMAGNINI (2001) such an interpretation is given.

[26] see the comments in BERTRAM (2005) on p. 291.

Yield Criteria

Let us first consider one particular elastic range $\{\mathscr{E}_p\ ,\ w_p\}$. We decompose the set $\mathscr{E}_p$ topologically into its interior $\mathscr{E}_p^{\,o}$ and its boundary $\partial\mathscr{E}_p$. The latter is called **yield surface** (in the configuration space). In order to describe it more easily, we introduce a real-valued tensor-function in the configuration space

$$\Phi_p\ :\ \mathscr{C}onf \to \mathscr{R}\ \ \big|\ \ \ (\mathbf{C}\ ,\boldsymbol{K})\ \mapsto\ \Phi_p(\mathbf{C}\ ,\boldsymbol{K})$$

the kernel of which coincides with the yield limit

$$\Phi_p(\mathbf{C}\,,\boldsymbol{K})\ =\ 0 \quad \Leftrightarrow \quad (\mathbf{C}\,,\boldsymbol{K}) \in \partial\mathscr{E}_p\,. \tag{2.95}$$

For distinguishing points in the interior and in the exterior of the elastic ranges, we postulate

$$\Phi_p(\mathbf{C}\,,\boldsymbol{K})\ <\ 0 \quad \Leftrightarrow \quad (\mathbf{C}\,,\boldsymbol{K}) \in \mathscr{E}_p^{\,o} \tag{2.96}$$

and, consequently,

$$\Phi_p(\mathbf{C}\,,\boldsymbol{K})\ >\ 0 \quad \Leftrightarrow \quad (\mathbf{C}\,,\boldsymbol{K}) \in \mathscr{C}onf \setminus \mathscr{E}_p\,. \tag{2.97}$$

We call such an indicator function or level set function a **yield criterion**, and assume further on for simplicity that Φ_p is differentiable, although there are also suggestions with corners and edges. One can always transform the yield criterion from the configuration space into the stress space by using the elastic laws using (2.90) and (2.91).

Instants of **yielding** are characterized by two facts.

- The configuration is currently on the yield limit and, thus, fulfils its **yield condition**

$$\Phi_p(\mathbf{C}\,,\boldsymbol{K})\ =\ 0\,. \tag{2.98}$$

- It is about to leave the current elastic range. This is expressed by the **loading condition**

$$\Phi_p^{\bullet}\ =\ \partial_{\mathbf{C}}\Phi_p \cdot\cdot\ \mathbf{C}^{\bullet} + \partial_{\boldsymbol{K}}\Phi_p \cdots \boldsymbol{K}^{\bullet}\ >\ 0\,. \tag{2.99}$$

Such a yield criterion is associated with some particular elastic range. In order to obtain a general yield criterion that holds for *all* elastic ranges in the same form, we introduce additional internal variables $\mathbf{Z}_p$ (here denoted as a dyadic) called **hardening variables** (although they could also describe softening). These can be tensors of arbitrary order or even a vector of such tensors and, thus, form elements of some finite dimensional linear space, the specification of which depends on the particular hardening model.

The general form of the yield criterion is assumed to be like

$$\varphi(\mathbf{P},\boldsymbol{P},\mathbf{C},\boldsymbol{K}\,,\mathbf{Z}_p) \tag{2.101}$$

such that

$$\Phi_p(\mathbf{C})\ =\ \varphi(\mathbf{P},\boldsymbol{P},\mathbf{C},\boldsymbol{K}\,,\mathbf{Z}_p) \tag{2.102}$$

holds for each particular elastic range.

With this extension we obtain for the yield condition (2.98)

(2.103) $$\varphi(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p) = 0$$

and for the loading condition (2.99)

(2.104) $$\partial_{\mathbf{C}} \varphi \cdot\cdot \mathbf{C}^{\bullet} + \partial_{\boldsymbol{K}} \varphi \cdots \boldsymbol{K}^{\bullet} > 0$$

where the plastic variables are kept constant.

Decomposition of the Stress Power

We will next consider the stress power again and specify it for our elastoplastic material. The specific stress power (2.24) is using (0.13) and (0.17)

(2.105) $\pi_i = 1/\rho_0\,(\tfrac{1}{2}\,\mathbf{S} \cdot\cdot \mathbf{C}^{\bullet} + \boldsymbol{H} \cdots \boldsymbol{K}^{\bullet})$

with (2.90), (2.91) $= 1/\rho_0\,(\tfrac{1}{2}\,k_p(\mathbf{C}, \boldsymbol{K}) \cdot\cdot \mathbf{C}^{\bullet} + K_p(\mathbf{C}, \boldsymbol{K}) \cdots \boldsymbol{K}^{\bullet})$

$= 1/\rho_0\,[\tfrac{1}{2}\,\mathbf{P} * k_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) \cdot\cdot \mathbf{C}^{\bullet}$

$+ \mathbf{P} \circ K_0\,(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) \cdots \boldsymbol{K}^{\bullet}]$

$= 1/\rho_0\,[\tfrac{1}{2}\,k_0(\mathbf{C}_e\,, \boldsymbol{K}_e) \cdot\cdot (\mathbf{P}^T * \mathbf{C}^{\bullet}) + K_0\,(\mathbf{C}_e\,, \boldsymbol{K}_e) \cdots (\mathbf{P}^T \circ \boldsymbol{K}^{\bullet})]$

with the abbreviations

(2.106) $\mathbf{C}_e := \mathbf{P}^T \cdot \mathbf{C} \cdot \mathbf{P} = \mathbf{P}^T * \mathbf{C}$

(2.107) $\boldsymbol{K}_e := \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}.$

This gives for the rates

(2.108) $\mathbf{C}_e^{\bullet} = (\mathbf{P}^T \cdot \mathbf{C} \cdot \mathbf{P})^{\bullet} = \mathbf{P}^T \cdot \mathbf{C}^{\bullet} \cdot \mathbf{P} + 2\,sym(\mathbf{P}^T \cdot \mathbf{C} \cdot \mathbf{P}^{\bullet})$

$= \mathbf{P}^T * \mathbf{C}^{\bullet} + 2\,sym(\mathbf{C}_e \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet})$

where *sym* stands for the symmetric part, and

(2.109) $\boldsymbol{K}_e^{\bullet} = \{\mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}\}^{\bullet} = \mathbf{P}^T \circ \boldsymbol{K}^{\bullet} + \boldsymbol{P}^{\bullet}$

$+ K_{ijk}\,\{(\mathbf{P}^{-1\bullet} \cdot \mathbf{e}_i) \otimes (\mathbf{P}^T \cdot \mathbf{e}_j) \otimes (\mathbf{P}^T \cdot \mathbf{e}_k)$

$+ (\mathbf{P}^{-1} \cdot \mathbf{e}_i) \otimes (\mathbf{P}^{T\bullet} \cdot \mathbf{e}_j) \otimes (\mathbf{P}^T \cdot \mathbf{e}_k) + (\mathbf{P}^{-1} \cdot \mathbf{e}_i) \otimes (\mathbf{P}^T \cdot \mathbf{e}_j) \otimes (\mathbf{P}^{T\bullet} \cdot \mathbf{e}_k)\}$

$= \mathbf{P}^T \circ \boldsymbol{K}^{\bullet} + \boldsymbol{P}^{\bullet}$

$+ K_{ijk}\,\{-(\mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet} \cdot \mathbf{P}^{-1} \cdot \mathbf{e}_i) \otimes (\mathbf{P}^T \cdot \mathbf{e}_j) \otimes (\mathbf{P}^T \cdot \mathbf{e}_k)$

$+ 2\,subsym[(\mathbf{P}^{-1} \cdot \mathbf{e}_i) \otimes (\mathbf{P}^T \cdot \mathbf{e}_j) \otimes (\mathbf{P}^T \cdot \mathbf{e}_k)\,\mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}]\}$

$= \mathbf{P}^T \circ \boldsymbol{K}^{\bullet} + \boldsymbol{P}^{\bullet} - \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet} \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\,subsym\,[(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}]$

the term with *subsym* being the symmetric part with respect to the right subsymmetry. We substitute this into (2.105) to obtain

$\pi_i = 1/\rho_0\,\{\tfrac{1}{2}\,k_0(\mathbf{C}_e\,, \boldsymbol{K}_e) \cdot\cdot [\mathbf{C}_e^{\bullet} - 2\,sym(\mathbf{C}_e \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet})]$

$+ K_0\,(\mathbf{C}_e\,, \boldsymbol{K}_e) \cdots [\boldsymbol{K}_e^{\bullet} - \boldsymbol{P}^{\bullet} + \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}\,(\boldsymbol{K}_e - \boldsymbol{P}) - 2\,subsym\,[(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}]\}$

and because of the symmetries of the stress tensors

$= 1/\rho_0\,\{\tfrac{1}{2}\,k_0(\mathbf{C}_e\,, \boldsymbol{K}_e) \cdot\cdot [\mathbf{C}_e^{\bullet} - 2\,sym(\mathbf{C}_e \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet})]$

$+ K_0\,(\mathbf{C}_e\,, \boldsymbol{K}_e) \cdots [\boldsymbol{K}_e^{\bullet} - \boldsymbol{P}^{\bullet} + \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}\,(\boldsymbol{K}_e - \boldsymbol{P}) - 2\,(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}\}$

$= 1/\rho_0\,\{\tfrac{1}{2}\,k_0(\mathbf{C}_e\,, \boldsymbol{K}_e) \cdot\cdot \mathbf{C}_e^{\bullet} + K_0\,(\mathbf{C}_e\,, \boldsymbol{K}_e) \cdots \boldsymbol{K}_e^{\bullet}$

$$
\begin{aligned}
&\quad - \tfrac{1}{2}\, k_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdot (2\, \mathbf{C}_e \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}) \\
&\quad - K_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdots [\boldsymbol{P}^{\bullet} - \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet} \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\, (\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}]\} \\
&= \partial_{\mathbf{C}}\, w_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdot\cdot\, \mathbf{C}_e^{\bullet} + \partial_{\boldsymbol{K}}\, w_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdots \boldsymbol{K}_e^{\bullet} \\
&\quad - \tfrac{1}{2}\, k_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdot\cdot\, (2\, \mathbf{C}_e \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}) \\
&\quad - K_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdots [\boldsymbol{P}^{\bullet} - \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet} \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\, (\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}] \\
&= w_0(\mathbf{C}_e, \boldsymbol{K}_e)^{\bullet} + \mathbf{S}_p \cdot\cdot\, \mathbf{P}^{\bullet} + \boldsymbol{H}_p \cdots [\boldsymbol{P}^{\bullet} - \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet} \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\, (\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}]
\end{aligned}
\tag{2.110}
$$

with the **plastic stress tensor**

$$
\mathbf{S}_p := -\mathbf{P}^{-T} \cdot \mathbf{C}_e \cdot k_0(\mathbf{C}_e, \boldsymbol{K}_e) = -\mathbf{P}^{-T} \cdot \mathbf{P}^{T} \cdot \mathbf{C} \cdot \mathbf{P} \cdot \mathbf{P}^{-1} \cdot \mathbf{S} \cdot \mathbf{P}^{-T} = -\mathbf{C} \cdot \mathbf{S} \cdot \mathbf{P}^{-T} \tag{2.111}
$$

and **plastic hyperstress tensor** defined as

$$
\boldsymbol{H}_p := -K_0(\mathbf{C}_e, \boldsymbol{K}_e) = -\mathbf{P}^{-1} \circ \boldsymbol{H}. \tag{2.112}
$$

According to (2.110) the stress power goes into a change of the elastic reference energy and a dissipative part that is only active during yielding, and works on the rates $\mathbf{P}^{\bullet}$ and $\boldsymbol{P}^{\bullet}$.

Flow and Hardening Rules

For the evolution of the internal plastic variables $\mathbf{P}$, $\boldsymbol{P}$, $\mathbf{Z}_p$ evolution equations are needed, namely two **flow rules**

(2.113) $$\mathbf{P}^\bullet = f(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p, \mathbf{C}^\bullet, \boldsymbol{K}^\bullet)$$

(2.114) $$\boldsymbol{P}^\bullet = F(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p, \mathbf{C}^\bullet, \boldsymbol{K}^\bullet)$$

and a **hardening rule**

(2.115) $$\mathbf{Z}_p^\bullet = h(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p, \mathbf{C}^\bullet, \boldsymbol{K}^\bullet)$$

all assumed to be in the form of rate-independent ODEs as customary in plasticity. This can be assured in the usual way by the introduction of a **plastic consistency parameter** $\lambda \geq 0$

(2.116) $$\mathbf{P}^\bullet = \lambda f^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p, \mathbf{C}^\circ, \boldsymbol{K}^\circ)$$

(2.117) $$\boldsymbol{P}^\bullet = \lambda F^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p, \mathbf{C}^\circ, \boldsymbol{K}^\circ)$$

(2.118) $$\mathbf{Z}_p^\bullet = \lambda h^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p, \mathbf{C}^\circ, \boldsymbol{K}^\circ)$$

where we normed the increments of the kinematical variables

(2.119) $$\mathbf{C}^\circ := \mathbf{C}^\bullet/\mu \quad \text{and} \quad \boldsymbol{K}^\circ := \boldsymbol{K}^\bullet/\mu$$

by a factor

(2.120) $$\mu := \sqrt{(|\mathbf{C}^\bullet|^2 + L^2 |\boldsymbol{K}^\bullet|^2)}$$

which is (only) positive during yielding. The positive constant L with the dimension of a length is necessary for dimensional reasons and controls the ratio of yielding due to $\mathbf{C}^\bullet$ and $\boldsymbol{K}^\bullet$.

We introduced three functions f°, F°, h°, which give the directions of the flow and hardening, while the amount is finally determined by the consistency parameter. The consistency parameter is zero during elastic processes. During yielding it can be calculated by the yield condition (2.103)

$$\begin{aligned} 0 &= \varphi(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p)^\bullet \\ &= \partial_{\mathbf{P}} \varphi \cdot\cdot \mathbf{P}^\bullet + \partial_{\boldsymbol{P}} \varphi \cdots \boldsymbol{P}^\bullet + \partial_{\mathbf{C}} \varphi \cdot\cdot \mathbf{C}^\bullet + \partial_{\boldsymbol{K}} \varphi \cdots \boldsymbol{K}^\bullet + \partial_{\mathbf{Z}_p} \varphi \cdot\cdot \mathbf{Z}_p^\bullet \end{aligned}$$

by (2.116) - (2.118)

(2.121) $$\begin{aligned} &= \partial_{\mathbf{P}} \varphi \cdot\cdot \lambda f^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p, \mathbf{C}^\circ, \boldsymbol{K}^\circ) + \partial_{\boldsymbol{P}} \varphi \cdots \lambda F^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p, \mathbf{C}^\circ, \boldsymbol{K}^\circ) \\ &+ \partial_{\mathbf{C}} \varphi \cdot\cdot \mathbf{C}^\bullet + \partial_{\boldsymbol{K}} \varphi \cdots \boldsymbol{K}^\bullet + \partial_{\mathbf{Z}_p} \varphi \cdot\cdot \lambda h^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p, \mathbf{C}^\circ, \boldsymbol{K}^\circ) \end{aligned}$$

which gives the quotient

(2.122) $$\begin{aligned} \lambda = &- [\partial_{\mathbf{C}} \varphi \cdot\cdot \mathbf{C}^\bullet + \partial_{\boldsymbol{K}} \varphi \cdots \boldsymbol{K}^\bullet] / \\ &[\partial_{\mathbf{P}} \varphi \cdot\cdot f^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p, \mathbf{C}^\circ, \boldsymbol{K}^\circ) + \partial_{\boldsymbol{P}} \varphi \cdots F^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}_p, \mathbf{Z}, \mathbf{C}^\circ, \boldsymbol{K}^\circ) \end{aligned}$$

$$+ \partial_{\mathbf{Z}_p} \varphi \cdot\cdot\, h^{\circ}(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}_p, \mathbf{C}^{\circ}, \boldsymbol{K}^{\circ})] \,.$$

Both, numerator and denominator of this ratio are always negative during yielding as a consequence of the loading condition (2.104), and, thus, λ is positive in this case.

If we substitute this value of λ into (2.116) - (2.118), we obtain the **consistent flow** and **hardening rules**. In all cases (elastic *and* plastic), the KUHN-TUCKER condition

(2.123) $$\lambda\, \varphi = 0 \qquad \text{with} \qquad \lambda \geq 0 \qquad \text{and} \qquad \varphi \leq 0$$

holds since at any time one of the two factors is zero.

Example. von MISES plasticity extended

As an example we use a generalization of the anisotropic v. MISES yield criterion[27] in the stress space[28]

(2.124) $$\varphi(\mathbf{S}, \boldsymbol{H}) = \tfrac{1}{2} (\mathbf{S} - \mathbf{S}_B) \cdot\cdot \overset{\langle 4 \rangle}{\boldsymbol{G}} (\mathbf{S} - \mathbf{S}_B) + \tfrac{1}{2} (\boldsymbol{H} - \boldsymbol{H}_B) \cdots \overset{\langle 6 \rangle}{\boldsymbol{G}} \cdots (\boldsymbol{H} - \boldsymbol{H}_B)$$
$$+ (\mathbf{S} - \mathbf{S}_B) \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{G}} \cdots (\boldsymbol{H} - \boldsymbol{H}_B) - \sigma_Y(\mathbf{Z}_p)^2$$

with three material tensors $\overset{\langle 4 \rangle}{\boldsymbol{G}}$, $\overset{\langle 5 \rangle}{\boldsymbol{G}}$, and $\overset{\langle 6 \rangle}{\boldsymbol{G}}$ reflecting the symmetry of the material, two back stresses $\mathbf{S}_B$ and $\boldsymbol{H}_B$, and a scalar yield stress σ_Y, which may depend on a hardening variable controlling isotropic hardening. This yield criterion lives in the stress space. However, by the elastic laws (2.90) and (2.91) it can be transformed into the strain space.

The **associated flow rules** are

(2.125) $$\mathbf{P}^{\bullet} = \lambda\, \partial_{\mathbf{S}} \varphi = \lambda \{ \overset{\langle 4 \rangle}{\boldsymbol{G}} \cdot\cdot [\mathbf{S} - \mathbf{S}_B] + \overset{\langle 5 \rangle}{\boldsymbol{G}} \cdots [\boldsymbol{H} - \boldsymbol{H}_B] \}$$

(2.126) $$\boldsymbol{P}^{\bullet} = \lambda\, \partial_{\boldsymbol{H}} \varphi = \lambda \{ \overset{\langle 6 \rangle}{\boldsymbol{G}} \cdots [\boldsymbol{H} - \boldsymbol{H}_B] + [\mathbf{S} - \mathbf{S}_B] \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{G}} \}.$$

[27] v. MISES (1928)
[28] see BERTRAM/ FOREST (2014)

2.3 Finite Gradient Thermoelasticity

This chapter is mainly based on

> Bertram, A.: *Finite gradient elasticity and plasticity: a constitutive thermodynamical framework*. Continuum Mech. Thermodyn., DOI 10.1007/s00161-015-0417-6 (in press)[29]

GURTIN (1965) has once shown that non-local elastic materials (other than simple elastic materials) cannot exist because of a violation of the CLAUSIUS-DUHEM inequality. However, this surprising conclusion is due to an inadequate format, in which higher-order stress tensors are not taken into account.

In the present work, we will show that for a more general format such materials can exist in a thermodynamical consistent form, and the second law gives the thermoelastic potentials and reasonable restrictions upon the yield and hardening rules in the case of plasticity.

There are not many papers yet concerning the thermodynamics of gradient elasticity and plasticity like, *e.g.*, POLIZZOTTO/ BORINO (1998), POLIZZOTTO (2003), (2011), HIRSCHBERGER/ STEINMANN (2009), GURTIN/ ANAND (2010), IEŞAN/ QUINTANILLA (2013) . In VÁN/ BEREZOVSKI/ PAPENFUSS (2013) a thermodynamic theory of weakly non-local continua is presented. It is specified for linear viscoelasticity and elasticity, an approach, however, that is rather different in assumptions and methodology from the present one.[30] CLAYTON/ MCDOWELL/ BAMMANN (2006) suggest a thermodynamic theory for gradient and polar plasticity that works on different scales, which is, however, beyond the scope of the present paper.

With respect to the mechanical properties we tried to find a format that is as wide as possible to cover essentially all kinds of gradient elasticity and unconstrained plasticity, isotropic or anisotropic. With respect to the thermodynamics, however, we follow the traditional lines of FOURIER and CLAUSIUS and DUHEM without introducing any non-classical concepts like, *e.g.*, CARDONA/ FOREST/ SIEVERT (1999) or FOREST/ AIFANTIS (2010) did.

The step from a purely mechanical theory in the preceding chapter to thermodynamics follows the lines of BERTRAM/ KRAWIETZ (2012) for classical thermoplasticity, and BERTRAM/ FOREST (2014) for the geometrically linear gradient plasticity (see next chapter).

This thermomechanical part of a gradient theory is organized as follows. After introducing the complete set of thermodynamical variables, we are able to define a general thermoelastic material of gradient type. This can be brought into a reduced form, thus allowing for the *Principle of Invariance under Rigid Body Modifications*. The CLAUSIUS-DUHEM inequality renders the potential relations for the stresses and the hyperstresses. After working out the

[29] see also BERTRAM (2013)
[30] see also PAPENFUSS/ FOREST (2006)

transformations of the constitutive equations under a change of the reference placement, we are able to define the symmetry transformations for such material models.

For a thermodynamical theory we introduce the following fields:

ε	the specific **internal energy**
Q	the **heat supply** per unit mass and time by irradiation and conduction
$\mathbf{q}$	the **spatial heat flux** per unit area in the current placement and unit time forming one part of Q
θ	the absolute **temperature**
$\mathbf{g}_0 := Grad\ \theta$	the material temperature gradient
η	the specific **entropy**

The **material heat flux** per unit time and per unit area in the reference placement is then

(2.127) $$\mathbf{q}_0 = J\,\mathbf{F}^{-1} \cdot \mathbf{q}_E .$$

A change of the reference placement will yield due to (2.36)

(2.128) $$\underline{\mathbf{g}}_0 = \mathbf{A}^T \cdot \mathbf{g}_0 = \mathbf{A}^T * \mathbf{g}_0$$

(2.129) $$\underline{\mathbf{q}}_0 = J_A\,\mathbf{A}^{-1} \cdot \mathbf{q}_0 = \mathbf{A}^{-1} * J_A\,\mathbf{q}_0$$

while the values of the scalar variables ε, Q, θ, η remain unaltered.

As usual we can substitute the internal energy by the **HELMHOLTZ free energy**

(2.130) $$\psi := \varepsilon - \theta\,\eta .$$

The **first law of thermodynamics** (energy balance) in the local form is assumed as

(2.131) $$Q + \pi_i = \varepsilon^{\bullet}.$$

The second law of thermodynamics is assumed in the form of the **CLAUSIUS-DUHEM inequality**

(2.132) $$\pi_i - \psi^{\bullet} - \theta^{\bullet}\,\eta - \frac{1}{\rho_0\theta}\,\mathbf{q}_0 \cdot \mathbf{g}_0 \geq 0 .$$

The last term in the inequality is the **thermal dissipation**, while the other three terms stand for the **mechanical dissipation**.

The independent material variables of this theory are given by the **thermo-kinematical processes** which we denote as

$$\{\boldsymbol{\chi}(\tau), \mathbf{F}(\tau), Grad\,\mathbf{F}(\tau), \theta(\tau), grad\,\theta(\tau)\Big|_{\tau=0}^{t}\}$$

where the time-variable τ runs over a finite closed time-interval $[0, t]$. In the general case of inelastic gradient materials it is assumed that such a process out of a particular initial state determines (by means of a *process functional*) the **caloro-dynamic state** at its end consisting of the stresses, the heat flux, the internal energy, and the entropy

$$\{\mathbf{T}(t), \boldsymbol{G}(t), \mathbf{q}(t), \varepsilon(t), \eta(t)\} \ .$$

Elasticity means that the current thermo-kinematical state already determines the current caloro-dynamic state, without any memory of the past process.

Definition 2.6. A **thermoelastic gradient material** is given by thermoelastic laws

$$\begin{aligned} \mathbf{T} &= T_E(\boldsymbol{\chi}, \mathbf{F}, \mathit{Grad}\,\mathbf{F}, \theta, \mathit{grad}\,\theta) \\ \boldsymbol{G} &= G_E(\boldsymbol{\chi}, \mathbf{F}, \mathit{Grad}\,\mathbf{F}, \theta, \mathit{grad}\,\theta) \\ \mathbf{q} &= q_E(\boldsymbol{\chi}, \mathbf{F}, \mathit{Grad}\,\mathbf{F}, \theta, \mathit{grad}\,\theta) \\ \varepsilon &= \varepsilon_E(\boldsymbol{\chi}, \mathbf{F}, \mathit{Grad}\,\mathbf{F}, \theta, \mathit{grad}\,\theta) \\ \eta &= \eta_E(\boldsymbol{\chi}, \mathbf{F}, \mathit{Grad}\,\mathbf{F}, \theta, \mathit{grad}\,\theta) \end{aligned} \tag{2.133}$$

where all variables are taken at the same material point at the same instant of time. The suffix E stands for elastic.

This format fulfils the *Principle of Equipresence* as it is usually claimed for.

These constitutive equations can be further reduced by means of the *Principle of Invariance under Rigid Body Modifications* [31]. As one can easily show, a reduced form for this set of constitutive equations is

$$\begin{aligned} \mathbf{S} &= k(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \\ \boldsymbol{H} &= K(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \\ \mathbf{q}_0 &= q(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \\ \varepsilon &= \varepsilon(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \\ \eta &= \eta(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \end{aligned} \tag{2.134}$$

with $(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \in \mathscr{C}onf \times \mathscr{R}^+ \times \mathscr{V}^3$, where exclusively material (or LAGRANGEan) variables have been used, which remain invariant under changes of observer and rigid body modifications.

The HELMHOLTZ free energy is after (2.130) also a function of the reduced thermo-kinematical state

$$\psi(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) := \varepsilon(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) - \theta\,\eta(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0). \tag{2.135}$$

We will next investigate the consequences of the second law of thermodynamics in the form of the CLAUSIUS-DUHEM inequality (2.132) for this class of elastic materials

$$\begin{aligned} 0 \geq -\frac{1}{\rho_0}\,(\tfrac{1}{2}\,\mathbf{S} \cdot\cdot \mathbf{C}^{\bullet} + \boldsymbol{H} \cdots \boldsymbol{K}^{\bullet}) + \partial_{\mathbf{C}}\,\psi \cdot\cdot\, \mathbf{C}^{\bullet} + \partial_{\boldsymbol{K}}\,\psi \cdots \boldsymbol{K}^{\bullet} + \partial_{\theta}\,\psi\,\theta^{\bullet} + \partial_{\mathbf{g}_0}\,\psi \cdot \mathbf{g}_0^{\bullet} \\ + \theta^{\bullet}\,\eta + \frac{1}{\rho_0\theta}\,\mathbf{q}_0 \cdot \mathbf{g}_0 \end{aligned} \tag{2.136}$$

[31] see BERTRAM/ SVENDSEN (2001), and BERTRAM (2005), therein called PISM

$$= (\partial_{\mathbf{C}}\,\psi - \frac{1}{2\rho_0}\,\mathbf{S}) \cdot\cdot\, \mathbf{C}^{\bullet} + (\partial_{\boldsymbol{K}}\,\psi - \frac{1}{\rho_0}\boldsymbol{H}) \cdot\cdot\cdot\, \boldsymbol{K}^{\bullet} + (\,\partial_{\theta}\,\psi + \eta)\,\theta^{\bullet} + \partial_{\mathbf{g}_0}\,\psi \cdot \mathbf{g}_0^{\bullet}$$

$$+ \frac{1}{\rho_0\theta}\,\mathbf{q}_0 \cdot \mathbf{g}_0\,.$$

This leads by standard arguments to the thermoelastic relations

(2.137) $\partial_{\mathbf{g}_0}\,\psi = \mathbf{o}$ (independence of the free energy of the temperature gradient)

(2.138) $k\,(\mathbf{C}, \boldsymbol{K}, \theta) = 2\rho_0\,\partial_{\mathbf{C}}\,\psi$ (potential for the stresses)

(2.139) $K\,(\mathbf{C}, \boldsymbol{K}, \theta) = \rho_0\,\partial_{\boldsymbol{K}}\,\psi$ (potential for the hyperstresses)

(2.140) $\eta = -\,\partial_{\theta}\,\psi$ (potential for the elastic part of the entropy)

(2.141) $0 \geq \mathbf{q}_0 \cdot \mathbf{g}_0$ (heat conduction inequality).

Theorem 2.3. *The CLAUSIUS-DUHEM inequality* (2.132) *is fulfilled for a thermoelastic gradient material during every thermo-kinematical process if and only if the following conditions hold.*

- *The free energy does not depend on the temperature gradient.*
- *The free energy is a potential for the stresses and the entropy after* (2.138) - (2.140).
- *The heat conduction inequality* (2.141) *holds at every instant.*

Thus in elasticity, the complete material model is determined if we only know the two functions $\psi(\mathbf{C}, \boldsymbol{K}, \theta)$ and $q\,(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0)$. The mechanical dissipation is here zero, while the thermal dissipation alone must be non-negative.

Material Isomorphy and Symmetry

We will next establish criteria to express the notion that two thermoelastic points consist of the same material. This is already the case if all the constitutive equations of the two points coincide. However, such a definition would be too restrictive, since we know that our variables depend on the choice of the reference placement. So we first have to admit an appropriate change of the reference placements, before we compare the constitutive equations. And since the mass density always influences the constitutive equations, we claim that the density in such reference placements must also be identical.

This leads to the concept of *material isomorphisms*. The basic idea behind this concept is the following[32]. We consider two thermoelastic points as isomorphic if their thermoelastic behavior shows no *measurable* difference during arbitrary processes. As measurable quantities we consider the stresses (as a result of balance of moments), the heat flux, and the rate of the internal energy (as a result of the energy balance), while the entropy or the free energy are certainly not measurable.

For thermoelastic materials the mechanical dissipation is zero

$$0 = \pi_i - \psi^\bullet - \theta^\bullet \eta = \pi_i - \varepsilon^\bullet + \theta \eta^\bullet = -Q + \theta \eta^\bullet \tag{2.142}$$

using (2.130) - (2.131). If the heat supply Q and the temperature θ are measurable quantities, then so is the rate of the entropy for thermoelastic materials. So the entropy of two isomorphic thermoelastic materials named X and Y can differ only by a constant, which can not be determined by any measurement, in principle,

$$\eta_Y(\mathbf{C}_Y, \boldsymbol{K}_Y, \theta) = \eta_X(\mathbf{C}_X, \boldsymbol{K}_X, \theta) + \eta_c \,. \tag{2.143}$$

By integrating this with respect to the temperature, we obtain for the free energy after (2.140)

$$\psi_Y(\mathbf{C}_Y, \boldsymbol{K}_Y, \theta) = \psi_X(\mathbf{C}_X, \boldsymbol{K}_X, \theta) - \eta_c \theta + \varepsilon_c \tag{2.144}$$

with another constant ε_c . The internal energy is then by (2.130)

$$\begin{aligned} \varepsilon_Y(\mathbf{C}_Y, \boldsymbol{K}_Y, \theta) &= \psi_Y(\mathbf{C}_Y, \boldsymbol{K}_Y, \theta) + \theta\, \eta_Y(\mathbf{C}_X, \boldsymbol{K}_X, \theta) \\ &= \psi_X(\mathbf{C}_X, \boldsymbol{K}_X, \theta) + \theta\, \eta_X(\mathbf{C}_X, \boldsymbol{K}_X, \theta) + \varepsilon_c \\ &= \varepsilon_X(\mathbf{C}_X, \boldsymbol{K}_X, \theta) + \varepsilon_c \,. \end{aligned} \tag{2.145}$$

In the context of plasticity we will see that these constants play a rather important role.

This leads to the following definition.

Definition 2.7. Two thermoelastic points X and Y are called **elastically isomorphic** if we can find reference placements κ_X for X and κ_Y for Y such that the following two conditions hold.

- In κ_X and κ_Y the mass densities are equal

[32] see BERTRAM/ KRAWIETZ (2012)

(2.146) $\rho_{0Y} = \rho_{0X}$.

- With respect to κ_X and κ_Y the thermoelastic laws are related by

(2.147) $\psi_Y(\mathbf{C}, \boldsymbol{K}, \theta) = \psi_X(\mathbf{C}, \boldsymbol{K}, \theta) - \eta_c\theta + \varepsilon_c \qquad \forall\ (\mathbf{C}, \boldsymbol{K}, \theta) \in \mathcal{C}onf \times \mathcal{R}^+$

(2.148) $q_Y(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) = q_X(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \quad \forall\ (\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \in \mathcal{C}onf \times \mathcal{R}^+ \times \mathcal{V}^3$

with two real constants η_c and ε_c.

As a consequence of (2.138) - (2.140) this leads also to identities of the other constitutive equations

(2.149) $k_Y(\mathbf{C}, \boldsymbol{K}, \theta) = k_X(\mathbf{C}, \boldsymbol{K}, \theta)$

(2.150) $K_Y(\mathbf{C}, \boldsymbol{K}, \theta) = K_X(\mathbf{C}, \boldsymbol{K}, \theta)$

(2.151) $\eta_Y(\mathbf{C}, \boldsymbol{K}, \theta) = \eta_X(\mathbf{C}, \boldsymbol{K}, \theta) + \eta_c$

(2.152) $\varepsilon_Y(\mathbf{C}, \boldsymbol{K}, \theta) = \varepsilon_X(\mathbf{C}, \boldsymbol{K}, \theta) + \varepsilon_c \qquad \forall\ (\mathbf{C}, \boldsymbol{K}, \theta) \in \mathcal{C}onf \times \mathcal{R}^+.$

If two points are isomorphic in the above sense, one would consider them as consisting of the same **material**. In fact, the isomorphy condition induces an equivalence relation on all thermoelastic gradient materials, the fibers of which constitute the different materials.

By arguments that have already been given for Theorem 2.1 in the mechanical case, one can then show that these conditions are equivalent to the following statement, which is much easier to handle than the one of the above definition.

Theorem 2.4. *Two thermoelastic points X and Y with thermoelastic laws ψ_X, q_X and ψ_Y, q_Y with respect to arbitrary reference placements are elastically isomorphic if and only if there exist two tensors $(\mathbf{P}, \boldsymbol{P}) \in \mathcal{I}nv\mathcal{C}omb$ and two real constants ε_c and η_c such that*

$$\rho_{0X} = \rho_{0Y}\ det\,\mathbf{P}$$

(2.153)
$$\psi_Y(\mathbf{C}, \boldsymbol{K}, \theta) = \psi_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}, \theta) - \eta_c\theta + \varepsilon_c$$
$$(det\,\mathbf{P})\, q_Y(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) = \mathbf{P} * q_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}, \theta, \mathbf{P}^T * \mathbf{g}_0)$$

hold for all $(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \in \mathcal{C}onf \times \mathcal{R}^+ \times \mathcal{V}^3$ with ρ_{0X} and ρ_{0Y} being the mass densities in the reference placements of X and Y, respectively.

Thus, the couple $(\mathbf{P}, \boldsymbol{P}) \in \mathcal{I}nv\mathcal{C}omb$ together with the reals ε_c and η_c determine the isomorphy transformation. If (2.153) hold, then we have also

$$det\,(\mathbf{P})\, k_Y(\mathbf{C}, \boldsymbol{K}, \theta) = \mathbf{P} * k_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}, \theta)$$
$$det\,(\mathbf{P})\, K_Y(\mathbf{C}, \boldsymbol{K}, \theta) = \mathbf{P} \circ K_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}, \theta)$$

(2.154)
$$\varepsilon_Y(\mathbf{C}, \boldsymbol{K}, \theta) = \varepsilon_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}, \theta) + \varepsilon_c$$
$$\eta_Y(\mathbf{C}, \boldsymbol{K}, \theta) = \eta_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}, \theta) + \eta_c$$

for all $(\mathbf{C}, \boldsymbol{K}, \theta) \in \mathcal{C}onf \times \mathcal{R}^+$.

If we particularize the concept of isomorphy to identical points $X \equiv Y$, it defines automorphy or **symmetry**. In this case, we use only one reference placement. Therefore, the isomorphism $\mathbf{P}$ must be unimodular, and the two constants in the free energy and the entropy can be omitted.

Definition 2.8. For a thermoelastic gradient material with material laws ψ and q, a **symmetry transformation** is a pair $(\mathbf{A}, \boldsymbol{A}) \in \mathcal{U}nim\mathcal{C}omb$ such that

$$\psi(\mathbf{C}, \boldsymbol{K}, \theta) = \psi(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A}, \theta) \tag{2.155}$$
$$q(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) = \mathbf{A} * q(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A}, \theta, \mathbf{A}^T * \mathbf{g}_0)$$

for all $(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \in \mathcal{C}onf \times \mathcal{R}^+ \times \mathcal{V}^3$.

As a consequence of (2.154) this leads to the symmetry transformations of the other constitutive equations

$$k(\mathbf{C}, \boldsymbol{K}, \theta) = \mathbf{A} * k(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A}, \theta)$$
$$K(\mathbf{C}, \boldsymbol{K}, \theta) = \mathbf{A} \circ K(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A}, \theta) \tag{2.156}$$
$$\varepsilon(\mathbf{C}, \boldsymbol{K}, \theta) = \varepsilon(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A}, \theta)$$
$$\eta(\mathbf{C}, \boldsymbol{K}, \theta) = \eta(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A}, \theta) \qquad \forall (\mathbf{C}, \boldsymbol{K}, \theta) \in \mathcal{C}onf \times \mathcal{R}^+.$$

The set of all such symmetry transformations $(\mathbf{A}, \boldsymbol{A}) \in \mathcal{U}nim\mathcal{C}omb$ represents the **symmetry group** of the material.

This group is used to define isotropy or anisotropy. If the symmetry group is a subgroup of $\mathcal{O}rth\mathcal{C}omb$ in the first entry and zero in the second, $(\mathbf{Q}, \boldsymbol{O})$, these transformations can be interpreted as rigid rotations, and we call the respective reference placement an **undistorted state**. If a material allows for such undistorted states, it is a **solid**. If it contains all orthogonal dyadics in the first entry, then the material is called **isotropic**. These definitions apply not only to gradient thermoelasticity, but also to any inelastic gradient material in an analogous way.

In all of these cases of orthogonal symmetry transformations, we obtain with respect to undistorted states using (2.44)

$$\mathbf{A} * k(\mathbf{C}, \boldsymbol{K}, \theta) = k(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K}, \mathbf{A} * \theta)$$
$$\mathbf{A} * K(\mathbf{C}, \boldsymbol{K}, \theta) = K(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K}, \mathbf{A} * \theta)$$
$$\mathbf{A} * q(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) = q(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K}, \mathbf{A} * \theta, \mathbf{A} * \mathbf{g}_0) \tag{2.157}$$
$$\mathbf{A} * \varepsilon(\mathbf{C}, \boldsymbol{K}, \theta) = \varepsilon(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K}, \mathbf{A} * \theta)$$
$$\mathbf{A} * \psi(\mathbf{C}, \boldsymbol{K}, \theta) = \psi(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K}, \mathbf{A} * \theta)$$
$$\mathbf{A} * \eta(\mathbf{C}, \boldsymbol{K}, \theta) = \eta(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K}, \mathbf{A} * \theta)$$

for all $(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \in \mathcal{C}onf \times \mathcal{R}^+ \times \mathcal{V}^3$. Thus, for an isotropic material in an undistorted state, the elastic laws are isotropic tensor functions.

2.4 Finite Gradient Thermoplasticity

After having provided the thermoelastic theory, we are able to consider plasticity, starting with the concept of thermoelastic ranges. After the assumption that the (measurable) thermo-elastic behavior is not altered by plastic yielding, we can introduce plastic variables in a natural way. It turns out that both a multiplicative and an additive split of the strain variables is obtained. The exploitation of the CLAUSIUS-DUHEM inequality leads to necessary and sufficient conditions for thermodynamical consistency. A residual dissipation inequality restricts the flow and hardening rules in connection with the yield condition. Finally, we can derive a rate-equation for the temperature evolution due to elastic and plastic deformations. The entire section is restricted to rate-independent behavior. And it is a second-order theory; higher-order gradients are not included.

For this class of materials, the concept of *elastic ranges* plays a fundamental role.

Definition 2.9. A **(thermo)elastic range** is a triple $\{\mathscr{E}_p, \psi_p, q_p\}$ consisting of

1) a non-empty and path-connected submanifold with boundary

$$\mathscr{E}_p \subset \mathscr{C}onf \times \mathscr{R}^+ \times \mathscr{V}^3$$

of the space of the thermo-kinematical variables,

2) and thermoelastic laws ψ_p, q_p that give for all thermo-kinematical processes out of some initial state

$$\{\mathbf{C}(\tau), \boldsymbol{K}(\tau), \theta(\tau), \mathbf{g}_0(\tau)\Big|_{t_0}^{t}\},$$

which remain at all times in $\mathscr{E}_p$, the caloro-dynamic state by thermoelastic laws

$$\begin{aligned} \psi(t) &= \psi_p(\mathbf{C}(t), \boldsymbol{K}(t), \theta(t)) \\ \mathbf{q}_0(t) &= q_p(\mathbf{C}(t), \boldsymbol{K}(t), \theta(t), \mathbf{g}_0(t)) \end{aligned} \tag{2.158}$$

and, consequently,

$$\begin{aligned} \mathbf{S}(t) &= 2\rho_0\, \partial_{\mathbf{C}}\, \psi_p(\mathbf{C}(t), \boldsymbol{K}(t), \theta(t)) \\ \boldsymbol{H}(t) &= \rho_0\, \partial_{\boldsymbol{K}}\, \psi_p(\mathbf{C}(t), \boldsymbol{K}(t), \theta(t)) \\ \varepsilon(t) &= \psi_p(\mathbf{C}(t), \boldsymbol{K}(t), \theta(t)) - \theta\, \partial_\theta\, \psi_p(\mathbf{C}(t), \boldsymbol{K}(t), \theta(t)) \\ \eta(t) &= -\partial_\theta\, \psi_p(\mathbf{C}(t), \boldsymbol{K}(t), \theta(t)). \end{aligned} \tag{2.159}$$

We assume further on that these functions are continuous and continuously differentiable on $\mathscr{E}_p$, and as such extendible to $\mathscr{C}onf \times \mathscr{R}^+ \times \mathscr{V}^3$.

Two assumptions are needed to specify elastoplastic behavior.

Assumption 2.3. *At the end of each thermo-kinematical process* $\{\mathbf{C}(\tau), \boldsymbol{K}(\tau), \theta(\tau), \mathbf{g}_0(\tau) \,\big|_{t_0}^{t}\}$ *of a thermoelastoplastic material point, there exists a thermoelastic range such that*

- *the terminate value of the process is contained in it*

$$\{\mathbf{C}(t), \boldsymbol{K}(t), \theta(t), \mathbf{g}_0(t)\} \in \mathscr{E}_p$$

- *and the caloro-dynamic state is determined by its thermoelastic laws* (2.158) *and* (2.159)*, the same as for any continuation of this process that remains in* $\mathscr{E}_p$ *at all times.*

For many materials it is a microphysically and experimentally well-substantiated fact that during yielding the elastic behavior hardly alters even under very large deformations, which will certainly not be the case for porous media, for foams, under damage, etc., but holds for most solid metals. This assumption reduces the effort for the identification tremendously, since otherwise one would have to identify the elastic constants at each step of the plastic deformation anew. We now give this assumption a precise form.

Assumption 2.4. *The thermoelastic laws of all elastic ranges are isomorphic.*

Note that this assumption only refers to the thermoelastic laws associated to the elastic range, and not to the set $\mathscr{E}_p$ itself. So nothing is said here with respect to changes of the latter due to hardening or softening or the like.

If all elastic laws belonging to different elastic ranges are mutually isomorphic, then they all are isomorphic to some appropriately chosen **elastic reference laws** q_0, ψ_0. While the current elastic laws q_p and ψ_p vary with time during yielding, these reference laws can always be chosen as constant in time. We express this useful fact in the following theorem using Theorem 2.4.

Theorem 2.5. *Let* ψ_0 and q_0 *be the elastic reference laws for an elasto-plastic gradient material. Then for each elastic range* $\{\mathscr{E}_p, \psi_p, q_p\}$ *there are two tensors* $(\mathbf{P}, \boldsymbol{P}) \in \mathscr{U}nim\mathscr{C}omb$ *and two real constants* ε_c *and* η_c *such that*

$$\psi_p(\mathbf{C}, \boldsymbol{K}, \theta) = \psi_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}, \theta) + \varepsilon_c - \theta\,\eta_c \tag{2.160}$$

$$q_p(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) = \mathbf{P} * q_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}, \theta, \mathbf{P}^T * \mathbf{g}_0)$$

hold for all $(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0) \in \mathscr{C}onf \times \mathscr{R}^+ \times \mathscr{V}^3$.

In our theory, the two tensors $(\mathbf{P}, \boldsymbol{P}) \in \mathscr{U}nim\mathscr{C}omb$ and the two real constants ε_c and η_c play the role of (plastic) internal variables. They do not have the interpretation of (plastic) deformations, but instead of (**plastic**) **transformations**.

Note that the additive constants ε_c and η_c can *not* depend on the current thermo-kinematical variables $\mathbf{C}$, $\boldsymbol{K}$, θ, and $\mathbf{g}_0$ after the above Assumption 2.4.

We introduce the following abbreviations

$$\begin{aligned} \mathbf{C}_e &:= \mathbf{P}^T \cdot \mathbf{C} \cdot \mathbf{P} = \mathbf{P}^T * \mathbf{C} \\ \boldsymbol{K}_e &:= \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P} \\ \mathbf{g}_e &:= \mathbf{P}^T \cdot \mathbf{g}_0 = \mathbf{P}^T * \mathbf{g}_0 . \end{aligned} \tag{2.161}$$

The time derivatives of these elastic variables will later be needed. They are

(2.162) $$\mathbf{C}_e^{\bullet} = \mathbf{P}^T * \mathbf{C}^{\bullet} + 2\, sym(\mathbf{C}_e \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet})$$

where *sym* stands for the symmetric part of a dyadic, and

(2.163) $$\boldsymbol{K}_e^{\bullet} = \mathbf{P}^T \circ \boldsymbol{K}^{\bullet} + \boldsymbol{P}^{\bullet} - \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet} \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\, subsym\, [(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}]$$

where *subsym* stands for the symmetric part of a triadic with respect to the right subsymmetry.

As consequences of (2.138) - (2.140) we obtain the isomorphic forms for all constitutive equations

(2.164)
$$\begin{aligned}
\psi &= \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) + \varepsilon_c - \theta \eta_c \\
\mathbf{S} &= \mathbf{P} * 2\rho_0 \partial_{\mathbf{C}_e} \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) \\
\boldsymbol{H} &= \mathbf{P} \circ \rho_0 \partial_{\boldsymbol{K}_e} \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) \\
\eta &= \eta_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) + \eta_c \\
\mathbf{q}_0 &= \mathbf{P} * q_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta, \mathbf{g}_e) \\
\varepsilon &= \varepsilon_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) + \varepsilon_c
\end{aligned}$$

with $$\eta_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) := -\partial_\theta \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta)$$

and $$\varepsilon_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) := \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) - \theta \partial_\theta \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta)\,.$$

Yielding and Hardening

The boundary $\partial\mathscr{E}_p$ of some elastic range $\mathscr{E}_p$ is called **yield limit** or **yield surface** of the thermo-elastic range.

However, there is no material known for which the yield limit depends on the temperature gradient, so that $\mathscr{E}_p$ is assumed to be trivial in its last component $\mathscr{V}^3$. In the sequel we will suppress this last component of $\mathscr{E}_p$, so that $\mathscr{E}_p$ is considered as a subset of only $\mathscr{C}onf \times \mathscr{R}^+$. In order to practically describe such subsets we introduce the yield criterion as an indicator function. More precisely, a **yield criterion** associated with some thermoelastic range is a mapping

$$\Phi_p \; : \; \mathscr{C}onf \times \mathscr{R}^+ \to \mathscr{R} \quad | \quad \{\mathbf{C}, \boldsymbol{K}, \theta\} \mapsto \Phi_p(\mathbf{C}, \boldsymbol{K}, \theta)$$

the kernel of which forms the yield surface

$$\Phi_p(\mathbf{C}, \boldsymbol{K}, \theta) = 0 \;\Leftrightarrow\; \{\mathbf{C}, \boldsymbol{K}, \theta\} \in \partial\mathscr{E}_p \,. \tag{2.165}$$

We refer to the equation (2.165) as the **yield condition**. For the distinction of states in the interior $\mathscr{E}_p^o$ and beyond the thermo-elastic range, we demand

$$\Phi_p(\mathbf{C}, \boldsymbol{K}, \theta) < 0 \;\Leftrightarrow\; (\mathbf{C}, \boldsymbol{K}, \theta) \in \mathscr{E}_p^o. \tag{2.166}$$

The **loading condition** is

$$\Phi_p(\mathbf{C}, \boldsymbol{K}, \theta)^\bullet = \partial_{\mathbf{C}}\Phi_p \cdot\cdot\, \mathbf{C}^\bullet + \partial_{\boldsymbol{K}}\Phi_p \cdots \boldsymbol{K}^\bullet + \partial_\theta \Phi_p \, \theta^\bullet > 0 \,. \tag{2.167}$$

Note that we defined the elastic ranges in the space of the independent variables, so that the yield criterion is expressed in terms of thermo-kinematical variables. If one prefers a description in the stress space, one can easily use the elastic laws (2.164) to transform them into the space of the caloro-dynamic variables.

Such a yield criterion is associated with a particular elastic range. A yield criterion can be found for every elastic range, but it is by no means unique. The differentiability my not be given for singular points like vertices of the yield surface. However, we will only refer to smooth yield surfaces in the rest of the text, for simplicity.

Up to now we only considered a yield criterion for one specific elastic range. We will next try to generalize this concept to all the other potential elastic ranges of the same material point. For the general yield criterion of *all* elastic ranges we use the ansatz $\varphi(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p)$ with **hardening variables** $\mathbf{Z}_p$ assumed to be differentiable in all arguments so that

$$\Phi_p(\mathbf{C}, \boldsymbol{K}, \theta) = \varphi(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p) \tag{2.168}$$

holds. $\mathbf{Z}_p$ is a tensor of arbitrary order or even a vector of such tensors. For convenience, we notated it as a dyadic. The general form of the yield condition is then

$$\varphi(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p) = 0 \tag{2.169}$$

and the loading condition

$$\partial_{\mathbf{C}}\varphi \cdot\cdot\, \mathbf{C}^\bullet + \partial_{\boldsymbol{K}}\varphi \cdots \boldsymbol{K}^\bullet + \partial_\theta \varphi \, \theta^\bullet > 0 \,, \tag{2.170}$$

which is *not* the complete time-derivative of φ .

For the rate-independent flow and hardening rules we make the following ansatz

$$\begin{aligned} \mathbf{P}^{\bullet} &= \lambda\, p(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0, \mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p, \mathbf{C}^{\circ}, \boldsymbol{K}^{\circ}, \theta^{\circ}) \\ \boldsymbol{P}^{\bullet} &= \lambda\, P(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0, \mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p, \mathbf{C}^{\circ}, \boldsymbol{K}^{\circ}, \theta^{\circ}) \\ \mathbf{Z}_p^{\bullet} &= \lambda\, h(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0, \mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p, \mathbf{C}^{\circ}, \boldsymbol{K}^{\circ}, \theta^{\circ}) \end{aligned} \tag{2.171}$$

with the increments of the thermo-kinematical variables

$$\mathbf{C}^{\circ} := \mathbf{C}^{\bullet}/\mu \qquad \boldsymbol{K}^{\circ} := \boldsymbol{K}^{\bullet}/\mu \qquad \theta^{\circ} := \theta^{\bullet}/\mu \tag{2.172}$$

normed by a positive factor

$$\mu := \sqrt{(\,|\,\mathbf{C}^{\bullet}\,|^2 + L^2\,|\,\boldsymbol{K}^{\bullet}\,|^2 + |\,\theta^{\bullet}\,|^2/\theta_0^2)} \tag{2.173}$$

with respect to a freely chosen reference temperature θ_0, and a parameter L of dimension "length". The **consistency parameter** λ is assumed to have a switcher, which sets its values to zero if not both the yield criterion and the loading condition are simultaneously fulfilled. We introduce the abbreviations for the yield directions and for the hardening direction

$$\begin{aligned} \mathbf{P}^{\circ} &:= \mathbf{P}^{\bullet}/\lambda = p(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0, \mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p, \mathbf{C}^{\circ}, \boldsymbol{K}^{\circ}, \theta^{\circ}) \\ \boldsymbol{P}^{\circ} &:= \boldsymbol{P}^{\bullet}/\lambda = P(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0, \mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p, \mathbf{C}^{\circ}, \boldsymbol{K}^{\circ}, \theta^{\circ}) \\ \mathbf{Z}_p^{\circ} &:= \mathbf{Z}_p^{\bullet}/\lambda = h(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0, \mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p, \mathbf{C}^{\circ}, \boldsymbol{K}^{\circ}, \theta^{\circ})\,. \end{aligned} \tag{2.174}$$

As the yield condition must permanently hold during yielding, we obtain the **consistency condition**

$$\begin{aligned} 0 &= \varphi(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p)^{\bullet} \\ &= \partial_{\mathbf{C}}\varphi \,\cdot\cdot\, \mathbf{C}^{\bullet} + \partial_{\boldsymbol{K}}\varphi \cdots \boldsymbol{K}^{\bullet} + \partial_{\theta}\varphi\, \theta^{\bullet} + \partial_{\mathbf{P}}\varphi \,\cdot\cdot\, \mathbf{P}^{\bullet} + \partial_{\boldsymbol{P}}\varphi \cdots \boldsymbol{P}^{\bullet} + \partial_{\mathbf{Z}_p}\varphi \,\cdot\cdot\, \mathbf{Z}_p^{\bullet} \\ &= \partial_{\mathbf{C}}\varphi \,\cdot\cdot\, \mathbf{C}^{\bullet} + \partial_{\boldsymbol{K}}\varphi \cdots \boldsymbol{K}^{\bullet} + \partial_{\theta}\varphi\, \theta^{\bullet} + \partial_{\mathbf{P}}\varphi \,\cdot\cdot\, \lambda\,\mathbf{P}^{\circ} + \partial_{\boldsymbol{P}}\varphi \cdots \lambda\,\boldsymbol{P}^{\circ} + \partial_{\mathbf{Z}_p}\varphi \,\cdot\cdot\, \lambda\,\mathbf{Z}_p^{\circ} \end{aligned} \tag{2.175}$$

which allows us to determine the consistency parameter as the quotient

$$\begin{aligned} &\lambda(\mathbf{C}, \boldsymbol{K}, \theta, \mathbf{g}_0, \mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p, \mathbf{C}^{\bullet}, \boldsymbol{K}^{\bullet}, \theta^{\bullet}) \\ &= -(\partial_{\mathbf{C}}\varphi \,\cdot\cdot\, \mathbf{C}^{\bullet} + \partial_{\boldsymbol{K}}\varphi \cdots \boldsymbol{K}^{\bullet} + \partial_{\theta}\varphi\, \theta^{\bullet}) \,/\, (\partial_{\mathbf{P}}\varphi \,\cdot\cdot\, \mathbf{P}^{\circ} + \partial_{\boldsymbol{P}}\varphi \cdots \boldsymbol{P}^{\circ} + \partial_{\mathbf{Z}_p}\varphi \,\cdot\cdot\, \mathbf{Z}_p^{\circ})\,. \end{aligned} \tag{2.176}$$

Due to the loading condition (2.170), λ is positive during yielding. If we substitute λ into the rules (2.171), we obtain the **consistent yield** and **hardening rules** as rate forms for the internal variables. Because of the switcher in λ these are incrementally nonlinear, which is typical for elastoplasticity. However, for cases of yielding, λ is linear in the increments $\mathbf{C}^{\bullet}$, $\boldsymbol{K}^{\bullet}$, $\theta^{\bullet}$, which assures rate-independence. The KUHN-TUCKER conditions hold

$$\lambda\,\varphi = 0 \qquad \text{with} \qquad \lambda \geq 0 \qquad \text{and} \qquad \varphi \leq 0 \tag{2.177}$$

since at any time one of the two factors is zero.

Thermodynamic Consistency

The additive constants in the free energy and in the entropy ε_c and η_c must remain constant during elastic processes because of the assumption of isomorphic thermo-elastic ranges, and thus cannot depend on $\mathbf{C}$, $\boldsymbol{K}$, θ, or $\mathbf{g}_0$. They can only depend on those state variables that are constant during elastic processes, namely $\mathbf{P}$, $\boldsymbol{P}$, and $\mathbf{Z}_p$. Consequently, after (2.164) we obtain

$$\begin{aligned} \psi &= \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) + \varepsilon_c(\mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p) - \theta\, \eta_c(\mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p) \\ \varepsilon &= \varepsilon_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) + \varepsilon_c(\mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p) \\ \eta &= \eta_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) + \eta_c(\mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p)\,. \end{aligned} \tag{2.178}$$

In the literature, an additive split of the free energy into *elastic* and *plastic* parts is often assumed[33]. In the present context this is a consequence of the isomorphy assumption 2.4. Note that the plastic parts of the internal energy and the entropy cannot depend on the temperature, while the plastic part of the free energy

$$\psi_c(\mathbf{P}, \boldsymbol{P}, \theta, \mathbf{Z}_p) := \varepsilon_c(\mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p) - \theta\, \eta_c(\mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p) \tag{2.179}$$

is linear in the temperature.

The material time-derivative of the free energy (2.178.1) is

$$\begin{aligned} \psi^\bullet &= \partial_{\mathbf{C}_e}\, \psi_0 \cdot\cdot\, \mathbf{C}_e^\bullet + \partial_{\boldsymbol{K}_e} \psi_0 \cdots \boldsymbol{K}_e^\bullet + \partial_\theta\, \psi_0\ \theta^\bullet \\ &+ (\partial_{\mathbf{P}}\, \varepsilon_c - \theta\, \partial_{\mathbf{P}}\, \eta_c) \cdot\cdot\, \mathbf{P}^\bullet + (\partial_{\boldsymbol{P}}\, \varepsilon_c - \theta\, \partial_{\boldsymbol{P}}\, \eta_c) \cdots \boldsymbol{P}^\bullet \\ &+ (\partial_{\mathbf{Z}_p}\, \varepsilon_c - \theta\, \partial_{\mathbf{Z}_p}\, \eta_c) \cdot\cdot\, \mathbf{Z}_p^\bullet - \theta^\bullet \eta_c\,. \end{aligned} \tag{2.180}$$

Without physical effect in the present setting, we assume the symmetry of $\partial_{\mathbf{C}_e} \psi_0$ and the right subsymmetry of $\partial_{\boldsymbol{K}_e} \psi_0$. By use of the rules (0.13) and (0.17) and of (2.162), (2.163), we continue

$$\begin{aligned} \psi^\bullet &= (\mathbf{P} * \partial_{\mathbf{C}_e}\, \psi_0) \cdot\cdot\, \mathbf{C}^\bullet + (\mathbf{P} \circ \partial_{\boldsymbol{K}_e}\, \psi_0) \cdots \boldsymbol{K}^\bullet + (\partial_\theta\, \psi_0 - \eta_c)\, \theta^\bullet \\ &+ \partial_{\mathbf{C}_e}\, \psi_0 \cdot\cdot\, (2\, \mathbf{C}_e\, \mathbf{P}^{-1}\, \mathbf{P}^\bullet) \\ &+ \partial_{\boldsymbol{K}_e}\, \psi_0 \cdots [\boldsymbol{P}^\bullet - \mathbf{P}^{-1} \cdot \mathbf{P}^\bullet \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\, (\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^\bullet] \\ &+ (\partial_{\mathbf{P}}\, \varepsilon_c - \theta\, \partial_{\mathbf{P}}\, \eta_c) \cdot\cdot\, \mathbf{P}^\bullet + (\partial_{\boldsymbol{P}}\, \varepsilon_c - \theta\, \partial_{\boldsymbol{P}}\, \eta_c) \cdots \boldsymbol{P}^\bullet + (\partial_{\mathbf{Z}_p}\, \varepsilon_c - \theta\, \partial_{\mathbf{Z}_p}\, \eta_c) \cdot\cdot\, \mathbf{Z}_p^\bullet. \end{aligned} \tag{2.181}$$

We substitute this and the stress power density (2.24) into the CLAUSIUS-DUHEM inequality (2.132) using (2.160), (2.161), (2.178) and (2.179)

$$\begin{aligned} 0 \geq\ & [-\frac{1}{2\rho_0}\, \mathbf{S} + \mathbf{P} * \partial_{\mathbf{C}_e} \psi_0] \cdot\cdot\, \mathbf{C}^\bullet + [-\frac{1}{\rho_0}\, \boldsymbol{H} + \mathbf{P} \circ \partial_{\boldsymbol{K}_e} \psi_0] \cdots \boldsymbol{K}^\bullet \\ &+ [\partial_\theta\, \psi_0 + \eta_0]\ \theta^\bullet + \frac{1}{\rho_0 \theta}\, \mathbf{q}_0 \cdot \mathbf{g}_0 \end{aligned} \tag{2.182}$$

[33] see, *e.g.*, EKH et al. (2007)

$$+ \partial_{\mathbf{C}_e}\psi_0 \cdot\cdot (2\,\mathbf{C}_e \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet})$$

$$+ \partial_{\boldsymbol{K}_e}\psi_0 \cdot\cdot\cdot [-\,\mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet} \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\,(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}]$$

$$+ \partial_{\mathbf{P}}\,\psi_c \cdot\cdot \mathbf{P}^{\bullet} + (\partial_{\boldsymbol{P}}\,\psi_c + \partial_{\boldsymbol{K}_e}\,\psi_0) \cdot\cdot\cdot \boldsymbol{P}^{\bullet} + \partial_{\mathbf{Z}_p}\,\psi_c \cdot\cdot \mathbf{Z}_p^{\bullet}.$$

If we exploit this inequality first for cases without yielding ($\mathbf{P}^{\bullet} \equiv \mathbf{0}$, $\boldsymbol{P}^{\bullet} \equiv \boldsymbol{0}$, $\mathbf{Z}_p^{\bullet} \equiv \mathbf{0}$), it leads again to the thermoelastic relations (2.138) - (2.141) in the form

(2.183) $\mathbf{S} = \mathbf{P} * 2\rho_0\,\partial_{\mathbf{C}_e}\psi_0$ (potential for the stresses)

(2.184) $\boldsymbol{H} = \mathbf{P} \circ \rho_0\,\partial_{\boldsymbol{K}_e}\psi_0$ (potential for the hyperstresses)

(2.185) $\eta_0 = -\,\partial_\theta\,\psi_0$ (potential for the elastic part of the entropy)

(2.186) $0 \geq \mathbf{q}_0 \cdot \mathbf{g}_0$ (heat conduction inequality).

In the case of yielding, the above findings must still hold because of continuity. Additionally, we obtain the **residual dissipation inequality**

$$0 \geq \partial_{\mathbf{C}_e}\psi_0 \cdot\cdot (2\,\mathbf{C}_e \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet})$$

(2.187) $$+ \partial_{\boldsymbol{K}_e}\,\psi_0 \cdot\cdot\cdot [-\,\mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet} \cdot\ (\boldsymbol{K}_e - \boldsymbol{P}) + 2\,(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}]$$

$$+ \partial_{\mathbf{P}}\,\psi_c \cdot\cdot \mathbf{P}^{\bullet} + (\partial_{\boldsymbol{P}}\,\psi_c + \partial_{\boldsymbol{K}_e}\,\psi_0) \cdot\cdot\cdot \boldsymbol{P}^{\bullet} + \partial_{\mathbf{Z}_p}\,\psi_c \cdot\cdot \mathbf{Z}_p^{\bullet}$$

or with a positive λ with (2.174)

$$0 \geq (2\,\mathbf{P}^{-T} \cdot \mathbf{C}_e \cdot \partial_{\mathbf{C}_e}\psi_0 + \partial_{\mathbf{P}}\,\psi_c) \cdot\cdot \mathbf{P}^{\circ}$$

(2.188) $$+ \partial_{\boldsymbol{K}_e}\,\psi_0 \cdot\cdot\cdot [2\,(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\circ} - \mathbf{P}^{-1} \cdot \mathbf{P}^{\circ} \cdot (\boldsymbol{K}_e - \boldsymbol{P})]$$

$$+ (\partial_{\boldsymbol{K}_e}\,\psi_0 + \partial_{\boldsymbol{P}}\,\psi_c) \cdot\cdot\cdot \boldsymbol{P}^{\circ} + \partial_{\mathbf{Z}_p}\,\psi_c \cdot\cdot \mathbf{Z}_p^{\circ}$$

posing a restriction on the flow rules and the hardening rule (2.171). Note that not each of these terms has to be negative, but only the sum of them. Thus, *yield against the stresses* is not excluded by the second law[34].

We state these findings in the following

Theorem 2.6. *The CLAUSIUS-DUHEM inequality* (2.132) *is fulfilled for a thermoelastoplastic gradient material during every thermo-kinematical process if and only if the free energy does not depend on the temperature gradient, and the conditions* (2.183) - (2.186) *and the residual inequality* (2.188) *hold.*

The first conditions are familiar from thermo-elasticity. They must hold for the thermo-elastic reference laws, and are then automatically valid for all isomorphic laws, including the current ones.

The specific stress power (2.24) becomes with the potentials (2.183) - (2.185) and (2.162) - (2.163)

$$\pi_i = \partial_{\mathbf{C}_e}\,\psi_0(\mathbf{C}_e\,,\boldsymbol{K}_e\,,\theta) \cdot\cdot (\mathbf{C}_e^{\bullet} - \mathbf{C}_e \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet})$$

[34] see the example in BERTRAM/ KRAWIETZ (2012) and BERTRAM/ FOREST (2014).

(2.189) $$+ \partial_{\boldsymbol{K}_e} \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) \cdots [\boldsymbol{K}_e^{\bullet} - \boldsymbol{P}^{\bullet} + \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet} \cdot (\boldsymbol{K}_e - \boldsymbol{P}) - 2\,(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}]$$
$$= \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta)^{\bullet} + \theta^{\bullet} \eta_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) + \mathbf{S}_p \cdot\cdot \mathbf{P}^{\bullet}$$
$$+ \boldsymbol{H}_p \cdots [\boldsymbol{P}^{\bullet} - \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet} \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\,(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^{\bullet}]$$

with the **plastic stress** and **plastic hyperstress tensor** defined as

(2.190) $$\mathbf{S}_p := -\mathbf{P}^{-T} \cdot \mathbf{C}_e \cdot \partial_{\mathbf{C}_e} \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) = -\frac{1}{2\rho_0} \mathbf{C} \cdot \mathbf{S} \cdot \mathbf{P}^{-T}$$

$$\boldsymbol{H}_p := -\partial_{\boldsymbol{K}_e} \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) = -\mathbf{P}^{-1} \circ \frac{1}{\rho_0} \boldsymbol{H}\,.$$

using (2.161), (2.183) and (2.184). Accordingly, the stress power is split into a part that is stored in the reference elastic free energy, and a dissipative part which works on the rates of the plastic transformations $\mathbf{P}^{\bullet}$ and $\boldsymbol{P}^{\bullet}$ (being linear in these rates), and is only active during yielding.

Temperature Changes

In order to determine the change of the temperature of the material point under consideration, we use the local form of the first law of thermodynamics with the heat supply Q, which results from irradiation and conduction. We substitute the internal energy (2.178) and the stress power (2.189) into the energy balance (2.131)

$$\begin{aligned} Q \;=\; & \varepsilon_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta)^\bullet + \varepsilon_c(\mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p)^\bullet \\ & - \psi_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta)^\bullet - \theta^\bullet \eta_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta) - \mathbf{S}_p \cdot\cdot \mathbf{P}^\bullet \\ & - \boldsymbol{H}_p \cdot\cdot\cdot [\boldsymbol{P}^\bullet - \mathbf{P}^{-1} \cdot \mathbf{P}^\bullet \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\,(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^\bullet] \,. \end{aligned} \tag{2.191}$$

By using (2.130) for the elastic parts we get

$$\begin{aligned} Q \;=\; & \theta\, \eta_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta)^\bullet + \varepsilon_c(\mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p)^\bullet \\ & - \mathbf{S}_p \cdot\cdot \mathbf{P}^\bullet - \boldsymbol{H}_p \cdot\cdot\cdot [\boldsymbol{P}^\bullet - \mathbf{P}^{-1} \cdot \mathbf{P}^\bullet \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\,(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^\bullet] \,. \end{aligned} \tag{2.192}$$

This can be split into a **thermoelastic heat generation**

$$\begin{aligned} Q_e \;:=\; & \theta\, \eta_0(\mathbf{C}_e, \boldsymbol{K}_e, \theta)^\bullet \\ =\; & -\theta\,(\mathbf{R} \cdot\cdot \mathbf{C}_e^\bullet + \boldsymbol{R} \cdot\cdot\cdot \boldsymbol{K}_e^\bullet) + c\,\theta^\bullet \end{aligned} \tag{2.193}$$

with the abbreviations for

- the *specific heat* $\quad c\,(\mathbf{C}_e, \boldsymbol{K}_e, \theta) \;:=\; \theta\, \partial_\theta \eta_0$
- the *2nd-order stress-temperature tensor* $\quad \mathbf{R}(\mathbf{C}_e, \boldsymbol{K}_e, \theta) \;:=\; -\partial_{\mathbf{C}_e} \eta_0$
- the *3rd-order stress-temperature tensor* $\quad \boldsymbol{R}(\mathbf{C}_e, \boldsymbol{K}_e, \theta) \;:=\; -\partial_{\boldsymbol{K}_e} \eta_0 \,,$

and a **plastic heat generation**

$$\begin{aligned} Q_p \;:=\; & \varepsilon_c(\mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p)^\bullet - \mathbf{S}_p \cdot\cdot \mathbf{P}^\bullet \\ & - \boldsymbol{H}_p \cdot\cdot\cdot [\boldsymbol{P}^\bullet - \mathbf{P}^{-1} \cdot \mathbf{P}^\bullet \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\,(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^\bullet] \,. \end{aligned} \tag{2.194}$$

This can be solved for the temperature change

$$\begin{aligned} \theta^\bullet \;=\; & \frac{1}{c} \{ Q + \theta\, \mathbf{R} \cdot\cdot \mathbf{C}_e^\bullet + \theta\, \boldsymbol{R} \cdot\cdot\cdot \boldsymbol{K}_e^\bullet - \varepsilon_c(\mathbf{P}, \boldsymbol{P}, \mathbf{Z}_p)^\bullet \\ & + \mathbf{S}_p \cdot\cdot \mathbf{P}^\bullet + \boldsymbol{H}_p \cdot\cdot\cdot [\boldsymbol{P}^\bullet - \mathbf{P}^{-1} \cdot \mathbf{P}^\bullet \cdot (\boldsymbol{K}_e - \boldsymbol{P}) + 2\,(\boldsymbol{K}_e - \boldsymbol{P}) \cdot \mathbf{P}^{-1} \cdot \mathbf{P}^\bullet] \} \,. \end{aligned} \tag{2.195}$$

By this equation, we can integrate the temperature along the process and so determine the final temperature after some arbitrary elasto-plastic process. Accordingly, temperature changes are caused by

- the heat supply Q from the outside;
- thermoelastic transformations due to the second and third term in (2.195);
- the heat Q_p generated by plastic yielding and hardening, which can be determined by use of the flow and hardening rules (2.171).

3. *N*th-order Gradient Materials

This chapter extends the results from

> Bertram, A.; S. Forest: *The thermodynamics of gradient elastoplasticity*. Continuum Mech. Thermodyn. **26**, 269-286 (2014)

where only first and second gradients are taken into account, to higher-order continua within the linear theory for small deformations.

In doing so, we generalize the approach of BERTRAM/ KRAWIETZ (2012) to such gradient materials. Within the frame of plasticity, we decompose the deformation tensor and the gradient of the deformation tensor into its elastic and its plastic parts. The theory is based on the usual assumption of identical thermoelastic behavior in all elastic ranges which means that all measurable thermoelastic properties are not affected by plastic deformations. This concept has been introduced by BERTRAM (1998, 2005) in the context of large deformations. In the present work, however, we limit ourselves to small deformations for the sake of simplicity and clearness.

This chapter is organized as follows. We start with providing appropriate notations and variables for such a theory. Then we consider the linear theory of a mechanical gradient elastoplasticity. This format will afterwards be extended to a thermodynamical theory of elastoplasticity. There the consequences of the second law of thermodynamics are worked out.

We start with the kinematics of small deformations. For an *N*th-order gradient material the following set of kinematical variables will be included as higher gradients of the displacement field $\mathbf{u}$

$$\overset{(\mathrm{I})}{\boldsymbol{U}} := grad^{\,i}\,\mathbf{u} = \mathbf{u} \otimes \nabla^{\,i} = \mathbf{u} \otimes \nabla \otimes \ldots \otimes \nabla \quad (i\text{-times}) \tag{3.1}$$

for $i = 1, 2, \ldots, N$, which forms a tensor field of order $i+1$. Within the format of small deformations, we do not have to distinguish between the spatial and the material nablas.

All these tensors are subsymmetric with respect to all entries except the first, like, *e.g.*, $U_{ijk} = U_{ikj}$ and $U_{ijkl} = U_{ikjl} = U_{ijlk}$, etc. For $i \equiv 1$ we have $\overset{(1)}{\boldsymbol{U}} \equiv grad\ \mathbf{u} = \mathbf{H}$, *i. e.*, the displacement gradient from the classical theory. Due to the postulated invariance of the material response under rigid rotations, only the symmetric part of this tensor can enter constitutive laws. This leads to the classical linear strain tensor $\mathbf{E}$, so that we identify $\overset{(1)}{\boldsymbol{U}} \equiv \mathbf{E}$.

In order to facilitate the notation, we introduce the following hyper-vector of tensors

$$\boldsymbol{U} := \{\overset{(1)}{\boldsymbol{U}}, \overset{(2)}{\boldsymbol{U}}, \ldots, \overset{(N)}{\boldsymbol{U}}\}\,. \tag{3.2}$$

We extend the scalar product on tensor spaces of different order to our hyper-vectors in the following way. For two hyper-vectors $\boldsymbol{U} := \{\overset{(1)}{\boldsymbol{U}}, \overset{(2)}{\boldsymbol{U}}, \ldots, \overset{(N)}{\boldsymbol{U}}\}$ and $\boldsymbol{N} := \{\overset{(1)}{\boldsymbol{N}}, \overset{(2)}{\boldsymbol{N}}, \ldots, \overset{(N)}{\boldsymbol{N}}\}$ the scalar product is defined as

$$(3.3) \qquad <\boldsymbol{U}, \boldsymbol{N}> := \overset{(1)}{\boldsymbol{U}} \cdot\cdot \overset{(1)}{\boldsymbol{N}} + \overset{(2)}{\boldsymbol{U}} \cdot\cdot\cdot \overset{(2)}{\boldsymbol{N}} + ... + \overset{(N)}{\boldsymbol{U}} \cdot ... \cdot \overset{(N)}{\boldsymbol{N}}$$

where "$\cdot ... \cdot$" stands for $N+1$ contractions.

The tensors $\overset{(i)}{\boldsymbol{U}}$ have the following number of independent variables

- for $i \equiv 1$ 6 as symmetric dyadics
- for $i \equiv 2$ 18 as triadics with right subsymmetry
- for $i \equiv 3$ 54 as tetradics with two right subsymmetries
- for i 2×3^i

so that the hyper-vector $\boldsymbol{U}$ has

$$(3.4) \qquad 6 + 18 + ... + 2 \times 3^N = 2 \times \sum_{i=1}^{N} 3^i$$

independent variables.

Dual to this kinematical set $\boldsymbol{U}$, we introduce the power conjugate **hyperstresses**

$$(3.5) \qquad \boldsymbol{T} := \{\overset{(1)}{\boldsymbol{T}}, \overset{(2)}{\boldsymbol{T}}, ..., \overset{(N)}{\boldsymbol{T}}\} = \{\mathbf{T}, \overset{(2)}{\boldsymbol{T}}, ..., \overset{(N)}{\boldsymbol{T}}\}$$

with the usual (symmetric) CAUCHY stress tensor $\mathbf{T}$ as its first component, such that the stress power density becomes

$$(3.6) \qquad \pi_i = <\boldsymbol{T}, \boldsymbol{U}^{\bullet}> = \overset{(1)}{\boldsymbol{T}} \cdot\cdot \overset{(1)}{\boldsymbol{U}}^{\bullet} + \overset{(2)}{\boldsymbol{T}} \cdot\cdot\cdot \overset{(2)}{\boldsymbol{U}}^{\bullet} + ... + \overset{(N)}{\boldsymbol{T}} \cdot ... \cdot \overset{(N)}{\boldsymbol{U}}^{\bullet}.$$

The first term on the right-hand side stands for the classical stress power of simple materials

$$(3.7) \qquad \overset{(1)}{\boldsymbol{T}} \cdot\cdot \overset{(1)}{\boldsymbol{U}}^{\bullet} = \mathbf{T} \cdot\cdot \mathbf{E}^{\bullet}.$$

At any instant, the mechanical balance laws have to be fulfilled after Theorem 1.21, namely

- the balance of linear momentum

$$(3.8) \qquad div \overset{(1)}{\boldsymbol{T}} - div^2 \overset{(2)}{\boldsymbol{T}} + div^3 \overset{(3)}{\boldsymbol{T}} - ... + (-1)^{N+1} div^N \overset{(N)}{\boldsymbol{T}} + \rho \mathbf{b} = \rho \mathbf{u}^{\bullet\bullet}$$

- the balance of angular momentum

$$(3.9) \qquad \mathbf{T} = \mathbf{T}^T.$$

The latter condition will be satisfied by appropriate constitutive equations.

The boundary conditions for higher gradient materials are less simple. For $N \equiv 1$ we have the classical displacement or traction boundary conditions (CAUCHY 1823). For $N \equiv 2$ we have displacement or traction boundary conditions after (1.142) and (1.143) (TOUPIN 1962). For $N \equiv 3$ we have displacement or traction boundary conditions (1.168) - (1.170) after MINDLIN (1962, eq. 18).

3.1 Linear *N*th-order Gradient Elasticity

The mechanical theory of gradient elasticity is constituted by a linear dependence between $\boldsymbol{T}$ and $\boldsymbol{U}$ which we denote as

(3.10) $$\boldsymbol{T} = \boldsymbol{C}[\boldsymbol{U}]$$

with a linear operator $\boldsymbol{C}$.

This elastic law contains all possible linear combinations between all $\overset{\langle i \rangle}{\boldsymbol{U}}$ and all $\overset{\langle j \rangle}{\boldsymbol{T}}$ and is, thus, in general highly coupled.

The material is called hyperelastic, if an elastic energy density $w(\boldsymbol{U})$ exists such that

(3.11) $$\pi_i = w(\boldsymbol{U})^{\bullet}$$

holds for all states of the material point. This leads to the potential relation for the hyperstresses

(3.12) $$\boldsymbol{T} = \partial_{\boldsymbol{U}} w(\boldsymbol{U}) .$$

In our linear theory, the elastic energy is given as a square form of the strain vector which we denote as

(3.13) $$w(\boldsymbol{U}) = \tfrac{1}{2}\, \boldsymbol{C}[\boldsymbol{U}, \boldsymbol{U}]$$

again leading to the linear elastic law (3.10) by means of (3.12). Here

(3.14) $$\boldsymbol{C} = \partial^2_{\boldsymbol{U}} w(\boldsymbol{U})$$

is the elastic operator. One can bring this operator in the following notation. Let $\overset{(i,k)}{\boldsymbol{C}}$ be a tensor of order $(i+k+2)$ which defines a bilinear form

(3.15) $$\overset{(i,k)}{\boldsymbol{C}}[\overset{(i)}{\boldsymbol{U}}, \overset{(k)}{\boldsymbol{U}}] .$$

Then one can organize $\boldsymbol{C}$ in an $N \times N$ quadratic hyper-matrix, the (i , k)-th element of which is such $\overset{(i,k)}{\boldsymbol{C}}$. This hyper-matrix has the major symmetry if for all $i , k = 1 , \ldots , N$

(3.16) $$\overset{(i,k)}{\boldsymbol{C}}[\overset{(i)}{\boldsymbol{U}}, \overset{(k)}{\boldsymbol{K}}] = \overset{(k,i)}{\boldsymbol{C}}[\overset{(k)}{\boldsymbol{K}}, \overset{(i)}{\boldsymbol{U}}]$$

holds for all tensors $\overset{(i)}{\boldsymbol{U}}$, $\overset{(k)}{\boldsymbol{K}}$ of order $i+1$ and $k+1$, respectively.

By standard arguments the symmetry of the elastic operator is a necessary and sufficient integrability condition of the elastic law (3.10) to the energy (3.13).

The number of independent variables of a symmetric $\boldsymbol{C}$ can be calculated after

(3.17) $$(M^2 + M) / 2 \qquad \text{with } M := 2 \times \sum_{i=1}^{N} 3^i .$$

This gives

- for $N \equiv 1$ 21
- for $N \equiv 2$ 300
- for $N \equiv 3$ 3,081

independent elastic constants.

Elastic Symmetry

The number of independent elastic constants can be further reduced by exploiting material symmetries of the elastic behavior. Since we are dealing here only with solids, we can restrict our considerations to orthogonal symmetry transformations, *i. e.*, to rotations and reflections.

Let $\mathbf{u}(\mathbf{x})$ be an arbitrary displacement field of the body. Elasticity is time-independent, so that we do not have to take the time-dependence of the displacements into account here. Further, let $\mathbf{Q}$ be some orthogonal tensor. We now create a second displacement field in the body region

$$\underline{\mathbf{u}}(\underline{\mathbf{x}}) := \mathbf{Q} \cdot \mathbf{u}(\mathbf{x}) \qquad \text{with } \underline{\mathbf{x}} := \mathbf{Q} \cdot \mathbf{x}, \tag{3.18}$$

which is simply the result of a rotation of the field $\mathbf{u}(\mathbf{x})$ by $\mathbf{Q}$ around the origin.

After the chain rule, we obtain for the displacement gradient for the rotated displacement field

$$\begin{aligned} \underline{\mathbf{H}} &= \mathit{grad}\, \underline{\mathbf{u}} = \partial_{\underline{\mathbf{x}}} \underline{\mathbf{u}} = \partial_{\mathbf{u}} \underline{\mathbf{u}} \cdot \partial_{\mathbf{x}} \mathbf{u} \cdot \partial_{\underline{\mathbf{x}}} \mathbf{x} \\ &= \mathbf{Q} \cdot \mathbf{H} \cdot \mathbf{Q}^T = \mathbf{Q} * \mathbf{H}\,. \end{aligned} \tag{3.19}$$

The same rotation applies also to its symmetric part

$$\underline{\mathbf{E}} = \mathbf{Q} * \mathbf{E} \tag{3.20}$$

and to all higher gradients

$$\underline{\overset{(i)}{\boldsymbol{U}}} = \mathbf{Q} * \overset{(i)}{\boldsymbol{U}}\,. \tag{3.21}$$

This gives rise to the following rotation for hyper-vectors

$$\underline{\boldsymbol{U}} = \mathbf{Q} * \boldsymbol{U} \quad \Leftrightarrow \quad \underline{\overset{(i)}{\boldsymbol{U}}} = \mathbf{Q} * \overset{(i)}{\boldsymbol{U}} \text{ for all } i = 1, \ldots, N\,. \tag{3.22}$$

Note that we achieve $(-\mathbf{Q}) * \overset{(i)}{\boldsymbol{U}} = \mathbf{Q} * \overset{(i)}{\boldsymbol{U}}$ for odd i, and $(-\mathbf{Q}) * \overset{(i)}{\boldsymbol{U}} = -(\mathbf{Q} * \overset{(i)}{\boldsymbol{U}})$ for even i. So it matters whether the orthogonal tensor $\mathbf{Q}$ is proper or improper.

An analogous rotation will now be applied to the corresponding stress fields, namely

$$\underline{\mathbf{T}} = \mathbf{Q} \cdot \mathbf{T} \cdot \mathbf{Q}^T = \mathbf{Q} * \mathbf{T} \tag{3.23}$$

for the second-order stress tensors,

(3.24) $$\underline{\overset{(\mathbf{i})}{\boldsymbol{T}}} = \mathbf{Q} * \overset{(\mathbf{i})}{\boldsymbol{T}}$$

for the *i*-th hyperstress tensor, and

(3.25) $$\underline{\boldsymbol{T}} = \mathbf{Q} * \boldsymbol{T}$$

for the hyper-vector of the stresses.

This leads to the following

Definition. An orthogonal tensor $\mathbf{Q}$ is called a **symmetry transformation** of the linear elastic law (3.10) if

(3.26) $$\mathbf{Q} * (\boldsymbol{C}[\boldsymbol{U}]) = \boldsymbol{C}[\mathbf{Q} * \boldsymbol{U}]$$

holds for all deformations $\boldsymbol{U}$. The set of all symmetry transformations of $\boldsymbol{C}$ is called the **symmetry group** $\mathscr{G}$ of the material.

Obviously, the set $\mathscr{G}$ fulfills the axioms of a group under composition in an algebraic sense, which justifies the name symmetry *group*.

It is easy to see that the symmetry condition (3.26) is equivalent to the following one

(3.27) $$\mathbf{Q} * \boldsymbol{C} = \boldsymbol{C}$$

which again implies a rotational invariance of all stiffness components.

We call the elastic law **centro-symmetric**, if $\mathscr{G}$ contains the negative identity $-\mathbf{I}$. As a consequence of the group axioms, we would then also have

(3.28) $$\mathbf{Q} \in \mathscr{G} \quad \Leftrightarrow \quad -\mathbf{Q} \in \mathscr{G}.$$

This transformation gives

(3.29) $$(-\mathbf{I}) * \overset{(\mathbf{i})}{\boldsymbol{U}} = (-1)^{i+1} \overset{(\mathbf{i})}{\boldsymbol{U}}$$

i.e., it changes the sign for odd-order tensors, while it is the identity on even-order tensors. This is why the coupling parts $\overset{(\mathbf{i,k})}{\boldsymbol{C}}$ disappear in the centro-symmetric case if $i + k$ is odd.

One calls the elastic law **isotropic** if the symmetry group contains all proper orthogonal tensors. The classical definitions of all the other symmetry classes for solid crystals[35] can now be applied in a strictly analogous way.

[35] see, *e.g.*, BERTRAM/ GLÜGE (2015)

3.2 *N*th-order Gradient Elastoplasticity

The mechanical theory of gradient plasticity consists of the following ingredients.

1.) an **additive decomposition** of all higher gradients

(3.30) $$\overset{(i)}{\boldsymbol{U}} = \overset{(i)}{\boldsymbol{U}}_e + \overset{(i)}{\boldsymbol{U}}_p$$

which gives rise to a decomposition of the hyper-vector

(3.31) $$\boldsymbol{U} = \boldsymbol{U}_e + \boldsymbol{U}_p$$

in a straightforward way. For all parts, elastic and plastic ones, we assume the same symmetry properties as we previously did for $\overset{(i)}{\boldsymbol{U}}$ itself.

It depends on the specific approach whether all $\overset{(i)}{\boldsymbol{U}}_p$ are considered as independent of each other (*unconstrained gradient plasticity*) or, alternatively, as $\overset{(i)}{\boldsymbol{U}}_p = grad \overset{(i-1)}{\boldsymbol{U}}_p$ in analogy to (3.1) (*constrained gradient plasticity*). Since the first case seems to be more general, we will consider the unconstrained case in what follows, and only eventually mention the constrained one.

2.) an **elastic energy** taken as a square form of the elastic strain vector

(3.32) $$w(\boldsymbol{U}_e) = \tfrac{1}{2}\, \boldsymbol{C}[\boldsymbol{U}_e , \boldsymbol{U}_e]$$

leading to the elastic law

(3.33) $$\boldsymbol{T} = \boldsymbol{C}[\boldsymbol{U}_e]$$

by use of a linear elasticity operator $\boldsymbol{C}$ reflecting the material symmetry properties (anisotropies) of the elastic behavior.

The underlying assumption of this law is that the stresses depend only on the elastic variables and are unaffected by plastic deformations.

3.) a **yield criterion**, which indicates the limit of the current elastic range. The general ansatz for the yield criterion in the strain space is

(3.34) $$\varphi(\boldsymbol{U}_e , \boldsymbol{U}_p , \boldsymbol{Z}_p)$$

where $\boldsymbol{Z}_p$ is the vector of additional scalar or tensorial *internal variables* such as hardening variables. In what follows we will also denote it as a hyper-vector thus leaving enough space to implement different hardening variables and mechanisms. Eventually a scalar-product in the space of the hardening variables is applied. We notate it again in the same way as for the other hyper-vectors. The index p tells us that this variable can only evolve during yielding.

The yield limit is the kernel of this function

(3.35) $$\varphi(\boldsymbol{U}_e , \boldsymbol{U}_p , \boldsymbol{Z}_p) = 0$$

(**yield condition**), while we assume

(3.36) $$\varphi(\boldsymbol{U}_e, \boldsymbol{U}_p, \boldsymbol{Z}_p) < 0$$

(only) in the interior of the elastic range.

One can always transform the yield criterion from the strain space into the stress space by using the elastic law (3.33).

Necessary and sufficient conditions for the material to yield are the yield condition (3.35) and the **loading condition**

(3.37) $$<\partial_{\boldsymbol{U}_e} \varphi, \boldsymbol{U}^\bullet> \; > 0$$

which states a violation of the yield criterion if the total deformation increments would be purely elastic.

This presentation is limited to rate-independent plasticity, but viscoplasticity can be introduced in a straightforward manner into the model.

4.) a **flow rule** which determines the evolution of $\boldsymbol{U}_p$. A general rate-independent unconstrained ansatz for it would be a first-order ODE depending on practically all variables and the rate of the total deformation

(3.38) $$\boldsymbol{U}_p^\bullet = U(\boldsymbol{U}_p, \boldsymbol{U}_e, \boldsymbol{Z}_p, \boldsymbol{U}^\bullet) .$$

Since we want to consider an *unconstraint* gradient plasticity, the evolution laws for the different components of the hyper-vector $\boldsymbol{U}_p$ can be chosen independently of each other.

5.) an evolution equation for the additional variable(s) $\boldsymbol{Z}_p$ called **hardening rule**, which is assumed to be of the same form as the flow rule above

(3.39) $$\boldsymbol{Z}_p^\bullet = Z(\boldsymbol{U}_p, \boldsymbol{U}_e, \boldsymbol{Z}_p, \boldsymbol{U}^\bullet) .$$

These constitutive laws establish a complete mechanical rate-independent format for a gradient elasto-plasticity.

We now specify the ansatz for the rate-independent evolution (3.38) and (3.39) for the plastic variables, namely the flow and the hardening rules in the following form

(3.40) $$\boldsymbol{U}_p^\bullet = \lambda \, U(\boldsymbol{U}_p, \boldsymbol{U}_e, \boldsymbol{Z}_p)$$

(3.41) $$\boldsymbol{Z}_p^\bullet = \lambda \, Z(\boldsymbol{U}_p, \boldsymbol{U}_e, \boldsymbol{Z}_p)$$

with a joint **plastic consistency parameter** λ and two functions U and Z of the listed arguments (by an abuse of notation we use the same symbols for the functions as before).

λ is introduced as zero if and only if no yielding occurs, *i.e.*, during elastic events. During yielding, however, λ is positive. In all cases the KUHN-TUCKER conditions hold in the form

(3.42) $$\lambda \, \varphi = 0 \qquad \text{with} \qquad \lambda \geq 0 \qquad \text{and} \qquad \varphi \leq 0 .$$

The plastic parameter can be determined during yielding by the **consistency condition** as a consequence of the yield condition (3.35)

(3.43) $$0 = <\partial_{\boldsymbol{U}_e} \varphi, \boldsymbol{U}^\bullet> + <\partial_{\boldsymbol{U}_p} \varphi, \lambda \, U(\boldsymbol{U}_p, \boldsymbol{U}_e, \boldsymbol{Z}_p)> + <\partial_{\boldsymbol{Z}_p} \varphi, \lambda \, Z(\boldsymbol{U}_p, \boldsymbol{U}_e, \boldsymbol{Z}_p)>$$

which can be solved for the plastic parameter

(3.44) $$\lambda = -\langle \partial_{\boldsymbol{U}_e} \varphi , \boldsymbol{U}^{\bullet} \rangle / [\langle \partial_{\boldsymbol{U}_p} \varphi , U(\boldsymbol{U}_p , \boldsymbol{U}_e , \boldsymbol{Z}_p) \rangle + \langle \partial_{\boldsymbol{Z}_p} \varphi , Z(\boldsymbol{U}_p , \boldsymbol{U}_e , \boldsymbol{Z}_p) \rangle] .$$

Due to the loading condition (3.37), $\lambda\alpha$ must be positive during plastic events. After (3.42), λ alone is positive, and so α must also be positive. After (3.44) this is a restriction to the flow rule U, the hardening rule Z, and the yield criterion φ.

3.3 *N*th-order Gradient Thermo-Elastoplasticity

We use the **energy balance** (first law of thermodynamics) in the local form

(3.45) $$\rho\,\varepsilon^{\bullet} = \rho\,Q + <\boldsymbol{T},\boldsymbol{U}^{\bullet}> .$$

with the stress power density $<\boldsymbol{T}, \boldsymbol{U}^{\bullet}>$ after (3.6) and the heat supply per unit mass and time Q, which results from irradiation r and conduction $\mathbf{q}$ in the usual form

(3.46) $$Q = r - (div\,\mathbf{q})/\rho .$$

By the introduction of the **HELMHOLTZ free energy**

(3.47) $$\psi := \varepsilon - \theta\,\eta$$

we assume as the second law the **CLAUSIUS-DUHEM inequality** in the form

(3.48) $$\frac{1}{\rho} <\boldsymbol{T},\boldsymbol{U}^{\bullet}> - \psi^{\bullet} - \eta\,\theta^{\bullet} - \frac{1}{\rho\theta}\mathbf{q}\cdot\mathbf{g} \geq 0 .$$

Thus, the specific **dissipation**, which consists of the mechanical dissipation

(3.49) $$\delta_{\mathrm{m}} := \frac{1}{\rho}<\boldsymbol{T},\boldsymbol{U}^{\bullet}> - \psi^{\bullet} - \eta\,\theta^{\bullet} = \theta\,\eta^{\bullet} - Q$$

by using (3.45) and (3.47), and the thermal dissipation

(3.50) $$\delta_{\mathrm{th}} := -\frac{1}{\rho\theta}\mathbf{q}\cdot\mathbf{g}$$

fulfill the *dissipation inequality*

(3.51) $$\delta = \delta_{\mathrm{m}} + \delta_{\mathrm{th}} \geq 0 .$$

In order to enlarge the mechanical plasticity theory to a thermo-mechanical one, we add the temperature and the temperature gradient to the list of independent variables called **thermo-kinematical variables**, and the heat flux, the entropy, and the internal energy or the free energy to the dependent variables called **caloro-dynamic variables**.

Thermoplastic materials can be understood as material models with *internal variables*. The set of the internal variables contains in the case of plasticity the plastic strains and, eventually, hardening variables. For all of these variables we assume rate-independent evolution equations in the general form

(3.52) $$\boldsymbol{U}_p^{\bullet} = U(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p, \boldsymbol{U}^{\bullet}, \theta^{\bullet}, \mathbf{g}^{\bullet})$$

(3.53) $$\boldsymbol{Z}_p^{\bullet} = Z(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p, \boldsymbol{U}^{\bullet}, \theta^{\bullet}, \mathbf{g}^{\bullet})$$

as generalizations of (3.38) and (3.39). The set of additional constitutive equations for a thermomechanical material with internal variables is assumed to be

$$\boldsymbol{T} = T(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p)$$

(3.54)
$$\mathbf{q} = q(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p)$$
$$\eta = \eta(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p)$$
$$\varepsilon = \varepsilon(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p)$$
or
$$\psi = \psi(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p).$$

Instead of elastic ranges of the mechanical theory, we will now have to deal with **thermoelastic ranges**. These are specified by a **yield criterion** assumed as a function

(3.55)
$$\varphi(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p)$$

which induces the **yield condition**

(3.56)
$$\varphi(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p) = 0$$

and the **loading condition**

(3.57)
$$< \partial_{\boldsymbol{U}_e} \varphi, \boldsymbol{U}^\bullet > + \partial_\theta \varphi \, \theta^\bullet + \partial_{\mathbf{g}} \varphi \cdot \mathbf{g}^\bullet > 0.$$

If not both conditions are simultaneously fulfilled, a (thermo) **elastic event** takes place, and hence the plastic variables do not evolve

(3.58)
$$\boldsymbol{U}_p^\bullet \equiv \boldsymbol{0} \qquad \boldsymbol{Z}_p^\bullet \equiv \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{U}_e^\bullet \equiv \boldsymbol{U}^\bullet$$

which are side conditions of the functions U and Z. Otherwise it is a **plastic event** or an event of **yielding**, in which the plastic variables necessarily have to evolve according to (3.52) - (3.53).

We will next investigate the restrictions imposed on the constitutive equations by the CLAUSIUS-DUHEM inequality. With the free energy from (3.54) we obtain for the dissipation inequality (3.48)

(3.59)
$$\begin{aligned} 0 \geq\ & < \partial_{\boldsymbol{U}_e} \varphi, \boldsymbol{U}_e^\bullet > + \partial_\theta \psi \, \theta^\bullet + \partial_{\mathbf{g}} \psi \cdot \mathbf{g}^\bullet + < \partial_{\boldsymbol{U}_p} \varphi, \boldsymbol{U}_p^\bullet > + < \partial_{\boldsymbol{Z}_p} \psi, \boldsymbol{Z}_p^\bullet > \\ & + \eta \, \theta^\bullet - \frac{1}{\rho} < \boldsymbol{T}, \boldsymbol{U}_e^\bullet + \boldsymbol{U}_p^\bullet > + \frac{1}{\rho\theta} \mathbf{q} \cdot \mathbf{g} \\ =\ & < \partial_{\boldsymbol{U}_e} \psi - \frac{1}{\rho} \boldsymbol{T}, \boldsymbol{U}_e^\bullet > + (\partial_\theta \psi + \eta) \, \theta^\bullet + \partial_{\mathbf{g}} \psi \cdot \mathbf{g}^\bullet \\ & < \partial_{\boldsymbol{U}_p} \psi - \frac{1}{\rho} \boldsymbol{T}, \boldsymbol{U}_p^\bullet > + < \partial_{\boldsymbol{Z}_p} \psi, \boldsymbol{Z}_p^\bullet > + \frac{1}{\rho\theta} \mathbf{q} \cdot \mathbf{g}. \end{aligned}$$

If we first consider elastic events, then due to (3.58) there remains only the inequality

(3.60)
$$0 \geq < \partial_{\boldsymbol{U}_e} \psi - \frac{1}{\rho} \boldsymbol{T}, \boldsymbol{U}^\bullet > + (\partial_\theta \psi + \eta) \, \theta^\bullet + \partial_{\mathbf{g}} \psi \cdot \mathbf{g}^\bullet + \frac{1}{\rho\theta} \mathbf{q} \cdot \mathbf{g}.$$

The exploitation of this inequality for states below the yield limit, where $\boldsymbol{U}^\bullet$ and $\theta^\bullet$ are not restricted, leads by standard arguments to the conditions from gradient thermoelasticity, namely the independence of the free energy of the temperature gradient

(3.61)
$$\psi = \psi(\boldsymbol{U}_e, \theta, \boldsymbol{U}_p, \boldsymbol{Z}_p)$$

instead of (3.54), and the *thermoelastic potentials*

(3.62) $$\boldsymbol{T} = \rho\, \partial_{\boldsymbol{U}_e} \psi = T(\boldsymbol{U}_e, \theta, \boldsymbol{U}_p, \boldsymbol{Z}_p)$$

(3.63) $$\eta = -\partial_\theta \psi = \eta(\boldsymbol{U}_e, \theta, \boldsymbol{U}_p, \boldsymbol{Z}_p)$$

as well as the *heat conduction inequality*

(3.64) $$\mathbf{q} \cdot \mathbf{g} \leq 0$$

as necessary and sufficient conditions for the second law to hold during elastic events. With these findings, the mechanical dissipation δ_m vanishes during elastic events.

Because of continuity, restrictions (3.61) - (3.64) must also hold when reaching the yield limit. If yielding occurs, however, then the additional terms of the CLAUSIUS-DUHEM inequality (3.59) must fulfill the **residual dissipation inequality**

(3.65) $$< \frac{1}{\rho} \boldsymbol{T} - \partial_{\boldsymbol{U}_p} \psi, \boldsymbol{U}_p^{\bullet} > - < \partial_{\boldsymbol{Z}_p} \psi, \boldsymbol{Z}_p^{\bullet} > \geq 0 .$$

This is a restriction on the yield criterion and the flow and hardening rules. The term $\rho\, \partial_{\boldsymbol{U}_p} \psi$ can be interpreted as *back stress*. We conclude the following

Theorem 3.1. *The CLAUSIUS-DUHEM inequality is fulfilled if and only if the following conditions hold:*

- *the representation of the free energy* (3.61)*,*
- *the thermoelastic potentials* (3.62) *for the stress tensors and the entropy,*
- *the heat-conduction inequality* (3.64)*,*
- *the residual dissipation inequality* (3.65)*.*

Identical Thermoelastic Behavior

Up to this point, our analysis has been rather general. In order to further specify this setting for gradient plastic materials, we introduce another important assumption, which is the generalization of the Assumption 2.4 of isomorphic elasticity for all elastic ranges.

Assumption 3.1. *The thermoelastic behavior within all thermoelastic ranges of the elasto-plastic material is identical.*

This means more precisely that during elastic events all measurable quantities like the stresses, the heat flux, and the heat supply depend only upon the elastic strains and the temperature and its gradient, but not upon the plastic variables $\boldsymbol{U}_p$ and $\boldsymbol{Z}_p$. The consequences of this assumption shall be investigated next.

We obtain for elastic events zero mechanical dissipation and hence from (3.49)

(3.66) $$\delta_m = 0 \qquad \Rightarrow \qquad Q = \theta \eta^{\bullet}$$

with the local heat supply Q , which we consider as a measurable quantity - at least in principle. With (3.63) the right-hand side becomes for elastic events with $\boldsymbol{U} \equiv \boldsymbol{U}_e$

(3.67) $\theta \dot{\eta} = \theta(<\partial_{\boldsymbol{U}_e} \eta \, , \dot{\boldsymbol{U}} > + \partial_\theta \eta \, \dot{\theta})$

which shall be independent of $\boldsymbol{U}_p$, and $\boldsymbol{Z}_p$. This can only be the case if the following decomposition of the entropy exists

(3.68) $\eta(\boldsymbol{U}_e \, , \theta, \boldsymbol{U}_p \, , \boldsymbol{Z}_p) = \eta_e(\boldsymbol{U}_e \, , \theta) + \eta_p(\boldsymbol{U}_p \, , \boldsymbol{Z}_p) \, .$

After the above assumption, also the stresses shall not depend on the plastic variables. Hence we have the reduced stress law

(3.69) $\boldsymbol{T} = T(\boldsymbol{U}_e \, , \theta)$

and instead of (3.62)

(3.70) $T(\boldsymbol{U}_e \, , \theta) = \rho \, \partial_{\boldsymbol{U}_e} \psi \, .$

Instead of (3.63) we obtain with (3.68)

(3.71) $\eta_e(\boldsymbol{U}_e \, , \theta) + \eta_p(\boldsymbol{U}_p, \boldsymbol{Z}_p) = - \partial_\theta \psi$

which leads to the split of the free energy with (3.47)

(3.72) $\psi = \psi_e(\boldsymbol{U}_e \, , \theta) - \theta \, \eta_p(\boldsymbol{U}_p \, , \boldsymbol{Z}_p) + \varepsilon_p(\boldsymbol{U}_p \, , \boldsymbol{Z}_p)$

such that

(3.73) $\eta_e(\boldsymbol{U}_e \, , \theta) = - \partial_\theta \psi_e(\boldsymbol{U}_e \, , \theta)$

with a (for the present) arbitrary function ε_p of the plastic variables $\boldsymbol{U}_p$ and $\boldsymbol{Z}_p$.

The internal energy follows from (3.47)

(3.74) $\varepsilon = \varepsilon_e(\boldsymbol{U}_e \, , \theta) + \varepsilon_p(\boldsymbol{U}_p \, , \boldsymbol{Z}_p)$

with

(3.75) $\varepsilon_e(\boldsymbol{U}_e \, , \theta) := \psi_e(\boldsymbol{U}_e \, , \theta) + \theta \, \eta_e(\boldsymbol{U}_e \, , \theta).$

After the above assumption, the heat conduction can neither depend on the plastic variables $\boldsymbol{U}_p$ and $\boldsymbol{Z}_p$, which gives rise to the reduced form of (23.3)

(3.76) $\mathbf{q} = q(\boldsymbol{U}_e \, , \theta, \mathbf{g}) \, .$

We state these findings in the following

Theorem 3.2. *The assumption of equal thermoelastic behavior in all elastic ranges is fulfilled if and only if the following representations hold for the*

- *entropy* $\eta = - \partial_\theta \psi_e(\boldsymbol{U}_e \, , \theta) + \eta_p(\boldsymbol{U}_p \, , \boldsymbol{Z}_p)$
- *internal energy* $\varepsilon = \psi_e(\boldsymbol{U}_e \, , \theta) - \theta \, \partial_\theta \psi_e(\boldsymbol{U}_e \, , \theta) + \varepsilon_p(\boldsymbol{U}_p \, , \boldsymbol{Z}_p)$
- *free energy* $\psi = \psi_e(\boldsymbol{U}_e \, , \theta) - \theta \, \eta_p(\boldsymbol{U}_p \, , \boldsymbol{Z}_p) + \varepsilon_p(\boldsymbol{U}_p \, , \boldsymbol{Z}_p)$
- *heat conduction* $\mathbf{q} = q(\boldsymbol{U}_e \, , \theta, \mathbf{g})$

- *hyperstresses* $\boldsymbol{T} = \rho\, \partial_{\boldsymbol{U}_e} \psi_e(\boldsymbol{U}_e, \theta)$

Accordingly, the following constitutive equations completely constitute the thermoplastic material model

- the elastic free energy $\psi_e(\boldsymbol{U}_e, \theta)$
- the plastic entropy $\eta_p(\boldsymbol{U}_p, \boldsymbol{Z}_p)$
- the plastic internal energy $\varepsilon_p(\boldsymbol{U}_p, \boldsymbol{Z}_p)$
- the heat conduction law $q(\boldsymbol{U}_e, \theta, \mathbf{g})$
- the yield criterion $\varphi(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p)$

together with the flow and hardening rules to be considered later.

(3.72) gives for the residual dissipation inequality (3.65)

$$0 \le < \frac{1}{\rho}\boldsymbol{T} - \partial_{\boldsymbol{U}_p}\varepsilon_p + \theta\,\partial_{\boldsymbol{U}_p}\eta_p\,,\, \boldsymbol{U}_p^{\bullet} >$$

(3.77)
$$+ < -\partial_{\boldsymbol{Z}_p}\varepsilon_p + \theta\,\partial_{\boldsymbol{Z}_p}\eta_p\,,\, \boldsymbol{Z}_p^{\bullet} >$$

$$= \frac{1}{\rho} < \boldsymbol{T}\,,\, \boldsymbol{U}_p^{\bullet} > + \theta\,\eta_p(\boldsymbol{U}_p, \boldsymbol{Z}_p)^{\bullet} - \varepsilon_p(\boldsymbol{U}_p, \boldsymbol{Z}_p)^{\bullet}.$$

In order to determine the change of the temperature of the material point under consideration, we use the first law of thermodynamics (3.45) together with (3.31) and (3.74)

(3.78)
$$Q = \varepsilon^{\bullet} - \frac{1}{\rho} < \boldsymbol{T}\,,\, \boldsymbol{U}^{\bullet} > = Q_e + Q_p$$

with a split of the heat supply into an elastic part

(3.79)
$$Q_e(\boldsymbol{U}_e, \theta, \boldsymbol{U}_e^{\bullet}, \theta^{\bullet}) := \varepsilon_e(\boldsymbol{U}_e, \theta)^{\bullet} - \frac{1}{\rho} < \boldsymbol{T}\,,\, \boldsymbol{U}_e^{\bullet} >$$

with (3.66)
$$= \theta\,\eta_e(\boldsymbol{U}_e, \theta)^{\bullet}$$

$$= c\,\theta^{\bullet} - \frac{\theta}{\rho} < \boldsymbol{R}\,,\, \boldsymbol{U}_e^{\bullet} >$$

with

- the *specific heat* $c(\boldsymbol{U}_e, \theta) := \theta\,\partial_{\theta}\eta_e$
- the *stress-temperature tensor of order (i+1)* $\overset{\langle i \rangle}{\boldsymbol{R}}(\boldsymbol{U}_e, \theta) := -\rho\,\partial_{\overset{(i)}{\boldsymbol{U}_e}}\eta_e$
- the *stress-temperature hyper-vector* $\boldsymbol{R}(\boldsymbol{U}_e, \theta) := \{\overset{\langle 1 \rangle}{\boldsymbol{R}}, \overset{\langle 2 \rangle}{\boldsymbol{R}}, \ldots, \overset{\langle N \rangle}{\boldsymbol{R}}\}$

and a plastic part

(3.80) $$Q_p := \varepsilon_p(\boldsymbol{U}_p, \boldsymbol{Z}_p)^\bullet - \frac{1}{\rho} < \boldsymbol{T}, \boldsymbol{U}_p^\bullet > .$$

(3.78) and (3.79) can be solved for the temperature rate

(3.81) $$c\,\theta^\bullet = Q - Q_p + \frac{\theta}{\rho} < \boldsymbol{R}, \boldsymbol{U}_e^\bullet > .$$

By this equation, we can integrate the temperature along the process and so determine the final temperature after some elasto-plastic process. Accordingly, temperature changes are caused by

1.) the heat supply Q from the outside

2.) the heat $-Q_p$ generated by plastic yielding and hardening, and

3.) thermoelastic transformations due to the last term in (3.81).

We now specify the ansatz for the rate-independent evolution (3.52) - (3.53) for the plastic variables, namely the flow and the hardening rules in the following form

(3.82) $$\boldsymbol{U}_p^\bullet = \lambda\, U(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p)$$

(3.83) $$\boldsymbol{Z}_p^\bullet = \lambda\, Z(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p)$$

with a joint **plastic consistency parameter** λ and two functions U and Z of the listed arguments. λ is zero if and only if no yielding occurs, *i.e.*, during thermo-elastic events.

During yielding, however, λ is positive. In all cases the KUHN-TUCKER-conditions hold in the form

(3.84) $$\lambda\,\varphi = 0 \qquad \text{with} \quad \lambda \geq 0 \quad \text{and} \quad \varphi \leq 0 .$$

The plastic parameter can be determined during yielding by the **consistency condition** as a consequence of the yield condition (3.56)

(3.85) $$\begin{aligned} 0 = {} & < \partial_{\boldsymbol{U}_e}\varphi, \boldsymbol{U}_e^\bullet > + \partial_\theta\varphi\,\theta^\bullet + \partial_{\mathbf{g}}\varphi \cdot \mathbf{g}^\bullet \\ & + < \partial_{\boldsymbol{U}_p}\varphi, \lambda\, U(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p) > \\ & + < \partial_{\boldsymbol{Z}_p}\varphi, \lambda\, Z(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p) > \end{aligned}$$

which can be solved for the plastic parameter

(3.86) $$\lambda = \alpha^{-1} [< \partial_{\boldsymbol{U}_e}\varphi, \boldsymbol{U}_e^\bullet > + \partial_\theta\varphi\,\theta^\bullet + \partial_{\mathbf{g}}\varphi \cdot \mathbf{g}^\bullet]$$

with a scalar denominator

(3.87) $$\begin{aligned} & \alpha(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p) \\ & := - < \partial_{\boldsymbol{U}_p}\varphi, U(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p) > - < \partial_{\boldsymbol{Z}_p}\varphi, Z(\boldsymbol{U}_e, \theta, \mathbf{g}, \boldsymbol{U}_p, \boldsymbol{Z}_p) > . \end{aligned}$$

Due to the loading condition (3.57), $\lambda\alpha$ must be positive during plastic events. After (3.84), λ alone is positively introduced, and so α must also be positive. After (3.87) this is a restriction to the functions U and Z, and the yield criterion φ.

Another restriction on these functions is obtained by the second law. We substitute (3.82) - (3.83) into the residual dissipation inequality (3.77)

$$(3.88) \qquad < \boldsymbol{T} - \rho\, \partial_{\boldsymbol{U}_p} \varepsilon_p + \rho\, \theta\, \partial_{\boldsymbol{U}_p} \eta_p \,,\, U(\boldsymbol{U}_e \,,\, \theta, \mathbf{g}, \boldsymbol{U}_p \,,\, \boldsymbol{Z}_p) >$$

$$+ < - \rho\, \partial_{\boldsymbol{Z}_p} \varepsilon_p + \rho\, \theta\, \partial_{\mathbf{Z}_p} \eta_p \,,\, Z(\boldsymbol{U}_e \,,\, \theta, \mathbf{g}, \boldsymbol{U}_p \,,\, \boldsymbol{Z}_p) > \;\geq\; 0.$$

In the example at the end of the next chapter it is demonstrated how this inequality can be satisfied for specific material functions.

4.1 2nd-Order Gradient Elasto-Plasticity

In this chapter we will particularize the approach of the preceding chapter to a second-order gradient theory. However, to make this chapter self-contained, we will repeat the entire deduction in a slightly reduced form so that it can be read independently. The notations also deviate from the ones in the preceding chapter whenever it seems to be appropriate.

We start with the kinematics of small deformations.

For small deformations of simple materials, on uses the displacement gradient

(4.1) $$\mathbf{H} := Grad\,\mathbf{u} = \mathbf{F} - \mathbf{I}$$

and takes its symmetric part, the **linear strain tensor**

(4.2) $$\mathbf{E} := sym\,\mathbf{H}\,.$$

In the linear theory, no distinction is made between material and spatial differentiation. So we set

(4.3) $$\mathbf{H}^{\bullet} = grad\,\mathbf{u}^{\bullet} = grad\,\mathbf{v} = \mathbf{L}$$

with its symmetric part

(4.4) $$\mathbf{D} = sym\,grad\,\mathbf{u}^{\bullet}.$$

For a second gradient material, we have different choices. MINDLIN/ ESHEL (1968) suggest three sets of second-gradient variables.

(*i*) The second displacement gradient

(4.5) $$grad\,\mathbf{H} = grad\,grad\,\mathbf{u}$$

with right subsymmetry and 18 DOFs;

(*ii*) The gradient of the strain tensor

(4.6) $$\boldsymbol{M} := grad\,\mathbf{E} = grad\,sym\,grad\,\mathbf{u}$$

with left subsymmetry and 18 DOFs;

(*iii*) The completely symmetric part of the gradient of the strain tensor

(4.7) $$sym\,grad\,\mathbf{E} = sym\,grad\,sym\,grad\,\mathbf{u}$$

with 10 DOFs, and the gradient of the curl of the displacement

(4.8) $$grad\,curl\,\mathbf{u}$$

with 8 DOFs.

In most cases, the second choice is made, and so do we here. As a consequence, all triadics in the rest of this chapter show the left subsymmetry.

The mechanical theory of gradient plasticity consists of the following ingredients.

1.) an **additive decomposition** of the linear strain tensor $\mathbf{E}$ into an elastic and a plastic part

(4.9) $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_p$

and an analogous one of the gradient of the strain tensor

(4.10) $\boldsymbol{M} = \boldsymbol{M}_e + \boldsymbol{M}_p$

with the following symmetry properties: $\mathbf{E} = \mathbf{E}^T$ and $M^{ijk} = M^{jik}$ (left subsymmetry), which also apply to their elastic and plastic parts and to the conjugate stresses.

It depends on the specific approach whether $\mathbf{E}_p$ and $\boldsymbol{M}_p$ are considered as independent of each other (*unconstrained gradient plasticity*) or, alternatively, as $\boldsymbol{M}_p = grad\, \mathbf{E}_p$ in analogy to (4.10) (*constrained gradient plasticity*). Since the first case seems to be more general, we will consider the unconstrained case in what follows, and only eventually mention the constrained one, as in the Appendix of this chapter.

2.) two **elastic laws** taken as linear mappings of the elastic strain tensors into the second and third-order stress tensors

(4.11) $\mathbf{T} = \overset{\langle 4 \rangle}{\boldsymbol{C}}_E \cdot\cdot\, \mathbf{E}_e + \overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM} \cdots \boldsymbol{M}_e$

(4.12) $\boldsymbol{G} = \overset{\langle 5 \rangle}{\boldsymbol{C}}_{ME} \cdot\cdot\, \mathbf{E}_e + \overset{\langle 6 \rangle}{\boldsymbol{C}}_M \cdots \boldsymbol{M}_e$

(see MINDLIN/ ESHEL, 1968) by use of a fourth-order elasticity tensor $\overset{\langle 4 \rangle}{\boldsymbol{C}}_E$, two fifth-order elasticity coupling tensors $\overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM}$ and $\overset{\langle 5 \rangle}{\boldsymbol{C}}_{ME}$, and a sixth-order elasticity tensor $\overset{\langle 6 \rangle}{\boldsymbol{C}}_M$. The underlying assumption of these laws is that the stresses depend only on the elastic variables and are unaffected by plastic deformations.

For $\mathbf{T}$ and $\mathbf{E}_e$ being symmetric and $\boldsymbol{G}$ and $\boldsymbol{M}_e$ having the left subsymmetry, these symmetries can also be assumed for the corresponding elasticity tensors.

Moreover, if the elastic laws are also hyperelastic, a major symmetry of the elasticity tensors can be imposed to $\overset{\langle 4 \rangle}{\boldsymbol{C}}_E$ and $\overset{\langle 6 \rangle}{\boldsymbol{C}}_M$, and

(4.13) $\mathbf{E}_e \cdot\cdot\, \overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM} \cdots \boldsymbol{M}_e = \boldsymbol{M}_e \cdots \overset{\langle 5 \rangle}{\boldsymbol{C}}_{ME} \cdot\cdot\, \mathbf{E}_e$

holds for all $\mathbf{E}_e$ and $\boldsymbol{M}_e$, so that $\overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM}$ is completely determined by $\overset{\langle 5 \rangle}{\boldsymbol{C}}_{ME}$.

These laws can be isotropic or anisotropic. In the central symmetric case, however, the fifth-order tensors disappear.

At any instant these variables have to fulfill the mechanical balance laws after Theorem 1.21, namely

- the balance of linear momentum $div\,(\mathbf{T} - div\,\boldsymbol{G}) + \rho\,\mathbf{b} = \rho\,\mathbf{u}^{\bullet\bullet}$
- the balance of angular momentum $\mathbf{T} = \mathbf{T}^T$.

The displacement or traction boundary conditions are (1.142) and (1.143) after MINDLIN (1965).

3.) a **yield limit** (yield criterion), which indicates the limit of the current elastic range. The general ansatz for the yield criterion in the strain space is

$$\varphi(\mathbf{E}_e\,,\,\boldsymbol{M}_e\,,\,\mathbf{E}_p\,,\,\boldsymbol{M}_p\,,\,\mathbf{Z}_p)$$

where $\mathbf{Z}_p$ is the vector of additional scalar or tensorial *internal variables* such as hardening variables. In what follows we will denote it as a second-order tensor, however, without the intention to constrain our considerations to this special case. The internal variables introduced in the formulation comprise the classical hardening or damage variables used in material modeling but also new variables related to the strain gradient or the elastic or plastic parts of the strain gradient. An internal variable is characterized by the fact that its evolution law is given by an ODE.

The yield limit is the kernel of this function

(4.14) $$\varphi(\mathbf{E}_e\,,\,\boldsymbol{M}_e\,,\,\mathbf{E}_p\,,\,\boldsymbol{M}_p\,,\,\mathbf{Z}_p) \;=\; 0$$

(**yield condition**), while we assume

(4.15) $$\varphi(\mathbf{E}_e\,,\,\boldsymbol{M}_e\,,\,\mathbf{E}_p\,,\,\boldsymbol{M}_p\,,\,\mathbf{Z}_p) < 0$$

(only) in the interior of the elastic range.

One can always transform the yield criterion from the strain space into the stress space by using the elastic laws (4.11) and (4.12).

Necessary and sufficient conditions for the material to yield are the yield condition and the **loading condition**

(4.16) $$\partial_{\mathbf{E}_e}\varphi \cdot\cdot\, \mathbf{E}^{\bullet} + \partial_{\boldsymbol{M}_e}\varphi \cdots \boldsymbol{M}^{\bullet} > 0$$

which states a violation of the yield criterion if the total deformation increments would be purely elastic.

The presentation is limited to rate-independent plasticity, but viscoplasticity can be introduced in a straightforward manner into the model, for instance, but not exclusively, by the introduction of a viscoplastic potential from which viscoplastic flow rule and evolution laws for hardening variables are derived, see FOREST/ SIEVERT (2003).

4.) **flow rules** which determine the evolution of $\mathbf{E}_p$ and $\boldsymbol{M}_p$. A general rate-independent unconstrained ansatz for them would be first-order ODEs depending on practically all variables and the rates of the total deformations

(4.17) $$\mathbf{E}_p^{\bullet} \;=\; E(\mathbf{E}_p\,,\,\mathbf{E}_e\,,\,\boldsymbol{M}_p\,,\,\boldsymbol{M}_e\,,\,\mathbf{Z}_p,\,\mathbf{E}^{\bullet},\,\boldsymbol{M}^{\bullet})$$

(4.18) $$\boldsymbol{M}_p^{\bullet} \;=\; M(\mathbf{E}_p\,,\,\mathbf{E}_e\,,\,\boldsymbol{M}_p\,,\,\boldsymbol{M}_e\,,\,\mathbf{Z}_p,\,\mathbf{E}^{\bullet},\,\boldsymbol{M}^{\bullet})\;.$$

If we, however, would assume $\boldsymbol{M}_p = \mathit{grad}\,\mathbf{E}_p$ (constrained gradient plasticity) then the second flow rule M is not needed since the evolution of $\boldsymbol{M}_p$ would be given by

(4.19) $$\boldsymbol{M}_p^{\bullet} \;=\; \mathit{grad}\,E(\mathbf{E}_p\,,\,\mathbf{E}_e\,,\,\boldsymbol{M}_p\,,\,\boldsymbol{M}_e\,,\,\mathbf{Z}_p,\,\mathbf{E}^{\bullet},\,\boldsymbol{M}^{\bullet})$$

In what follows we will not use (4.19) but instead (4.17) and (4.18) which we consider as describing the more general case.

5.) an evolution equation called **hardening rule** for the additional variable(s) $\mathbf{Z}_p$, which is assumed to be of the same form as the flow rules above

$$\mathbf{Z}_p^{\bullet} = Z(\mathbf{E}_p, \mathbf{E}_e, \boldsymbol{M}_p, \boldsymbol{M}_e, \mathbf{Z}_p, \mathbf{E}^{\bullet}, \boldsymbol{M}^{\bullet}) . \tag{4.20}$$

These constitutive laws establish a complete mechanical rate-independent format for a gradient elasto-plasticity.

We now specify the ansatz for the rate-independent evolution (4.17), (4.18), (4.20) for the plastic variables, namely the flow and the hardening rules in the following form

$$\mathbf{E}_p^{\bullet} = \lambda\, E(\mathbf{E}_e, \boldsymbol{M}_e, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \tag{4.21}$$

$$\boldsymbol{M}_p^{\bullet} = \lambda\, M(\mathbf{E}_e, \boldsymbol{M}_e, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \tag{4.22}$$

$$\mathbf{Z}_p^{\bullet} = \lambda\, Z(\mathbf{E}_e, \boldsymbol{M}_e, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \tag{4.23}$$

with a joint **plastic consistency parameter** λ and three functions E, M, and Z of the listed arguments. λ is zero if and only if no yielding occurs, *i.e.* during elastic events. During yielding, however, λ is positive. In all cases the KUHN-TUCKER-conditions hold in the form

$$\lambda\, \varphi = 0 \qquad \text{with} \qquad \lambda \geq 0 \qquad \text{and} \qquad \varphi \leq 0 . \tag{4.24}$$

The plastic parameter can be determined during yielding by the **consistency condition** as a consequence of the yield condition (4.14)

$$\begin{aligned} 0 &= \partial_{\mathbf{E}_e} \varphi \cdot\cdot\, \mathbf{E}_e^{\bullet} + \partial_{\boldsymbol{M}_e} \varphi \cdots \boldsymbol{M}_e^{\bullet} \\ &+ \partial_{\mathbf{E}_p} \varphi \cdot\cdot\, \lambda\, E(\mathbf{E}_e, \boldsymbol{M}_e, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\ &+ \partial_{\boldsymbol{M}_p} \varphi \cdots \lambda\, M(\mathbf{E}_e, \boldsymbol{M}_e, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\ &+ \partial_{\mathbf{Z}_p} \varphi \cdot\cdot\, \lambda\, Z(\mathbf{E}_e, \boldsymbol{M}_e, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \end{aligned} \tag{4.25}$$

which can be solved for the plastic parameter

$$\lambda = \alpha^{-1} (\partial_{\mathbf{E}_e} \varphi \cdot\cdot\, \mathbf{E}_e^{\bullet} + \partial_{\boldsymbol{M}_e} \varphi \cdots \boldsymbol{M}_e^{\bullet}) \tag{4.26}$$

with a scalar denominator

$$\begin{aligned} \alpha(\mathbf{E}_e, \boldsymbol{M}_e, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) :=\ & - \partial_{\mathbf{E}_p} \varphi \cdot\cdot\, E(\mathbf{E}_e, \boldsymbol{M}_e, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\ & - \partial_{\boldsymbol{M}_p} \varphi \cdots M(\mathbf{E}_e, \boldsymbol{M}_e, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) - \partial_{\mathbf{Z}_p} \varphi \cdot\cdot\, Z(\mathbf{E}_e, \boldsymbol{M}_e, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p). \end{aligned} \tag{4.27}$$

Due to the loading condition (4.16), $\lambda\alpha$ must be positive during plastic events. After (4.24), λ alone is positive, and so α must also be positive. After (4.27) this is a restriction to the functions E, M, Z, and the yield criterion.

4.2 2nd-Order Gradient Thermo-Elastoplasticity

We use the **energy balance** (first law of thermodynamics) in the local form

$$\rho\,\varepsilon^{\bullet} = \rho\,Q + \mathbf{T} \cdot\cdot\, \mathbf{E}^{\bullet} + \boldsymbol{G} \cdots \boldsymbol{M}^{\bullet}. \tag{4.28}$$

with the stress power density $\mathbf{T} \cdot\cdot\, \mathbf{E}^{\bullet} + \boldsymbol{G} \cdots \boldsymbol{M}^{\bullet}$ and the heat supply per unit mass and time Q, which results from irradiation r and conduction $\mathbf{q}$ in the usual form

$$Q = r - (div\,\mathbf{q})/\rho. \tag{4.29}$$

By the introduction of the **HELMHOLTZ free energy**

$$\psi := \varepsilon - \theta\,\eta \tag{4.30}$$

we assume the second law as the **CLAUSIUS-DUHEM inequality** in the form

$$\frac{1}{\rho}\mathbf{T} \cdot\cdot\, \mathbf{E}^{\bullet} + \frac{1}{\rho}\boldsymbol{G} \cdots \boldsymbol{M}^{\bullet} - \psi^{\bullet} - \eta\,\theta^{\bullet} - \frac{1}{\rho\theta}\mathbf{q}\cdot\mathbf{g} \geq 0. \tag{4.31}$$

Thus, the specific **dissipation**, which consists of the mechanical dissipation

$$\delta_{\mathrm{m}} := \frac{1}{\rho}\mathbf{T} \cdot\cdot\, \mathbf{E}^{\bullet} + \frac{1}{\rho}\boldsymbol{G} \cdots \boldsymbol{M}^{\bullet} - \psi^{\bullet} - \eta\,\theta^{\bullet} = \theta\,\eta^{\bullet} - Q \tag{4.32}$$

by using (4.28) and (4.30), and the thermal dissipation

$$\delta_{\mathrm{th}} := -\frac{1}{\rho\theta}\mathbf{q}\cdot\mathbf{g} \tag{4.33}$$

fulfill the *dissipation inequality*

$$\delta = \delta_{\mathrm{m}} + \delta_{\mathrm{th}} \geq 0. \tag{4.34}$$

In order to enlarge the mechanical plasticity theory to a thermo-mechanical one, we add the temperature and the temperature gradient to the list of independent variables called **thermo-kinematical variables**, and the heat flux, the entropy, and the internal energy or the free energy to the dependent variables called **caloro-dynamic variables**.

Thermoplastic materials can be understood as material models with *internal variables*. The set of the internal variables contains in the case of plasticity the first and second-order plastic strains and, eventually, hardening variables. For all of these variables we assume as generalizations of (4.17), (4.18), (4.20) rate-independent evolution equations in the general form

$$\mathbf{E}_p^{\bullet} = E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p, \mathbf{E}^{\bullet}, \boldsymbol{M}^{\bullet}, \theta^{\bullet}, \mathbf{g}^{\bullet}) \tag{4.35}$$

$$\boldsymbol{M}_p^{\bullet} = M(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p, \mathbf{E}^{\bullet}, \boldsymbol{M}^{\bullet}, \theta^{\bullet}, \mathbf{g}^{\bullet}) \tag{4.36}$$

$$\mathbf{Z}_p^{\bullet} = Z(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p, \mathbf{E}^{\bullet}, \boldsymbol{M}^{\bullet}, \theta^{\bullet}, \mathbf{g}^{\bullet}). \tag{4.37}$$

Again, M will only be needed if we consider $\boldsymbol{M}_p$ as an internal variable independent of $\mathbf{E}_p$. The set of additional constitutive equations for a thermomechanical material with internal variables is assumed to be

$$
\begin{aligned}
\mathbf{T} &= T(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\
\boldsymbol{G} &= G(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\
\mathbf{q} &= q(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\
\eta &= \eta(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\
\varepsilon &= \varepsilon(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)
\end{aligned} \tag{4.38}
$$

or $\quad \psi = \psi(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$.

Instead of elastic ranges we will now have to deal with **thermoelastic ranges**. These are specified by a **yield criterion**

$$\varphi(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \tag{4.39}$$

which induces the **yield condition**

$$\varphi(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) = 0 \tag{4.40}$$

and the **loading condition**

$$\partial_{\mathbf{E}_e} \varphi \cdot\cdot \mathbf{E}^\bullet + \partial_{\boldsymbol{M}_e} \varphi \cdot\cdot\cdot \boldsymbol{M}^\bullet + \partial_\theta \varphi \, \theta^\bullet + \partial_{\mathbf{g}} \varphi \cdot \mathbf{g}^\bullet > 0. \tag{4.41}$$

If not both conditions are simultaneously fulfilled, we consider it as a (thermo) **elastic event** and, hence, the plastic variables do not evolve

$$\mathbf{E}_p^\bullet \equiv \mathbf{0} \qquad \boldsymbol{M}_p^\bullet \equiv \boldsymbol{0} \qquad \mathbf{Z}_p^\bullet \equiv \mathbf{0} \tag{4.42}$$

$$\Rightarrow \qquad \mathbf{E}_e^\bullet \equiv \mathbf{E}^\bullet \qquad \text{and} \qquad \boldsymbol{M}_e^\bullet \equiv \boldsymbol{M}^\bullet$$

which are side conditions of the functions E, M, and Z. Otherwise it is a **plastic event** or an event of **yielding**, in which the plastic variables necessarily have to evolve according to (4.35) - (4.37).

We will next investigate the restrictions imposed on the constitutive equations by the CLAUSIUS-DUHEM inequality. With the free energy (4.30) we obtain for the inequality (4.31)

$$
\begin{aligned}
0 \geq\ & \partial_{\mathbf{E}_e} \psi \cdot\cdot \mathbf{E}_e^\bullet + \partial_{\boldsymbol{M}_e} \psi \cdot\cdot\cdot \boldsymbol{M}_e^\bullet + \partial_\theta \psi \, \theta^\bullet + \partial_{\mathbf{g}} \psi \cdot \mathbf{g}^\bullet + \partial_{\mathbf{E}_p} \psi \cdot\cdot \mathbf{E}_p^\bullet + \partial_{\boldsymbol{M}_p} \psi \cdot\cdot\cdot \boldsymbol{M}_p^\bullet \\
& + \partial_{\mathbf{Z}_p} \psi \cdot\cdot \mathbf{Z}_p^\bullet + \eta \, \theta^\bullet - \frac{1}{\rho} \mathbf{T} \cdot\cdot (\mathbf{E}_e^\bullet + \mathbf{E}_p^\bullet) - \frac{1}{\rho} \boldsymbol{G} \cdot\cdot\cdot (\boldsymbol{M}_e^\bullet + \boldsymbol{M}_p^\bullet) + \frac{1}{\rho\theta} \mathbf{q} \cdot \mathbf{g} \\
=\ & (\partial_{\mathbf{E}_e} \psi - \frac{\mathbf{T}}{\rho}) \cdot\cdot \mathbf{E}_e^\bullet + (\partial_{\boldsymbol{M}_e} \psi - \frac{1}{\rho} \boldsymbol{G}) \cdot\cdot\cdot \boldsymbol{M}_e^\bullet + (\partial_\theta \psi + \eta) \, \theta^\bullet + \partial_{\mathbf{g}} \psi \cdot \mathbf{g}^\bullet \\
& + (\partial_{\mathbf{E}_p} \psi - \frac{\mathbf{T}}{\rho}) \cdot\cdot \mathbf{E}_p^\bullet + (\partial_{\boldsymbol{M}_p} \psi - \frac{1}{\rho} \boldsymbol{G}) \cdot\cdot\cdot \boldsymbol{M}_p^\bullet + \partial_{\mathbf{Z}_p} \psi \cdot\cdot \mathbf{Z}_p^\bullet + \frac{1}{\rho\theta} \mathbf{q} \cdot \mathbf{g}.
\end{aligned} \tag{4.43}
$$

By standard arguments which can be found in the preceding chapter, we obtain the following results.

Theorem 3.1. *The CLAUSIUS-DUHEM inequality is fulfilled if and only if the following conditions hold:*

- *the representation of the free energy*

(4.44) $$\psi = \psi(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$$

- *the thermoelastic potentials* (4.45) - (4.47) *for the stress tensors and the entropy*

(4.45) $$\mathbf{T} = \rho\, \partial_{\mathbf{E}_e} \psi = T(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$$

(4.46) $$\boldsymbol{G} = \rho\, \partial_{\boldsymbol{M}_e} \psi = G(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$$

(4.47) $$\eta = -\partial_\theta \psi = \eta(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$$

- *the heat-conduction inequality*

(4.48) $$\mathbf{q} \cdot \mathbf{g} \leq 0$$

- *the residual dissipation inequality*

(4.49) $$(\frac{\mathbf{T}}{\rho} - \partial_{\mathbf{E}_p} \psi) \cdot\cdot\, \mathbf{E}_p^{\bullet} + (\frac{1}{\rho}\boldsymbol{G} - \partial_{\boldsymbol{M}_p} \psi) \cdot\cdot\cdot\, \boldsymbol{M}_p^{\bullet} - \partial_{\mathbf{Z}_p} \psi \cdot\cdot\, \mathbf{Z}_p^{\bullet} \geq 0\,.$$

Identical Thermoelastic Behavior

If we again apply the Assumption 3.1 of identical thermoelastic behavior in all thermoelastic ranges, we can conclude that during elastic events all measurable quantities like the stresses, the heat flux, and the heat supply depend only upon the elastic strains and the temperature and its gradient, but not upon the plastic variables $\mathbf{E}_p$, $\boldsymbol{M}_p$, and $\mathbf{Z}_p$. The consequences of this assumption have already been investigated in the preceding chapter. Here we just summarize the results in analogy to Theorem 3.2.

Theorem 4.2. *The assumption of equal thermoelastic behavior in all elastic ranges is fulfilled if and only if the following material functions*

with the following material functions

- the elastic free energy $\psi_e(\mathbf{E}_e, \boldsymbol{M}_e, \theta)$
- the plastic entropy $\eta_p(\mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$
- the plastic internal energy $\varepsilon_p(\mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$

[36] see also the formulations by FOREST/ AMESTOY (2008) and VOYIADJIS/ FAGHIHI (2012, 2013).

- the heat conduction law $q(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g})$

give the representations for the total

- *entropy* $\eta = -\partial_\theta \psi_e(\mathbf{E}_e, \boldsymbol{M}_e, \theta) + \eta_p(\mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$
- *internal energy* $\varepsilon = \psi_e(\mathbf{E}_e, \boldsymbol{M}_e, \theta) - \theta\,\partial_\theta \psi_e(\mathbf{E}_e, \boldsymbol{M}_e, \theta) + \varepsilon_p(\mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$
- *free energy* $\psi = \psi_e(\mathbf{E}_e, \boldsymbol{M}_e, \theta) - \theta\,\eta_p(\mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) + \varepsilon_p(\mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$
- *heat conduction* $\mathbf{q} = q(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g})$
- *2nd-order stresses* $\mathbf{T} = \rho\ \partial_{\mathbf{E}e}\, \psi_e(\mathbf{E}_e, \boldsymbol{M}_e, \theta)$
- *3rd-order stresses* $\boldsymbol{G} = \rho\ \partial_{\boldsymbol{M}e}\, \psi_e(\mathbf{E}_e, \boldsymbol{M}_e, \theta)$.

These material functions constitute together with the yield criterion (4.39) and the flow and hardening rules the complete material model for a second-gradient elastoplastic material.

(4.59) gives for the residual dissipation inequality (4.49)

$$
\begin{aligned}
0 \le\ & \left(\frac{1}{\rho}\mathbf{T} - \partial_{\mathbf{E}_p}\varepsilon_p + \theta\,\partial_{\mathbf{E}_p}\eta_p\right) \cdot\cdot\ \mathbf{E}_p^{\bullet} + \left(\frac{1}{\rho}\boldsymbol{G} - \partial_{\boldsymbol{M}_p}\varepsilon_p + \theta\,\partial_{\boldsymbol{M}_p}\eta_p\right) \cdots \boldsymbol{M}_p^{\bullet} \\
& + \left(-\partial_{\mathbf{Z}_p}\varepsilon_p + \theta\,\partial_{\mathbf{Z}_p}\eta_p\right) \cdot\cdot\ \mathbf{Z}_p^{\bullet} \qquad (4.50) \\
=\ & \frac{1}{\rho}\mathbf{T} \cdot\cdot\ \mathbf{E}_p^{\bullet} + \frac{1}{\rho}\boldsymbol{G} \cdots \boldsymbol{M}_p^{\bullet} + \theta\,\eta_p(\mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)^{\bullet} - \varepsilon_p(\mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)^{\bullet}.
\end{aligned}
$$

In order to determine the change of the temperature of the material point, we use the first law of thermodynamics (4.28) together with (4.9) and (4.10) and (4.60) to obtain the split

$$Q = Q_e + Q_p \qquad (4.51)$$

into an elastic part

$$Q_e(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{E}_e^{\bullet}, \boldsymbol{M}_e^{\bullet}, \theta^{\bullet}) := c\ \theta^{\bullet} - \frac{\theta}{\rho}(\mathbf{R} \cdot\cdot\ \mathbf{E}_e^{\bullet} + \boldsymbol{R} \cdots \boldsymbol{M}_e^{\bullet}) \qquad (4.52)$$

with

- the *specific heat* $c(\mathbf{E}_e, \boldsymbol{M}_e, \theta) := \theta\,\partial_\theta \eta_e$
- the *2nd-order stress-temperature tensor* $\mathbf{R}(\mathbf{E}_e, \boldsymbol{M}_e, \theta) := -\rho\,\partial_{\mathbf{E}_e}\eta_e$
- the *3rd-order stress-temperature tensor* $\boldsymbol{R}(\mathbf{E}_e, \boldsymbol{M}_e, \theta) := -\rho\,\partial_{\boldsymbol{M}_e}\eta_e$

and a plastic part

$$Q_p := \varepsilon_p(\mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)^{\bullet} - \frac{\mathbf{T}}{\rho} \cdot\cdot\ \mathbf{E}_p^{\bullet} - \frac{1}{\rho}\boldsymbol{G} \cdots \boldsymbol{M}_p^{\bullet}. \qquad (4.53)$$

This can be solved for the temperature rate

$$c\ \theta^{\bullet} = Q - Q_p + \frac{\theta}{\rho}\ (\mathbf{R} \cdot\cdot\ \mathbf{E}_e^{\bullet} + \boldsymbol{R} \cdots \boldsymbol{M}_e^{\bullet})\ . \qquad (4.54)$$

By this equation, we can integrate the temperature along the process and so determine the final temperature after some elasto-plastic process. Accordingly, temperature changes are caused by

1.) the heat supply Q from the outside

2.) the heat $-Q_p$ generated by plastic yielding and hardening, and

3.) thermoelastic transformations due to the last term in (4.54).

We now specify the ansatz for the rate-independent evolution (4.35) - (4.37) for the plastic variables, namely the flow and the hardening rules in the following form

$$\mathbf{E}_p^{\bullet} = \lambda\, E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \tag{4.55}$$

$$\boldsymbol{M}_p^{\bullet} = \lambda\, M(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \tag{4.56}$$

$$\mathbf{Z}_p^{\bullet} = \lambda\, Z(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \tag{4.57}$$

with a joint **plastic consistency parameter** λ and three functions E, M, and Z of the listed arguments. λ is zero if and only if no yielding occurs, *i.e.* during thermo-elastic events.

During yielding, however, λ is positive. In all cases the KUHN-TUCKER-conditions hold in the form

$$\lambda\, \varphi = 0 \qquad \text{with} \quad \lambda \geq 0 \qquad \text{and} \quad \varphi \leq 0\,. \tag{4.58}$$

The plastic parameter can be determined during yielding by the **consistency condition** as a consequence of the yield condition (4.40) in the form $\varphi^{\bullet} = 0$ as

$$\lambda = \alpha^{-1} \left(\partial_{\mathbf{E}_e} \varphi \cdot\cdot\, \mathbf{E}_e^{\bullet} + \partial_{\boldsymbol{M}_e} \varphi \cdots \boldsymbol{M}_e^{\bullet} + \partial_{\theta} \varphi\, \theta^{\bullet} + \partial_{\mathbf{g}} \varphi \cdot \mathbf{g}^{\bullet}\right) \tag{4.59}$$

with a scalar denominator

$$\begin{aligned} &\alpha(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\ &:= -\partial_{\mathbf{E}_p} \varphi \cdot\cdot\, E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\ &\quad -\partial_{\boldsymbol{M}_p} \varphi \cdots M(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\ &\quad -\partial_{\mathbf{Z}_p} \varphi \cdot\cdot\, Z(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)\,. \end{aligned} \tag{4.60}$$

Due to the loading condition (4.41), $\lambda\alpha$ must be positive during plastic events. After (4.58), λ alone is positively introduced, and so α must also be positive. After (4.60) this is a restriction to the functions $E, M, Z,$ and the yield criterion.

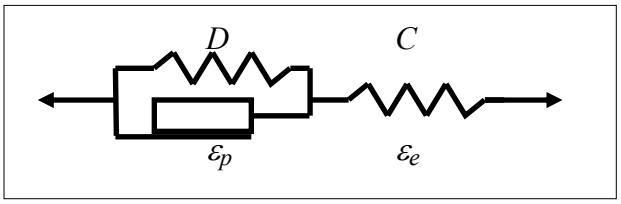
Another restriction on these functions is obtained by the second law. We substitute (4.55) - (4.57) into the residual dissipation inequality (4.50)

$$\begin{aligned} &(\mathbf{T} - \rho\, \partial_{\mathbf{E}_p} \varepsilon_p + \rho\, \theta\, \partial_{\mathbf{E}_p} \eta_p) \cdot\cdot\, E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\ &+ (\boldsymbol{G} - \rho\, \partial_{\boldsymbol{M}_p} \varepsilon_p + \rho\, \theta\, \partial_{\boldsymbol{M}_p} \eta_p) \cdots M(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\ &+ (-\rho\, \partial_{\mathbf{Z}_p} \varepsilon_p + \rho\, \theta\, \partial_{\mathbf{Z}_p} \eta_p) \cdot\cdot\, Z(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \geq 0. \end{aligned} \tag{4.61}$$

This restriction on the functions E, M, Z, and the yield criterion will be considered in the following example.

Example

We will next discuss the foregoing framework for gradient plasticity by means of a simple example, which is a generalization of the one in BERTRAM/ KRAWIETZ (2012). It is based on a one-dimensional PRAGER model with two springs and a ST.VENANT element which stands for dry friction. This model performs an elastoplastic behavior with linear kinematical hardening induced by the spring D, while the stresses can be determined by the strains in the spring C. We will generalize this model into three dimensions first with full anisotropy, and finally particularize it to the isotropic case.



Within the format of a linear theory, the elastic part of the free energy has a quadratic form familiar from linear thermoelasticity

$$\rho\,\psi_e(\mathbf{E}_e, \boldsymbol{M}_e, \theta) = \tfrac{1}{2}\,\mathbf{E}_e \cdot\cdot \overset{\langle 4\rangle}{\boldsymbol{C}}_E \cdot\cdot\, \mathbf{E}_e + \tfrac{1}{2}\,\boldsymbol{M}_e \cdots \overset{\langle 6\rangle}{\boldsymbol{C}}_M \cdots \boldsymbol{M}_e + \mathbf{E}_e \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM} \cdots \boldsymbol{M}_e + c\,\rho\,(\Delta\theta - \theta\, ln\,\frac{\theta}{\theta_0}) + \Delta\theta\,(\mathbf{R} \cdot\cdot\, \mathbf{E}_e + \boldsymbol{R} \cdots \boldsymbol{M}_e) \tag{4.62}$$

with higher-order elasticity tensors $\overset{\langle 4\rangle}{\boldsymbol{C}}_E$, $\overset{\langle 6\rangle}{\boldsymbol{C}}_M$, $\overset{\langle 5\rangle}{\boldsymbol{C}}_{EM}$ with the same symmetry properties as in (4.13), two constant stress-temperature tensors $\mathbf{R}$ and $\boldsymbol{R}$, a constant specific heat c after (4.51), a reference temperature θ_0, and the deviation from it $\Delta\theta := \theta - \theta_0$.

Since all material parameters are taken as constant, this model applies only to moderate temperature changes and small deformations.

We obtain with (4.56) - (4.57) the thermoelastic laws for the stresses

$$\mathbf{T} = \overset{\langle 4\rangle}{\boldsymbol{C}}_E \cdot\cdot\, \mathbf{E}_e + \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM} \cdots \boldsymbol{M}_e + \Delta\theta\,\mathbf{R} \tag{4.63}$$

$$\boldsymbol{G} = \mathbf{E}_e \cdot\cdot\, \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM} + \overset{\langle 6\rangle}{\boldsymbol{C}}_M \cdots \boldsymbol{M}_e + \Delta\theta\,\boldsymbol{R} \tag{4.64}$$

in analogy to (4.11) - (4.12).

By the results of Theorem 4.2 we get for the elastic entropy

$$\rho\,\eta_e(\mathbf{E}_e, \mathbf{K}_e, \theta) = -\mathbf{R} \cdot\cdot\, \mathbf{E}_e - \boldsymbol{R} \cdots \boldsymbol{M}_e + \rho\, c\, ln\,\frac{\theta}{\theta_0} \tag{4.65}$$

and for the elastic internal energy

$$\rho\,\varepsilon_e(\mathbf{E}_e\,,\boldsymbol{M}_e\,,\theta) \;=\; \tfrac{1}{2}\,\mathbf{E}_e \cdot\cdot \overset{\langle 4\rangle}{\boldsymbol{C}}_E \cdot\cdot\, \mathbf{E}_e + \tfrac{1}{2}\,\boldsymbol{M}_e \cdots \overset{\langle 6\rangle}{\boldsymbol{C}}_M \cdots \boldsymbol{M}_e \\ + \mathbf{E}_e \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM} \cdots \boldsymbol{M}_e + c\,\rho\,\Delta\theta - \theta_0\,(\mathbf{R} \cdot\cdot\, \mathbf{E}_e + \boldsymbol{R} \cdots \boldsymbol{M}_e)\,. \tag{4.66}$$

For the heat flux we choose a FOURIER-type law

$$\mathbf{q} \;=\; -\,\mathbf{K} \cdot \mathbf{g} \tag{4.67}$$

with a second-order positive semi-definite tensor $\mathbf{K}$ so that the heat conduction inequality (4.48) is always fulfilled.

In accordance with the PRAGER model we write for the back stresses two elastic laws in the plastic strains

$$\mathbf{T}_B \;=\; \overset{\langle 4\rangle}{\boldsymbol{D}}_E \cdot\cdot\, \mathbf{E}_p + \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM} \cdots \boldsymbol{M}_p \tag{4.68}$$

$$\boldsymbol{G}_B \;=\; \mathbf{E}_p \cdot\cdot\, \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM} + \overset{\langle 6\rangle}{\boldsymbol{D}}_M \cdots \boldsymbol{M}_p \tag{4.69}$$

with material tensors $\overset{\langle 4\rangle}{\boldsymbol{D}}_E$, $\overset{\langle 6\rangle}{\boldsymbol{D}}_M$, $\overset{\langle 5\rangle}{\boldsymbol{D}}_{EM}$ with the usual symmetry properties (4.13).

We introduce the specific **plastic work** in some time interval $[t_0\,,\,t_1]$ as the work of the effective stresses upon the plastic deformations

$$w_{plast} := \int_{t_0}^{t_1} \frac{1}{\rho}\,[(\mathbf{T} - \mathbf{T}_B) \cdot\cdot\, \mathbf{E}_p^{\bullet} + (\boldsymbol{G} - \boldsymbol{G}_B) \cdots \boldsymbol{M}_p^{\bullet}]\,dt \tag{4.70}$$

$$\Rightarrow \qquad w_{plast}^{\bullet} \;=\; \frac{1}{\rho}\,(\mathbf{T} - \mathbf{T}_B) \cdot\cdot\, \mathbf{E}_p^{\bullet} + \frac{1}{\rho}\,(\boldsymbol{G} - \boldsymbol{G}_B) \cdots \boldsymbol{M}_p^{\bullet}$$

as the only hardening variable in our example, so $\mathbf{Z}_p \equiv w_{plast}$.

The plastic part of the entropy is set to zero

$$\eta_p(\mathbf{E}_p, \boldsymbol{M}_p\,, \mathbf{Z}_p) \;=\; 0\,. \tag{4.71}$$

We will later show under which restrictions this ansatz will satisfy the residual dissipation inequality, and also that other non-trivial choices for the plastic entropy will work as well.

We also assume a quadratic form for the plastic part of the internal energy

$$\rho\,\varepsilon_p(\mathbf{E}_p, \boldsymbol{M}_p\,, \mathbf{Z}_p) \;=\; \tfrac{1}{2}\,\mathbf{E}_p \cdot\cdot \overset{\langle 4\rangle}{\boldsymbol{D}}_E \cdot\cdot\, \mathbf{E}_p + \tfrac{1}{2}\,\boldsymbol{M}_p \cdots \overset{\langle 6\rangle}{\boldsymbol{D}}_M \cdots \boldsymbol{M}_p \\ + \mathbf{E}_p \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM} \cdots \boldsymbol{M}_p + \rho\,\mu\,w_{plast} \tag{4.72}$$

with a scalar coefficient μ.

In accordance with (4.49) and (4.71) we can in fact verify the ansatz (4.68) and (4.69) for the back stresses.

We use a generalization of the anisotropic v. MISES yield criterion (1928) in the stress space

$$\varphi \;=\; (\mathbf{T} - \mathbf{T}_B) \cdot\cdot \overset{\langle 4\rangle}{\boldsymbol{W}} \cdot\cdot\, (\mathbf{T} - \mathbf{T}_B) + (\boldsymbol{G} - \boldsymbol{G}_B) \cdots \overset{\langle 6\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G} - \boldsymbol{G}_B) \tag{4.73}$$

$$+ (\mathbf{T} - \mathbf{T}_B) \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G} - \boldsymbol{G}_B) - \sigma_Y(w_{plast}\,,\, \theta, \mathbf{g})^2$$

with three material tensors $\overset{\langle 4 \rangle}{\boldsymbol{W}}$, $\overset{\langle 5 \rangle}{\boldsymbol{W}}$, and $\overset{\langle 6 \rangle}{\boldsymbol{W}}$ with the same symmetry properties as in (4.13), and reflecting the symmetry of the material, and a scalar yield stress σ_Y depending on the plastic work, the temperature, and the temperature gradient. However, to the best of our knowledge a direct dependence of the yield function on the temperature gradient has not yet been proposed in the literature.

Uniqueness and regularity of the model are ensured if the yield function is convex with respect to all its arguments, including the higher-order strain (or stress). In the proposed quadratic potential (4.73), convexity of the yield surface is ensured if the linear elastic mapping is positive-definite.

If the stresses and the back stresses are substituted by (4.63), (4.64), (4.68), (4.69), the assumed form of the yield criterion can in fact be transformed into the strain space

$$\begin{aligned}
&\varphi(\mathbf{E}_e\,,\, \boldsymbol{M}_e\,,\, \theta, \mathbf{g}, \mathbf{E}_p\,,\, \boldsymbol{M}_p\,,\, w_{plast}) \\
&= (\overset{\langle 4 \rangle}{\boldsymbol{C}}_E \cdot\cdot\, \mathbf{E}_e + \overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM} \cdots \boldsymbol{M}_e + \Delta\theta\, \mathbf{R} - \overset{\langle 4 \rangle}{\boldsymbol{D}}_E \cdot\cdot\, \mathbf{E}_p - \overset{\langle 5 \rangle}{\boldsymbol{D}}_{EM} \cdots \boldsymbol{M}_p) \cdot\cdot \overset{\langle 4 \rangle}{\boldsymbol{W}} \\
&\cdot\cdot (\overset{\langle 4 \rangle}{\boldsymbol{C}}_E \cdot\cdot\, \mathbf{E}_e + \overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM} \cdots \boldsymbol{M}_e + \Delta\theta\, \mathbf{R} - \overset{\langle 4 \rangle}{\boldsymbol{D}}_E \cdot\cdot\, \mathbf{E}_p - \overset{\langle 5 \rangle}{\boldsymbol{D}}_{EM} \cdots \boldsymbol{M}_p) \\
&+ (\mathbf{E}_e \cdot\cdot\, \overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM} + \overset{\langle 6 \rangle}{\boldsymbol{C}}_M \cdots \boldsymbol{M}_e + \Delta\theta\, \boldsymbol{R} - \overset{\langle 6 \rangle}{\boldsymbol{D}}_M \cdots \boldsymbol{M}_p - \mathbf{E}_p \cdot\cdot\, \overset{\langle 5 \rangle}{\boldsymbol{D}}_{EM}) \cdots \overset{\langle 6 \rangle}{\boldsymbol{W}} \\
&\cdots (\mathbf{E}_e \cdot\cdot\, \overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM} + \overset{\langle 6 \rangle}{\boldsymbol{C}}_M \cdots \boldsymbol{M}_e + \Delta\theta\, \boldsymbol{R} - \overset{\langle 6 \rangle}{\boldsymbol{D}}_M \cdots \boldsymbol{M}_p - \mathbf{E}_p \cdot\cdot\, \overset{\langle 5 \rangle}{\boldsymbol{D}}_{EM}) \\
&+ (\overset{\langle 4 \rangle}{\boldsymbol{C}}_E \cdot\cdot\, \mathbf{E}_e + \overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM} \cdots \boldsymbol{M}_e + \Delta\theta\, \mathbf{R} - \overset{\langle 4 \rangle}{\boldsymbol{D}}_E \cdot\cdot\, \mathbf{E}_p - \overset{\langle 5 \rangle}{\boldsymbol{D}}_{EM} \cdots \boldsymbol{M}_p) \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{W}} \\
&\cdots (\mathbf{E}_e \cdot\cdot\, \overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM} + \overset{\langle 6 \rangle}{\boldsymbol{C}}_M \cdots \boldsymbol{M}_e + \Delta\theta\, \boldsymbol{R} - \overset{\langle 6 \rangle}{\boldsymbol{D}}_M \cdots \boldsymbol{M}_p - \mathbf{E}_p \cdot\cdot\, \overset{\langle 5 \rangle}{\boldsymbol{D}}_{EM}) \\
&- \sigma_Y\,(w_{plast}\,,\, \theta, \mathbf{g})^2
\end{aligned} \tag{4.74}$$

such that the loading condition (4.41) becomes

$$\begin{aligned}
0 \;<\; &[2(\mathbf{T} - \mathbf{T}_B) \cdot\cdot \overset{\langle 4 \rangle}{\boldsymbol{W}} + \overset{\langle 5 \rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G} - \boldsymbol{G}_B)] \cdot\cdot [\overset{\langle 4 \rangle}{\boldsymbol{C}}_E \cdot\cdot\, \mathbf{E}^\bullet + \overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM} \cdots \boldsymbol{M}^\bullet + \theta^\bullet\, \mathbf{R}] \\
&+ [2(\boldsymbol{G} - \boldsymbol{G}_B) \cdots \overset{\langle 6 \rangle}{\boldsymbol{W}} + (\mathbf{T} - \mathbf{T}_B) \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{W}}] \cdots [\mathbf{E}^\bullet \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM} + \overset{\langle 6 \rangle}{\boldsymbol{C}}_M \cdots \boldsymbol{M}^\bullet + \theta^\bullet\, \boldsymbol{R}] \\
&- 2\sigma_Y\, (\partial_\theta \sigma_Y\, \theta^\bullet + \partial_{\mathbf{g}} \sigma_Y \cdot \mathbf{g}^\bullet)\,.
\end{aligned} \tag{4.75}$$

The associated flow rules are

$$\mathbf{E}_p^\bullet \;=\; \lambda\, \partial_{\mathbf{T}}\, \varphi \;=\; \lambda\, [2 \overset{\langle 4 \rangle}{\boldsymbol{W}} \cdot\cdot (\mathbf{T} - \mathbf{T}_B) + \overset{\langle 5 \rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G} - \boldsymbol{G}_B)] \tag{4.76}$$

$$\boldsymbol{M}_p^\bullet \;=\; \lambda\, \partial_{\mathbf{G}}\, \varphi \;=\; \lambda\, [2 \overset{\langle 6 \rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G} - \boldsymbol{G}_B) + (\mathbf{T} - \mathbf{T}_B) \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{W}}] \tag{4.77}$$

with a joint plastic consistency parameter λ. The consistency condition requires with (4.9), (4.10), (4.63), (4.64), (4.68), (4.69), (4.74), (4.76), (4.77)

$$0 \;=\; \varphi(\mathbf{E}_e\,,\, \boldsymbol{M}_e\,,\, \theta, \mathbf{g}, \mathbf{E}_p\,,\, \boldsymbol{M}_p\,,\, w_{plast})^\bullet$$

$$= 2\,(\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 4\rangle}{\boldsymbol{W}} \cdot\cdot (\mathbf{T}-\mathbf{T}_B)^{\bullet} + 2\,(\boldsymbol{G}-\boldsymbol{G}_B) \cdots \overset{\langle 6\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B)^{\bullet}$$

$$+ (\mathbf{T}-\mathbf{T}_B)^{\bullet} \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B) + (\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B)^{\bullet}$$

$$- 2\sigma_Y \{\partial_{w_{plast}} \sigma_Y [\frac{1}{\rho}(\mathbf{T}-\mathbf{T}_B) \cdot\cdot \mathbf{E}_p^{\bullet} + \frac{1}{\rho}(\boldsymbol{G}-\boldsymbol{G}_B) \cdots \boldsymbol{M}_p^{\bullet}] + \partial_\theta \sigma_Y\, \theta^{\bullet} + \partial_{\mathbf{g}} \sigma_Y \cdot \mathbf{g}^{\bullet}\}$$

$$= [2\,(\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 4\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B)]$$

$$\cdot\cdot [\overset{\langle 4\rangle}{\boldsymbol{C}}_E \cdot\cdot (\mathbf{E}-\mathbf{E}_p)^{\bullet} + \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM} \cdots (\boldsymbol{M}-\boldsymbol{M}_p)^{\bullet} + \Delta\theta^{\bullet}\mathbf{R} - \overset{\langle 4\rangle}{\boldsymbol{D}}_E \cdot\cdot \mathbf{E}_p^{\bullet} - \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM} \cdots \boldsymbol{M}_p^{\bullet}]$$

$$+ [2\,(\boldsymbol{G}-\boldsymbol{G}_B) \cdots \overset{\langle 6\rangle}{\boldsymbol{W}} + (\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{W}}\,]$$

$$\cdots [(\mathbf{E}-\mathbf{E}_p)^{\bullet} \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM} + \overset{\langle 6\rangle}{\boldsymbol{C}}_M \cdots (\boldsymbol{M}-\boldsymbol{M}_p)^{\bullet} + \Delta\theta^{\bullet}\boldsymbol{R} - \overset{\langle 6\rangle}{\boldsymbol{D}}_M \cdots \boldsymbol{M}_p^{\bullet} - \mathbf{E}_p^{\bullet} \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM}]$$

$$- 2\sigma_Y \{\partial_{w_{plast}} \sigma_Y [\frac{1}{\rho}(\mathbf{T}-\mathbf{T}_B) \cdot\cdot \mathbf{E}_p^{\bullet} + \frac{1}{\rho}(\boldsymbol{G}-\boldsymbol{G}_B) \cdots \boldsymbol{M}_p^{\bullet}] + \partial_\theta \sigma_Y\, \theta^{\bullet} + \partial_{\mathbf{g}} \sigma_Y \cdot \mathbf{g}^{\bullet}\}$$

(4.78)
$$= [2\,(\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 4\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B)]$$

$$\cdot\cdot [\overset{\langle 4\rangle}{\boldsymbol{C}}_E \cdot\cdot (\mathbf{E}^{\bullet} - \lambda \{2\overset{\langle 4\rangle}{\boldsymbol{W}} \cdot\cdot (\mathbf{T}-\mathbf{T}_B) + \overset{\langle 5\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B)\})$$

$$+ \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM} \cdots (\boldsymbol{M}^{\bullet} - \lambda \{2\overset{\langle 6\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B) + (\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{W}}\}) + \Delta\theta^{\bullet}\,\mathbf{R}$$

$$- \overset{\langle 4\rangle}{\boldsymbol{D}}_E \cdot\cdot \lambda \{2\overset{\langle 4\rangle}{\boldsymbol{W}} \cdot\cdot (\mathbf{T}-\mathbf{T}_B) + \overset{\langle 5\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B)\}$$

$$- \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM} \cdots \lambda \{2\overset{\langle 6\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B) + (\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{W}}\}]$$

$$+ [2\,(\boldsymbol{G}-\boldsymbol{G}_B) \cdots \overset{\langle 6\rangle}{\boldsymbol{W}} + (\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{W}}\,]$$

$$\cdots [\,(\mathbf{E}^{\bullet} - \lambda \{2\overset{\langle 4\rangle}{\boldsymbol{W}} \cdot\cdot (\mathbf{T}-\mathbf{T}_B) + \overset{\langle 5\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B)\}) \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM}$$

$$+ \overset{\langle 6\rangle}{\boldsymbol{C}}_M \cdots (\boldsymbol{M}^{\bullet} - \lambda \{2\overset{\langle 6\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B) + (\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{W}}\}) + \Delta\theta^{\bullet}\,\boldsymbol{R}$$

$$- \overset{\langle 6\rangle}{\boldsymbol{D}}_M \cdots \lambda \{2\overset{\langle 6\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B) + (\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{W}}\}$$

$$- \lambda \{2\overset{\langle 4\rangle}{\boldsymbol{W}} \cdot\cdot (\mathbf{T}-\mathbf{T}_B) + \overset{\langle 5\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B)\} \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM}]$$

$$- 2\sigma_Y [\partial_{w_{plast}} \sigma_Y (\frac{\mathbf{T}-\mathbf{T}_B}{\rho} \cdot\cdot \lambda \{2\overset{\langle 4\rangle}{\boldsymbol{W}} \cdot\cdot (\mathbf{T}-\mathbf{T}_B) + \overset{\langle 5\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B)\}$$

$$+ \frac{1}{\rho}(\boldsymbol{G}-\boldsymbol{G}_B) \cdots \lambda \{2\overset{\langle 6\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B) + (\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 5\rangle}{\boldsymbol{W}}\}) + \partial_\theta \sigma_Y\, \theta^{\bullet} + \partial_{\mathbf{g}} \sigma_Y \cdot \mathbf{g}^{\bullet}]$$

or

$$[2\,(\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 4\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G}-\boldsymbol{G}_B)] \cdot\cdot [\overset{\langle 4\rangle}{\boldsymbol{C}}_E \cdot\cdot \mathbf{E}^{\bullet} + \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM} \cdots \boldsymbol{M}^{\bullet} + \Delta\theta^{\bullet}\mathbf{R}]$$

$$
\begin{aligned}
&+ [2\,(\boldsymbol{G}-\boldsymbol{G}_B)\cdots\overset{\langle 6\rangle}{\boldsymbol{W}} + (\mathbf{T}-\mathbf{T}_B)\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{W}}]\cdots[\mathbf{E}^{\bullet}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{C}}_{EM} + \overset{\langle 6\rangle}{\boldsymbol{C}}_{M}\cdots\boldsymbol{M}^{\bullet} + \Delta\theta^{\bullet}\,\boldsymbol{R}] \\
&- 2\sigma_Y\,(\partial_\theta\sigma_Y\,\theta^{\bullet} + \partial_{\mathbf{g}}\sigma_Y\cdot\mathbf{g}^{\bullet}) \\
(4.79)\quad &= \lambda\Big[(\mathbf{T}-\mathbf{T}_B)\cdot\cdot\Big\{2\overset{\langle 4\rangle}{\boldsymbol{W}}\cdot\cdot[\overset{\langle 4\rangle}{\boldsymbol{C}}_{E}\cdot\cdot 2\overset{\langle 4\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM}\cdots\overset{\langle 5\rangle}{\boldsymbol{W}}^{*} + \overset{\langle 4\rangle}{\boldsymbol{D}}_{E}\cdot\cdot 2\overset{\langle 4\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM}\cdots\overset{\langle 5\rangle}{\boldsymbol{W}}^{*}] \\
&\quad + \overset{\langle 5\rangle}{\boldsymbol{W}}\cdots[2\overset{\langle 4\rangle}{\boldsymbol{W}}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{C}}_{EM} + \overset{\langle 6\rangle}{\boldsymbol{C}}_{M}\cdots\overset{\langle 5\rangle}{\boldsymbol{W}}^{*} + \overset{\langle 6\rangle}{\boldsymbol{D}}_{M}\cdots\overset{\langle 5\rangle}{\boldsymbol{W}}^{*} + \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM}^{*}\cdot\cdot 2\overset{\langle 4\rangle}{\boldsymbol{W}}] \\
&\quad + 4\sigma_Y\,\frac{1}{\rho}\partial_{w_{plast}}\sigma_Y\,\overset{\langle 4\rangle}{\boldsymbol{W}}\Big\}\cdot\cdot(\mathbf{T}-\mathbf{T}_B) \\
&\quad + (\boldsymbol{G}-\boldsymbol{G}_B)\cdots\Big\{\overset{\langle 5\rangle}{\boldsymbol{W}}^{*}\cdot\cdot[\overset{\langle 4\rangle}{\boldsymbol{C}}_{E}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM}\cdots 2\overset{\langle 6\rangle}{\boldsymbol{W}} + \overset{\langle 4\rangle}{\boldsymbol{D}}_{E}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM}\cdots 2\overset{\langle 6\rangle}{\boldsymbol{W}}] \\
&\quad + 2\,\overset{\langle 6\rangle}{\boldsymbol{W}}\cdots[\overset{\langle 5\rangle}{\boldsymbol{C}}_{EM}^{*}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{W}} + \overset{\langle 6\rangle}{\boldsymbol{C}}_{M}\cdots 2\overset{\langle 6\rangle}{\boldsymbol{W}} + \overset{\langle 6\rangle}{\boldsymbol{D}}_{M}\cdots 2\overset{\langle 6\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM}^{*}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{W}}] \\
&\quad + 4\sigma_Y\,\frac{1}{\rho}\partial_{w_{plast}}\sigma_Y\,\overset{\langle 6\rangle}{\boldsymbol{W}}\Big\}\cdots(\boldsymbol{G}-\boldsymbol{G}_B) \\
&\quad + (\mathbf{T}-\mathbf{T}_B)\cdot\cdot\Big\{2\overset{\langle 4\rangle}{\boldsymbol{W}}\cdot\cdot[\overset{\langle 4\rangle}{\boldsymbol{C}}_{E}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM}\cdots 2\overset{\langle 6\rangle}{\boldsymbol{W}} + \overset{\langle 4\rangle}{\boldsymbol{D}}_{E}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM}\cdots 2\overset{\langle 6\rangle}{\boldsymbol{W}}] \\
&\quad + \overset{\langle 5\rangle}{\boldsymbol{W}}\cdots[\overset{\langle 5\rangle}{\boldsymbol{W}}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{C}}_{EM} + \overset{\langle 6\rangle}{\boldsymbol{C}}_{M}\cdots 2\overset{\langle 6\rangle}{\boldsymbol{W}} + \overset{\langle 6\rangle}{\boldsymbol{D}}_{M}\cdots 2\overset{\langle 6\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM}^{*}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{W}}] \\
&\quad + 2\sigma_Y\,\frac{1}{\rho}\partial_{w_{plast}}\sigma_Y\,\overset{\langle 5\rangle}{\boldsymbol{W}}\Big\}\cdots(\boldsymbol{G}-\boldsymbol{G}_B) \\
&\quad + (\boldsymbol{G}-\boldsymbol{G}_B)\cdots\Big\{\overset{\langle 5\rangle}{\boldsymbol{W}}^{*}\cdot\cdot[\overset{\langle 4\rangle}{\boldsymbol{C}}_{E}\cdot\cdot 2\overset{\langle 4\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{C}}_{EM}\cdots\overset{\langle 5\rangle}{\boldsymbol{W}}^{*} + \overset{\langle 4\rangle}{\boldsymbol{D}}_{E}\cdot\cdot 2\overset{\langle 4\rangle}{\boldsymbol{W}} + \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM}\cdots\overset{\langle 5\rangle}{\boldsymbol{W}}^{*}] \\
&\quad + 2\overset{\langle 6\rangle}{\boldsymbol{W}}\cdots[2\overset{\langle 4\rangle}{\boldsymbol{W}}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{C}}_{EM}^{*} + \overset{\langle 6\rangle}{\boldsymbol{C}}_{M}\cdots\overset{\langle 5\rangle}{\boldsymbol{W}}^{*} + \overset{\langle 6\rangle}{\boldsymbol{D}}_{M}\cdots\overset{\langle 5\rangle}{\boldsymbol{W}}^{*} + \overset{\langle 5\rangle}{\boldsymbol{D}}_{EM}^{*}\cdot\cdot 2\overset{\langle 4\rangle}{\boldsymbol{W}}] \\
&\quad + 2\sigma_Y\,\frac{1}{\rho}\partial_{w_{plast}}\sigma_Y\,\overset{\langle 5\rangle}{\boldsymbol{W}}^{*}\Big\}\cdot\cdot(\mathbf{T}-\mathbf{T}_B)\Big]
\end{aligned}
$$

where the superimposed asterisk indicates that particular transposition of a fifth-order tensor $\overset{\langle 5\rangle}{\boldsymbol{W}}$ which fulfils

$$\boldsymbol{G}\cdots\overset{\langle 5\rangle}{\boldsymbol{W}}^{*}\cdot\cdot\,\mathbf{T} = \mathbf{T}\cdot\cdot\overset{\langle 5\rangle}{\boldsymbol{W}}\cdots\boldsymbol{G}$$

for every second-order tensor $\mathbf{T}$ and every third-order tensor $\boldsymbol{G}$.

This linear equation can be uniquely solved for the consistency parameter λ. The left-hand side of this equation is positive due to the loading condition (4.75). If the elasticities have the usual positive definiteness properties, and if hardening occurs with $\partial_{w_{plast}}\sigma_Y > 0$, then the terms in {}-brackets are always positive, so that λ will in fact be positive during yielding, as it should be. If we substitute λ into the flow rules (4.76) - (4.77), we obtain the *consistent flow rules*, which is straightforward but not done here for brevity.

If we substitute (4.76) - (4.77) into the definition of the plastic work (4.70), we obtain

$$w_{plast}^{\bullet} = \frac{\lambda}{\rho} \{ (\mathbf{T} - \mathbf{T}_B) \cdot\cdot [2 \overset{\langle 4 \rangle}{\boldsymbol{W}} \cdot\cdot (\mathbf{T} - \mathbf{T}_B) + \overset{\langle 5 \rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G} - \boldsymbol{G}_B)] \tag{4.80}$$

$$+ (\boldsymbol{G} - \boldsymbol{G}_B) \cdots [2 \overset{\langle 6 \rangle}{\boldsymbol{W}} \cdots (\boldsymbol{G} - \boldsymbol{G}_B) + (\mathbf{T} - \mathbf{T}_B) \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{W}} \} .$$

By an appropriate choice of $\overset{\langle 4 \rangle}{\boldsymbol{W}}$, $\overset{\langle 5 \rangle}{\boldsymbol{W}}$, and $\overset{\langle 6 \rangle}{\boldsymbol{W}}$ (positive semi-definiteness) we can assure that the plastic work is non-negative.

The heat generated by yielding $-Q_p$ after (4.53) is then according to (4.68), (4.69), (4.72) determined by

$$-Q_p = -\frac{1}{\rho} (\mathbf{E}_p \cdot\cdot \overset{\langle 4 \rangle}{\boldsymbol{D}}_E \cdot\cdot \mathbf{E}_p^{\bullet} + \boldsymbol{M}_p \cdots \overset{\langle 6 \rangle}{\boldsymbol{D}}_M \cdots \boldsymbol{M}_p^{\bullet} + \mathbf{E}_p^{\bullet} \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{D}}_{EM} \cdots \boldsymbol{M}_p \tag{4.81}$$

$$+ \mathbf{E}_p \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{D}}_{EM} \cdots \boldsymbol{M}_p^{\bullet}) - \mu \, w_{plast}^{\bullet} + \frac{\mathbf{T}}{\rho} \cdot\cdot \mathbf{E}_p^{\bullet} + \frac{\mathbf{M}}{\rho} \cdots \boldsymbol{M}_p^{\bullet}$$

$$= [\frac{\mathbf{T} - \mathbf{T}_B}{\rho} \cdot\cdot \mathbf{E}_p^{\bullet} - \theta \, \partial_{\mathbf{E}_p} \eta_p] \cdot\cdot \mathbf{E}_p^{\bullet} + [\frac{1}{\rho} (\boldsymbol{G} - \boldsymbol{G}_B) - \rho \, \theta \, \partial_{\boldsymbol{M}_p} \eta_p] \cdot\cdot \boldsymbol{M}_p^{\bullet} - \mu \, w_{plast}^{\bullet}$$

$$= (1 - \mu) \, w_{plast}^{\bullet} - \theta (\partial_{\mathbf{E}_p} \eta_p \cdot\cdot \mathbf{E}_p^{\bullet} + \partial_{\boldsymbol{M}_p} \eta_p \cdot\cdot \boldsymbol{M}_p^{\bullet}) .$$

$(1-\mu)$ can be interpreted as a TAYLOR-QUINNEY factor. The residual dissipation inequality (4.50) becomes with (4.53), (4.71), and (4.81)

$$0 \le -Q_p + \theta \, \eta_p^{\bullet} = (1 - \mu) \, w_{plast}^{\bullet} . \tag{4.82}$$

This can always be satisfied by choosing $0 \le \mu \le 1$. We see that a non-trivial ansatz for the plastic part of the entropy instead of (4.71)

$$\eta_p \equiv \gamma \, w_{plast} \tag{4.83}$$

with any non-negative real constant γ would also satisfy the dissipation inequality. So the plastic part of the entropy is only weakly restricted by the second law and by no means unique.

We finally obtain for the free energy for our model with (4.62), (4.72) and (4.83)

$$\psi = \frac{1}{2\rho} \mathbf{E}_e \cdot\cdot \overset{\langle 4 \rangle}{\boldsymbol{C}}_E \cdot\cdot \mathbf{E}_e + \frac{1}{2\rho} \boldsymbol{M}_e \cdot\cdot \overset{\langle 6 \rangle}{\boldsymbol{C}}_M \cdots \boldsymbol{M}_e + \frac{1}{\rho} \mathbf{E}_e \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{C}}_{EM} \cdots \boldsymbol{M}_e$$

$$+ c \, \rho \, (\Delta\theta - \theta \, ln \, \frac{\theta}{\theta_0}) + \Delta\theta \, \frac{1}{\rho} (\mathbf{R} \cdot\cdot \mathbf{E}_e + \boldsymbol{R} \cdots \mathbf{K}_e) \tag{4.84}$$

$$+ \frac{1}{2\rho} \mathbf{E}_p \cdot\cdot \overset{\langle 4 \rangle}{\boldsymbol{D}}_E \cdot\cdot \mathbf{E}_p + \frac{1}{2\rho} \boldsymbol{M}_p \cdots \overset{\langle 6 \rangle}{\boldsymbol{D}}_M \cdots \boldsymbol{M}_p$$

$$+ \frac{1}{\rho} \mathbf{E}_p \cdot\cdot \overset{\langle 5 \rangle}{\boldsymbol{D}}_{EM} \cdots \boldsymbol{M}_p + (1 - \mu) \, w_{plast}$$

which completes the model equations for the general anisotropic example.

Isotropic Example

For isotropic centro-symmetric materials we use the isotropic representations by CAUCHY and MINDLIN/ ESHEL (1968) for the constitutive equations. For the free energy we obtain after (4.84)

$$\begin{aligned}\rho\,\psi \;=\; & a_1\,(\mathbf{E}_e \cdot\cdot\, \mathbf{I})^2 + a_2\,\mathbf{E}_e \cdot\cdot\, \mathbf{E}_e + a_3\,\mathbf{I} \cdot\cdot\, \boldsymbol{M}_e \cdot \boldsymbol{M}_e \cdot\cdot\, \mathbf{I} + a_4\,(\boldsymbol{M}_e \cdot\cdot\, \mathbf{I}) \cdot (\boldsymbol{M}_e \cdot\cdot\, \mathbf{I}) \\ & + a_5\,(\mathbf{I} \cdot\cdot\, \boldsymbol{M}_e) \cdot (\mathbf{I} \cdot\cdot\, \boldsymbol{M}_e) + a_6\,\boldsymbol{M}_e \cdots \boldsymbol{M}_e + a_7\,\boldsymbol{M}_e \cdots \boldsymbol{M}_e^{\,t} + c\,\rho\,(\Delta\theta - \theta\,ln\,\frac{\theta}{\theta_0}) \\ & + \Delta\theta\; a_8\,\mathbf{I} \cdot\cdot\, \mathbf{E}_e + \rho\,(\mu - \theta\,\gamma)\,w_{plast} \\ & + b_1\,(\mathbf{E}_p \cdot\cdot\, \mathbf{I})^2 + b_2\,\mathbf{E}_p \cdot\cdot\, \mathbf{E}_p + b_3\,\mathbf{I} \cdot\cdot\, \boldsymbol{M}_p \cdot \boldsymbol{M}_p \cdot\cdot\, \mathbf{I} + b_4\,(\boldsymbol{M}_p \cdot\cdot\, \mathbf{I}) \cdot (\boldsymbol{M}_p \cdot\cdot\, \mathbf{I}) \\ & + b_5\,(\mathbf{I} \cdot\cdot\, \boldsymbol{M}_p) \cdot (\mathbf{I} \cdot\cdot\, \boldsymbol{M}_p) + b_6\,\boldsymbol{M}_p \cdots \boldsymbol{M}_p + b_7\,\boldsymbol{M}_p \cdots \boldsymbol{M}_p^{\,t}\end{aligned} \tag{4.85}$$

where the following particular transposition for a triadic $[\mathbf{M}^t]_{ijk} := M_{kji}$ is used. a_i and b_i are scalar material constants.

This gives the following stresses after (4.63) and (4.64)

$$\mathbf{T} \;=\; 2a_1\,(\mathbf{E}_e \cdot\cdot\, \mathbf{I})\,\mathbf{I} \;+ 2a_2\,\mathbf{E}_e + \Delta\theta\,a_8\,\mathbf{I} \tag{4.86}$$

$$\boldsymbol{G} \;=\; 2a_3\,\mathbf{I} \otimes \boldsymbol{M}_e \cdot\cdot\, \mathbf{I} + 2a_4\,(\boldsymbol{M}_e \cdot\cdot\, \mathbf{I}) \otimes \mathbf{I} + 2a_5\,\mathbf{I} \otimes \mathbf{I} \cdot\cdot\, \boldsymbol{M}_e + 2a_6\,\boldsymbol{M}_e + 2a_7\,\boldsymbol{M}_e^{\,t}. \tag{4.87}$$

The back stresses of (4.68) and (4.69) reduce in the isotropic case to

$$\mathbf{T}_B \;=\; 2d_1\,(\mathbf{E}_p \cdot\cdot\, \mathbf{I})\,\mathbf{I} \;+ 2d_2\,\mathbf{E}_p \tag{4.88}$$

$$\boldsymbol{G}_B \;=\; 2d_3\,\mathbf{I} \otimes \boldsymbol{M}_p \cdot\cdot\, \mathbf{I} + 2d_4\,(\boldsymbol{M}_p \cdot\cdot\, \mathbf{I}) \otimes \mathbf{I} + 2d_5\,\mathbf{I} \otimes \mathbf{I} \cdot\cdot\, \boldsymbol{M}_p + 2d_6\,\boldsymbol{M}_p + 2d_7\,\boldsymbol{M}_p^{\,t} \tag{4.89}$$

with scalar material constants d_i.

The FOURIER law for the heat flux (4.67) obtains the usual form

$$\mathbf{q} \;=\; -\kappa\,\mathbf{g} \tag{4.90}$$

with a non-negative coefficient κ.

For the entropy after (4.65) and (4.83) we obtain

$$\eta \;=\; -\,\frac{1}{\rho}\;a_8\,\mathbf{I} \cdot\cdot\, \mathbf{E}_e + c\;ln\;\frac{\theta}{\theta_0} + \gamma\,w_{plast}\,. \tag{4.91}$$

The v. MISES-type yield criterion (4.73) becomes in the isotropic case

$$\begin{aligned}\varphi \;=\; & g_1\,[(\mathbf{T} - \mathbf{T}_B) \cdot\cdot\, \mathbf{I}]^2 + g_2\,(\mathbf{T} - \mathbf{T}_B) \cdot\cdot\, (\mathbf{T} - \mathbf{T}_B) \\ & + g_3\,\mathbf{I} \cdot\cdot\, (\boldsymbol{G} - \boldsymbol{G}_B) \cdot (\boldsymbol{G} - \boldsymbol{G}_B) \cdot\cdot\, \mathbf{I} + g_4\,[(\boldsymbol{G} - \boldsymbol{G}_B) \cdot\cdot\, \mathbf{I}] \cdot [(\boldsymbol{G} - \boldsymbol{G}_B) \cdot\cdot\, \mathbf{I}] \\ & + g_5\,[\mathbf{I} \cdot\cdot\, (\boldsymbol{G} - \boldsymbol{G}_B)] \cdot [\mathbf{I} \cdot\cdot\, (\boldsymbol{G} - \boldsymbol{G}_B)] + g_6\,(\boldsymbol{G} - \boldsymbol{G}_B) \cdots (\boldsymbol{G} - \boldsymbol{G}_B) \\ & + g_7\,(\boldsymbol{G} - \boldsymbol{G}_B) \cdots (\boldsymbol{G} - \boldsymbol{G}_B)^t - \sigma_Y^2(w_{plast},\,\theta,\,|\,\mathbf{g}\,|)\end{aligned} \tag{4.92}$$

with scalar material constants g_i one of which can be generally normalized to one.

The associated flow rules (4.76) - (4.77) are accordingly

$$\mathbf{E}_p^{\bullet} = \lambda \{g_1 [(\mathbf{T} - \mathbf{T}_B) \cdot\cdot \mathbf{I}] \mathbf{I} + g_2 (\mathbf{T} - \mathbf{T}_B)\} \tag{4.93}$$

$$\begin{aligned} \boldsymbol{M}_p^{\bullet} = \lambda \{&g_3 \mathbf{I} \otimes (\boldsymbol{G} - \boldsymbol{G}_B) \cdot\cdot \mathbf{I} + g_4 [(\boldsymbol{G} - \boldsymbol{G}_B) \cdot\cdot \mathbf{I}] \otimes \mathbf{I} + g_5 \mathbf{I} \otimes \mathbf{I} \cdot\cdot (\boldsymbol{G} - \boldsymbol{G}_B) \\ &+ g_6 \mathbf{K}_e + g_7 (\boldsymbol{G}^t - \boldsymbol{G}_B{}^t)\} \end{aligned} \tag{4.94}$$

where the factor 2 has been drawn into the plastic parameter.

If the flow criterion is density-insensitive then $g_1 = g_2/3$, and $(\mathbf{T} - \mathbf{T}_B)$ is deviatoric, the same as $\mathbf{E}_p$.

Appendix: On Constrained Gradient Elastoplasticity

In the preceding parts we considered *unconstrained gradient elastoplasticity*. Since many authors in the field of gradient materials prefer a constrained format, we want to give some remarks on this class of models.

As in the unconstrained case, we assume an additive decomposition of the strain tensor into an elastic and a plastic part

$$\mathbf{E} = \mathbf{E}_e + \mathbf{E}_p \tag{4.1}$$

and an analogous one of the gradient of the strain tensor

$$\boldsymbol{M} := \mathit{grad}\,\mathbf{E} = \mathbf{E} \otimes \nabla = \boldsymbol{M}_e + \boldsymbol{M}_p \tag{4.2}$$

However, in contrast to the procedure before, we assume the constraint

$$\boldsymbol{M}_p = \mathit{grad}\,\mathbf{E}_p \tag{4A.1}$$

and consequently

$$\boldsymbol{M}_e = \mathit{grad}\,\mathbf{E}_e\,. \tag{4A.2}$$

According to this ansatz, we need only one flow rule

$$\mathbf{E}_p^{\bullet} = \lambda\, E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \tag{4A.3}$$

and a hardening rule

$$\mathbf{Z}_p^{\bullet} = \lambda\, Z(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)\,. \tag{4A.4}$$

This gives after the chain rule for the plastic increment

$$\begin{aligned}
&\boldsymbol{M}_p^{\bullet} = \mathit{grad}\,\mathbf{E}_p^{\bullet} = \mathit{grad}\,[\lambda\, E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)] \\
&= E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \otimes \mathit{grad}\,\lambda \\
&+ \lambda\,[\partial_{\mathbf{E}_e} E \cdot\cdot \mathit{grad}\,\mathbf{E}_e + \partial_{\boldsymbol{M}_e} E \cdots \mathit{grad}\,\boldsymbol{M}_e + \partial_\theta E \otimes \mathbf{g} + \partial_{\mathbf{g}} E \cdot \mathit{grad}\,\mathbf{g} \\
&+ \partial_{\mathbf{E}_p} E \cdot\cdot \mathit{grad}\,\mathbf{E}_p + \partial_{\boldsymbol{M}_p} E \cdots \mathit{grad}\,\boldsymbol{M}_p + \partial_{\mathbf{Z}_p} E \cdot\cdot \mathit{grad}\,\mathbf{Z}_p]\,.
\end{aligned} \tag{4A.5}$$

The consistency condition after (4.72) becomes in this case

$$\begin{aligned}
&0 = \varphi(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)^{\bullet} \\
&= \partial_{\mathbf{E}_e}\varphi \cdot\cdot \mathbf{E}_e^{\bullet} + \partial_{\boldsymbol{M}_e}\varphi \cdots \boldsymbol{M}_e^{\bullet} + \partial_\theta \varphi\, \theta^{\bullet} + \partial_{\mathbf{g}}\varphi \cdot \mathbf{g}^{\bullet} \\
&+ \partial_{\mathbf{E}_p}\varphi \cdot\cdot \mathbf{E}_p^{\bullet} + \partial_{\boldsymbol{M}_p}\varphi \cdots \boldsymbol{M}_p^{\bullet} + \partial_{\mathbf{Z}_p}\varphi \cdot\cdot \mathbf{Z}_p^{\bullet} \\
&= \partial_{\mathbf{E}_e}\varphi \cdot\cdot \mathbf{E}_e^{\bullet} + \partial_{\boldsymbol{M}_e}\varphi \cdots \boldsymbol{M}_e^{\bullet} + \partial_\theta \varphi\, \theta^{\bullet} + \partial_{\mathbf{g}}\varphi \cdot \mathbf{g}^{\bullet} \\
&+ \partial_{\mathbf{E}_p}\varphi \cdot\cdot \lambda\, E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \\
&+ \partial_{\boldsymbol{M}_p}\varphi \cdots \{E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \otimes \mathit{grad}\,\lambda \\
&+ \lambda\,[\partial_{\mathbf{E}_e} E \cdot\cdot \boldsymbol{M}_e + \partial_{\boldsymbol{M}_e} E \cdots \mathit{grad}\,\boldsymbol{M}_e + \partial_\theta E \otimes \mathbf{g} + \partial_{\mathbf{g}} E \cdot \mathit{grad}\,\mathbf{g}
\end{aligned} \tag{4A.6}$$

$$+ \partial_{\mathbf{E}_p} E \cdot\cdot \boldsymbol{M}_p + \partial_{\boldsymbol{M}_p} E \cdots grad\, \boldsymbol{M}_p + \partial_{\mathbf{Z}_p} E \cdot\cdot grad\, \mathbf{Z}_p]\}$$

$$+ \partial_{\mathbf{Z}_p} \varphi \cdot\cdot \lambda\, Z(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) .$$

This is no longer a linear equation for the plastic parameter λ, which can be easily solved like (4.59), but a PDE instead.

The CLAUSIUS-DUHEM inequality gives with a free energy (4.38)

$$0 \geq \psi^{\bullet} + \eta\, \theta^{\bullet} - \frac{\mathbf{T}}{\rho} \cdot\cdot \mathbf{E}^{\bullet} - \frac{1}{\rho} \boldsymbol{G} \cdots \boldsymbol{M}^{\bullet} + \frac{1}{\rho\, \theta} \mathbf{q} \cdot \mathbf{g}$$

$$= \partial_{\mathbf{E}_e} \psi \cdot\cdot \mathbf{E}_e^{\bullet} + \partial_{\boldsymbol{M}_e} \psi \cdots \boldsymbol{M}_e^{\bullet} + \partial_{\theta} \psi\, \theta^{\bullet} + \partial_{\mathbf{g}} \psi \cdot \mathbf{g}^{\bullet} + \partial_{\mathbf{E}_p} \psi \cdot\cdot \mathbf{E}_p^{\bullet} + \partial_{\boldsymbol{M}_p} \psi \cdots \boldsymbol{M}_p^{\bullet}$$

(4A.7) $$+ \partial_{\mathbf{Z}_p} \psi \cdot\cdot \mathbf{Z}_p^{\bullet} + \eta\, \theta^{\bullet} - \frac{\mathbf{T}}{\rho} \cdot\cdot \mathbf{E}^{\bullet} - \frac{1}{\rho} \boldsymbol{G} \cdots grad\, \mathbf{E}^{\bullet} + \frac{1}{\rho\, \theta} \mathbf{q} \cdot \mathbf{g}$$

$$= \partial_{\mathbf{E}_e} \psi \cdot\cdot (\mathbf{E}^{\bullet} - \mathbf{E}_p^{\bullet}) + \partial_{\boldsymbol{M}_e} \psi \cdots (grad\, \mathbf{E}^{\bullet} - grad\, \mathbf{E}_p^{\bullet}) + \partial_{\theta} \psi\, \theta^{\bullet} + \partial_{\mathbf{g}} \psi \cdot \mathbf{g}^{\bullet}$$

$$+ \partial_{\mathbf{E}_p} \psi \cdot\cdot \mathbf{E}_p^{\bullet} + \partial_{\boldsymbol{M}_p} \psi \cdots \boldsymbol{M}_p^{\bullet}$$

$$+ \partial_{\mathbf{Z}_p} \psi \cdot\cdot \mathbf{Z}_p^{\bullet} + \eta\, \theta^{\bullet} - \frac{\mathbf{T}}{\rho} \cdot\cdot \mathbf{E}^{\bullet} - \frac{1}{\rho} \boldsymbol{G} \cdots grad\, \mathbf{E}^{\bullet} + \frac{1}{\rho\, \theta} \mathbf{q} \cdot \mathbf{g}$$

with (4A.5) $$= (\partial_{\mathbf{E}_e} \psi - \frac{\mathbf{T}}{\rho}) \cdot\cdot \mathbf{E}^{\bullet} + (\partial_{\boldsymbol{M}_e} \psi - \frac{1}{\rho} \boldsymbol{G}) \cdots grad\, \mathbf{E}^{\bullet} + (\partial_{\theta} \psi + \eta)\, \theta^{\bullet} + \partial_{\mathbf{g}} \psi \cdot \mathbf{g}^{\bullet}$$

$$+ (\partial_{\mathbf{E}_p} \psi - \partial_{\mathbf{E}_e} \psi) \cdot\cdot \mathbf{E}_p^{\bullet}$$

$$+ (\partial_{\boldsymbol{M}_p} \psi - \partial_{\boldsymbol{M}_e} \psi) \cdots \{E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \otimes grad\, \lambda$$

$$+ \lambda\, [\partial_{\mathbf{E}_e} E \cdot\cdot grad\, \mathbf{E}_e + \partial_{\boldsymbol{M}_e} E \cdots grad\, \boldsymbol{M}_e + \partial_{\theta} E \otimes \mathbf{g} + \partial_{\mathbf{g}} E \cdot grad\, \mathbf{g}$$

$$+ \partial_{\mathbf{E}_p} E \cdot\cdot grad\, \mathbf{E}_p + \partial_{\boldsymbol{M}_p} E \cdots grad\, \boldsymbol{M}_p + \partial_{\mathbf{Z}_p} E \cdot\cdot grad\, \mathbf{Z}_p]\}$$

$$+ \partial_{\mathbf{Z}_p} \psi \cdot\cdot \mathbf{Z}_p^{\bullet} + \frac{1}{\rho\, \theta} \mathbf{q} \cdot \mathbf{g} .$$

If we first consider elastic events, then there remains only the inequality

(4A.8) $$0 \geq (\partial_{\mathbf{E}_e} \psi - \frac{\mathbf{T}}{\rho}) \cdot\cdot \mathbf{E}^{\bullet} + (\partial_{\boldsymbol{M}_e} \psi - \frac{1}{\rho} \boldsymbol{G}) \cdots grad\, \mathbf{E}^{\bullet} + (\partial_{\theta} \psi + \eta)\, \theta^{\bullet}$$

$$+ \partial_{\mathbf{g}} \psi \cdot \mathbf{g}^{\bullet} + \frac{1}{\rho\, \theta} \mathbf{q} \cdot \mathbf{g} .$$

The exploitation of this inequality for states below the yield limit, where $\mathbf{E}^{\bullet}$, $grad\, \mathbf{E}^{\bullet}$, and $\theta^{\bullet}$ can locally be varied independently, leads by standard arguments to the conditions from gradient thermoelasticity, namely the independence of the free energy of the temperature gradient

(4A.9) $$\psi = \psi(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$$

instead of (4.38), and the *thermoelastic potentials*

(4A.10) $$\mathbf{T} = \rho\, \partial_{\mathbf{E}_e} \psi = T(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$$

(4A.11) $$\boldsymbol{G} = \rho\, \partial_{\boldsymbol{M}_e} \psi = G(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$$

(4A.12) $$\eta = -\partial_\theta \psi = \eta(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$$

as well as the *heat conduction inequality*

(4A.13) $$-\mathbf{q} \cdot \mathbf{g} \geq 0$$

as necessary and sufficient conditions for the second law to hold during elastic events.

Because of continuity, restrictions (4A.10) - (4A.13) must also hold when reaching the yield limit. If yielding occurs, however, then the additional terms of the CLAUSIUS-DUHEM inequality (4A.7) must fulfill the **residual inequality**

$$0 \geq (\partial_{\mathbf{E}_p} \psi - \frac{\mathbf{T}}{\rho}) \cdot\cdot\, \mathbf{E}_p^{\bullet} + (\partial_{\boldsymbol{M}_p} \psi - \frac{1}{\rho} \boldsymbol{G}) \cdots grad\, \mathbf{E}_p^{\bullet} - \partial_{\mathbf{Z}_p} \psi \cdot\cdot\, \mathbf{Z}_p^{\bullet} .$$

$$= (\partial_{\mathbf{E}_p} \psi - \frac{\mathbf{T}}{\rho}) \cdot\cdot\, \lambda\, E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p)$$

(4A.14) $$+ (\partial_{\boldsymbol{M}_p} \psi - \frac{1}{\rho} \boldsymbol{G}) \cdots \{E(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) \otimes grad\, \lambda$$

$$+ \lambda\, [\partial_{\mathbf{E}_e} E \cdot\cdot\, grad\, \mathbf{E}_e + \partial_{\boldsymbol{M}_e} E \cdots grad\, \boldsymbol{M}_e + \partial_\theta E \otimes \mathbf{g} + \partial_{\mathbf{g}} E \cdot grad\, \mathbf{g}$$

$$+ \partial_{\mathbf{E}_p} E \cdot\cdot\, grad\, \mathbf{E}_p + \partial_{\boldsymbol{M}_p} E \cdots grad\, \boldsymbol{M}_p + \partial_{\mathbf{Z}_p} E \cdot\cdot\, grad\, \mathbf{Z}_p]\}$$

$$- \lambda\, \partial_{\mathbf{Z}_p} \psi \cdot\cdot\, Z(\mathbf{E}_e, \boldsymbol{M}_e, \theta, \mathbf{g}, \mathbf{E}_p, \boldsymbol{M}_p, \mathbf{Z}_p) .$$

using (4A.4) and (4A.5). This is also a PDE and can only be evaluated as a result of the boundary value problem.

We must conclude that the *constraint gradient plasticity* leads to complicated PDEs for both the consistency condition and for the CLAUSIUS-DUHEM inequality which can be hardly exploited in general, in contrast to the unconstrained case.

5. On Isotropic Stiffness Hexadics

The following chapter is based on the article

> R. Glüge, J. Kalisch, A. Bertram: *The eigenmodes in isotropic strain gradient elasticity*, to appear in *Generalized Continua as Models for Classical and Advanced Materials*. Edts. H. Altenbach, S. Forest. Springer-Verlag (in press)

It aims at investigating the stiffness tensors for linear isotropic second gradient elasticity. In doing so, we refer to the notations introduced in Chapter 3, specified for the case $N \equiv 2$.

In particular, we use the following set of kinematical variables

$$\mathbf{E} := sym\ grad\ \mathbf{u}$$

$$\overset{(2)}{\boldsymbol{U}} := grad\ grad\ \mathbf{u}$$

with the following symmetry properties by definition

$$E_{ij} = E_{ji} \qquad \text{and} \qquad U_{ijk} = U_{ikj} . \tag{5.1}$$

The work-conjugate stress tensors $\mathbf{T}$ and $\overset{(2)}{\boldsymbol{T}}$ are assumed to show the same symmetries.

According to the notations of Chapter 3, the linear elastic law within this format is constituted by the following tensors: $\overset{(1,1)}{\boldsymbol{C}}, \overset{(1,2)}{\boldsymbol{C}}, \overset{(2,1)}{\boldsymbol{C}}, \overset{(2,2)}{\boldsymbol{C}}$ such that

$$\begin{aligned} \mathbf{T} &= \overset{(1,1)}{\boldsymbol{C}} \cdot\cdot\, \mathbf{E} + \overset{(1,2)}{\boldsymbol{C}} \cdots \overset{(2)}{\boldsymbol{U}} \\ \overset{(2)}{\boldsymbol{T}} &= \overset{(2,1)}{\boldsymbol{C}} \cdot\cdot\, \mathbf{E} + \overset{(2,2)}{\boldsymbol{C}} \cdots \overset{(2)}{\boldsymbol{U}} . \end{aligned} \tag{5.2}$$

The stiffness tensors inherit the following subsymmetries from the above variables

$$\begin{aligned} \overset{(1,1)}{C}_{ijkl} &= \overset{(1,1)}{C}_{jikl} = \overset{(1,1)}{C}_{ijlk} \\ \overset{(1,2)}{C}_{ijklm} &= \overset{(1,2)}{C}_{jiklm} = \overset{(1,2)}{C}_{ijkml} \\ \overset{(2,1)}{C}_{ijklm} &= \overset{(2,1)}{C}_{ikjlm} = \overset{(2,1)}{C}_{ijkml} \\ \overset{(2,2)}{C}_{ijklmn} &= \overset{(2,2)}{C}_{ikjlmn} = \overset{(2,2)}{C}_{ijklnm} . \end{aligned} \tag{5.3}$$

The material is hyperelastic, if an elastic energy exists of the following form as a special case of (3.13)

$$w(\mathbf{E}, \overset{(2)}{\boldsymbol{U}}) = \tfrac{1}{2}\, \mathbf{E} \cdot\cdot \overset{(1,1)}{\boldsymbol{C}} \cdot\cdot\, \mathbf{E} + \tfrac{1}{2}\, \overset{(2)}{\boldsymbol{U}} \cdots \overset{(2,2)}{\boldsymbol{C}} \cdots \overset{(2)}{\boldsymbol{U}} + \mathbf{E} \cdot\cdot \overset{(1,2)}{\boldsymbol{C}} \cdots \overset{(2)}{\boldsymbol{U}} \tag{5.4}$$

so we can assume the additional symmetries

$$\begin{aligned} \overset{(1,1)}{C}_{ijkl} &= \overset{(1,1)}{C}_{klij} \\ \overset{(2,2)}{C}_{ijklmn} &= \overset{(2,2)}{C}_{lmnijk} \end{aligned} \tag{5.5}$$

while $\overset{(1,2)}{\boldsymbol{C}}$ is completely determined by $\overset{(2,1)}{\boldsymbol{C}}$ according to (3.16), and *vice versa*.

Moreover, if the material is centro-symmetric, $\overset{(1,2)}{\boldsymbol{C}}$ and $\overset{(2,1)}{\boldsymbol{C}}$ vanish because of (3.29).

If we restrict our concern in this chapter to the centro-symmetric isotropic case, the tensors $\overset{(1,1)}{\boldsymbol{C}}$ and $\overset{(2,2)}{\boldsymbol{C}}$ must obey

$$\mathbf{Q} * \overset{(1,1)}{\boldsymbol{C}} = \overset{(1,1)}{\boldsymbol{C}} \qquad \text{and} \qquad \mathbf{Q} * \overset{(2,2)}{\boldsymbol{C}} = \overset{(2,2)}{\boldsymbol{C}} \tag{5.6}$$

for all orthogonal symmetry transformations $\mathbf{Q}$ after (3.27).

The representation of an isotropic tetradic like $\overset{(1,1)}{\boldsymbol{C}}$ is in principle known since two centuries and enjoys a clear physical interpretation. For isotropic hexadics like $\overset{(2,2)}{\boldsymbol{C}}$ we find them in MINDLIN/ ESHEL (1968), see (4.84). However, a physical interpretation for the latter is still lacking. In what follows we want to further the understanding and knowledge of such isotropic hexadics.

An isotropic hexadic can be represented by

$$\overset{(2,2)}{\boldsymbol{C}} = \sum_{i=1}^{5} c_i \, \boldsymbol{B}_i \tag{5.7}$$

as a linear combination of five base hexadics with components

$$\begin{aligned}
B_{1\,ijklmn} &= \delta_{jk}\,\delta_{im}\,\delta_{nl} + \delta_{\,jk}\,\delta_{in}\,\delta_{ml} + \delta_{ji}\,\delta_{kl}\,\delta_{mn} + \delta_{jl}\,\delta_{ik}\,\delta_{mn} \\
B_{2\,ijklmn} &= \delta_{ji}\,\delta_{km}\,\delta_{nl} + \delta_{jm}\,\delta_{ki}\,\delta_{nl} + \delta_{ji}\,\delta_{kn}\,\delta_{ml} + \delta_{jn}\,\delta_{ik}\,\delta_{ml} \\
B_{3\,ijklmn} &= \delta_{jm}\,\delta_{kl}\,\delta_{in} + \delta_{jl}\,\delta_{in}\,\delta_{km} + \delta_{jn}\,\delta_{im}\,\delta_{kl} + \delta_{jl}\,\delta_{im}\,\delta_{nk} \\
B_{4\,ijklmn} &= \delta_{jn}\,\delta_{il}\,\delta_{km} + \delta_{jm}\,\delta_{kn}\,\delta_{il} \\
B_{5\,ijklmn} &= \delta_{jk}\,\delta_{il}\,\delta_{mn}
\end{aligned} \tag{5.8}$$

with respect to any orthonormal basis $\{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_m \otimes \mathbf{e}_n\}$.

The metric of $\{\boldsymbol{B}_i\}$ is

$$\boldsymbol{B}_i \cdots \cdots \boldsymbol{B}_j = \begin{bmatrix} 168 & 96 & 96 & 24 & 36 \\ 96 & 192 & 72 & 48 & 12 \\ 96 & 72 & 192 & 48 & 12 \\ 24 & 48 & 48 & 72 & 18 \\ 36 & 12 & 12 & 18 & 27 \end{bmatrix} \tag{5.9}$$

We observe that

- $\frac{1}{2}\,\boldsymbol{B}_4$ maps every subsymmetric triadic onto itself
- $\frac{1}{3}\,\boldsymbol{B}_5$ maps every tensor of the form $\mathbf{v} \otimes \mathbf{I}$ onto itself
- $\frac{1}{8}\,\boldsymbol{B}_2$ maps every tensor of the form $\mathbf{I} \otimes \mathbf{v}$ onto $sym_{23}\,(\mathbf{I} \otimes \mathbf{v})$, *i.e.*, its right subsymmetric part.

Before turning to the spectral decomposition, a more suitable basis is introduced by

$$\underline{\boldsymbol{B}}_1 := -\frac{1}{15}(\boldsymbol{B}_1 + \boldsymbol{B}_2 + \boldsymbol{B}_5) + \frac{1}{6}(\boldsymbol{B}_3 + \boldsymbol{B}_4)$$

$$\underline{\boldsymbol{B}}_2 := {}^1\!/_{12}\,(2\boldsymbol{B}_1 - \boldsymbol{B}_2 - 2\boldsymbol{B}_3 + 4\boldsymbol{B}_4 - 4\boldsymbol{B}_5)$$

(5.10)
$$\underline{\boldsymbol{B}}_3 := {}^1\!/_{60}\,(6\boldsymbol{B}_1 - 9\boldsymbol{B}_2 + 16\boldsymbol{B}_5)$$

$$\underline{\boldsymbol{B}}_4 := \frac{1}{6\sqrt{5}}\,(3\boldsymbol{B}_1 - 4\boldsymbol{B}_5)$$

$$\underline{\boldsymbol{B}}_5 := {}^1\!/_{20}\,(-2\boldsymbol{B}_1 + 3\boldsymbol{B}_2 + 8\boldsymbol{B}_5)\,.$$

The metric of this basis is diagonal with

(5.11)
$$\boldsymbol{B}_i \cdots \cdots \boldsymbol{B}_j = \{7, 5, 6, 6, 6\}.$$

The components of $\overset{(2,2)}{\boldsymbol{C}}$ with respect to this base are

$$\underline{c}_1 = 2\,(c_4 - c_3)$$

$$\underline{c}_2 = 4\,c_3 + 2\,c_4$$

(5.12)
$$\underline{c}_3 = {}^1\!/_6\,(12\,c_1 - 16\,c_2 + 2\,c_3 + 9\,c_5)$$

$$\underline{c}_4 = \frac{2\sqrt{5}}{3}\,(3c_1 + 2\,c_2 + 2\,c_3)$$

$$\underline{c}_5 = {}^1\!/_2\,(4\,c_1 + 8\,c_2 + 2\,c_3 + 4\,c_4 + 3\,c_5)\,.$$

The spectral representation of $\overset{(2,2)}{\boldsymbol{C}}$ can be expressed with respect to this basis as

(5.13)
$$\overset{(2,2)}{\boldsymbol{C}} = \sum_{i=1}^{4} \lambda_i \boldsymbol{P}_i$$

with the eigenvalues

(5.14)
$$\begin{aligned} \lambda_1 &= \underline{c}_1 & c_r &= {}^1\!/_2\,(\lambda_3 - \lambda_4) \\ \lambda_2 &= \underline{c}_2 & \underline{c}_3 &= c_r \cos\kappa \\ \lambda_3 &= \underline{c}_5 + c_r & \underline{c}_4 &= c_r \sin\kappa \\ \lambda_4 &= \underline{c}_5 - c_r & \underline{c}_5 &= {}^1\!/_2\,(\lambda_3 + \lambda_4) \end{aligned}$$

with $c_r = \sqrt{\underline{c}_3^2 + \underline{c}_4^2}$ and the eigenprojectors

$$\boldsymbol{P}_1 = \underline{\boldsymbol{B}}_1$$

$$\boldsymbol{P}_2 = \underline{\boldsymbol{B}}_2$$

(5.15)
$$\boldsymbol{P}_3 = {}^1\!/_2\,(\underline{\boldsymbol{B}}_5 + \frac{\underline{c}_3}{c_r}\underline{\boldsymbol{B}}_3 + \frac{\underline{c}_4}{c_r}\,\underline{\boldsymbol{B}}_4)$$

$$\boldsymbol{P}_4 = {}^1\!/_2\,(\underline{\boldsymbol{B}}_5 - \frac{\underline{c}_3}{c_r}\underline{\boldsymbol{B}}_3 - \frac{\underline{c}_4}{c_r}\,\underline{\boldsymbol{B}}_4)\,.$$

We introduce a dimensionless parameter κ by

$$\cos\kappa = \frac{c_3}{c_r} \quad \Leftrightarrow \quad \sin\kappa = \frac{c_4}{c_r} \tag{5.16}$$

which allows us to use the four eigenvalues $\lambda_1, \ldots, \lambda_4$ and this parameter as the five independent material constants, where κ determines the last two eigenprojectors. One can express the dependence of the projectors and eigenvalues on this angle like

$$\begin{aligned} \boldsymbol{P}_3(\kappa) &= \boldsymbol{P}_4(\kappa+\pi) \\ \lambda_3(\kappa) &= \lambda_4(\kappa+\pi) \end{aligned} \tag{5.17}$$

so that it is reasonable to restrict κ to the interval $[0, \pi)$. The metric of this basis is diagonal with

$$\boldsymbol{P}_i \cdots \boldsymbol{P}_j = \{7, 5, 3, 3\}, \tag{5.18}$$

i.e., the multiplicities of the eigenvalues are *5*, *7*, *3*, and *3*. The projectors fulfill the usual conditions like the bi-orthogonality

$$\boldsymbol{P}_i \cdots \boldsymbol{P}_j = \delta_{ij} \boldsymbol{P}_i \tag{5.19}$$

and the completeness since $\sum_{i=1}^{4} \boldsymbol{P}_i$ gives the sixth-order identity on triadics with right subsymmetry. These equations resemble those of the spectral decomposition of a transversely isotropic stiffness tetradic, which also contains in general five independent components and four distinct eigenvalues[37].

The previous formulae are convenient if one knows the parameters $c_1, \ldots, c_5$ and wants to determine the eigenvalues and the third and fourth eigenprojector. The inversion of these equations is

$$\begin{aligned} c_1 &= [10\,\lambda_1 - 4\,\lambda_2 - 3\,(\lambda_3+\lambda_4) + 3\,(\lambda_3-\lambda_4)\,(\cos\kappa + \sqrt{5}\sin\kappa)]/60 \\ c_2 &= [-10\,\lambda_1 - 8\,\lambda_2 + 3\,(\lambda_3+\lambda_4) - 3\,(\lambda_3-\lambda_4)\cos\kappa]/120 \\ c_3 &= (\lambda_2 - \lambda_1)/6 \\ c_4 &= (2\lambda_1 + \lambda_2)/6 \\ c_5 &= [-5\,\lambda_1 - \lambda_2 + 3\,(\lambda_3+\lambda_4) + (\lambda_3-\lambda_4)\,(2\cos\kappa + \sqrt{5}\sin\kappa)]/15. \end{aligned} \tag{5.20}$$

Harmonic Decomposition

These results become clearer from the point of view of the harmonic decomposition of a third-order tensor with one subsymmetry[38]. The projectors, or more precisely, the parameter κ distinguishes a specific decomposition of the first-order harmonic contribution, which is discussed next[39].

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[37] see Appendix A of KALISCH/ GLÜGE (2015).
[38] see OLIVE/ AUFFRAY (2014), ZHENG/ ZOU (2000).
[39] see GOLUBITSKY/ STEWART/ SCHAEFFER (1988),

We will denote the set of all triadics with right subsymmetry by $\mathscr{Triad}$. In this space live $\overset{(2)}{\boldsymbol{T}}$ and $\overset{(2)}{\boldsymbol{U}}$. By virtue of the harmonic decomposition, a triadic $\boldsymbol{H}$ is decomposed into a sum of mutually orthogonal tensors

$$\boldsymbol{H} = \sum_{i=1}^{N} \boldsymbol{H}_i \qquad \text{with} \quad \boldsymbol{H}_i \cdots \boldsymbol{H}_j = 0 \ \text{ for } \ i \neq j \tag{5.21}$$

which can correspond to the eigentensors of $\overset{(2,2)}{\boldsymbol{C}}$. Here N is the number of different eigenvalues. Each $\boldsymbol{H}_i$ is related to a different harmonic tensor $\boldsymbol{H}^{\boldsymbol{n}}{}_i$ by an isotropic linear mapping $\overset{\langle 3+n \rangle}{\boldsymbol{L}}_i$ so that

$$\boldsymbol{H}_i = \overset{\langle 3+n \rangle}{\boldsymbol{L}}_i \cdot\ldots\cdot \boldsymbol{H}^{\boldsymbol{n}}{}_i \tag{5.22}$$

with an n-fold contraction. The order n of the harmonic tensors does not exceed that of the decomposed tensor. The harmonic tensor spaces are denoted by $\mathscr{H}_i$ with dimension $2i+1$ due to the fact that harmonic tensors are completely symmetric and traceless, *i.e.*, zero for all possible index contractions.

$\mathscr{Triad}$ is decomposed into the direct sum ($\oplus$) of mutually orthogonal subspaces $\mathscr{H}_i$. These subspaces are closed under the action of the RAYLEIGH product (0.4) with an element of $\mathscr{Orth}^+$, *i.e.*,

$$\boldsymbol{H} \in \mathscr{H}_i \qquad \Leftrightarrow \qquad \mathbf{Q} * \boldsymbol{H} \in \mathscr{H}_i .$$

A further decomposition without loss of this property is not possible. For this reason this decomposition is irreducible.

The harmonic decomposition can be thought of as the diagonalization of a matrix. The matrix originates from the action of the group $\mathscr{Orth}^+$ on the tensor space as rotations by means of the RAYLEIGH product. Subspaces for harmonic spaces of equal order form block matrices on the main diagonal, unless we define additional orthogonal decompositions.

The respective tensor $\boldsymbol{H}$ can be represented by a linear combination of products of the form

$$\boldsymbol{H} = \sum_{i=1}^{3} \sum_{j=1}^{6} C_{ij}\, \mathbf{e}_i \otimes \mathbf{E}_j \tag{5.23}$$

with two orthonormal bases $\{\mathbf{e}_i\}$ and $\{\mathbf{E}_i\}$ in the three-dimensional EUCLIDean space and the space of symmetric second-order tensors, respectively. The harmonic decomposition of these spaces is given by $\mathscr{H}_1$ and $\mathscr{H}_0 \oplus \mathscr{H}_2$, respectively. The three-dimensional space cannot be decomposed into harmonic subspaces, hence it is represented by the three-dimensional space $\mathscr{H}_1$. The six-dimensional space of symmetric second order tensors is decomposed into the well-known spherical and deviatoric symmetric parts, the first of which is one-dimensional and corresponds to $\mathscr{H}_0$, while the second one is five-dimensional and corresponds to the symmetric and traceless second-order tensors of $\mathscr{H}_2$.

Similar to the decomposition (5.23), one can construct the space $\mathscr{Triad}$ as the dyadic product of the form

$$\mathcal{T}_{triad} = \mathcal{H}_1 \otimes (\mathcal{H}_0 \oplus \mathcal{H}_2) \,. \tag{5.24}$$

With the CLEBSCH-GORDAN rule

$$\mathcal{H}_m \otimes \mathcal{H}_n = \bigoplus_{k=|m-n|}^{m+n} \mathcal{H}_k \tag{5.25}$$

we achieve

$$\begin{aligned} \mathcal{T}_{triad} &= \mathcal{H}_1 \otimes (\mathcal{H}_0 \oplus \mathcal{H}_2) \\ &= (\mathcal{H}_1 \otimes \mathcal{H}_0) \oplus (\mathcal{H}_1 \otimes \mathcal{H}_2) \\ &= \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \\ &= \mathcal{H}_3 \oplus \mathcal{H}_2 \oplus \mathcal{H}_1^{\oplus 2} . \end{aligned} \tag{5.26}$$

Thus we get two three-dimensional, one five-dimensional, and one seven-dimensional subspace, altogether forming the 18-dimensional space $\mathcal{T}_{triad}$ of third-order tensors with right subsymmetry.

The harmonic decomposition is unique regarding the number and the dimensionality of the subspaces. However, when two subspaces of equal dimension appear, then there is an arbitrariness in the isomorphisms that connect $\boldsymbol{H}_i$ and $\boldsymbol{H}^n{}_i$ after (5.22). In our representation, this arbitrariness corresponds to the angle κ that determines the direction of the two eigenprojectors $\boldsymbol{P}_3$ and $\boldsymbol{P}_4$ of the eigenvalues λ_3 and λ_4, each having the multiplicity of three. The relations (5.22) are here

$$\begin{aligned} \boldsymbol{H}_1 &= \boldsymbol{H}^3{}_1 \\ \boldsymbol{H}_2 &= \boldsymbol{\mathcal{E}} \cdot \boldsymbol{H}^2{}_2 \quad \text{with the permutation triadic } \boldsymbol{\mathcal{E}} \\ \boldsymbol{H}_3 &= \boldsymbol{H}^1{}_3 \cdot [\cos(\kappa/2)\, \boldsymbol{P}_{4/1} + \sqrt{5}\, \sin(\kappa/2)\, \boldsymbol{P}_{4/2}] \\ \boldsymbol{H}_4 &= \boldsymbol{H}^1{}_4 \cdot [-\sin(\kappa/2)\, \boldsymbol{P}_{4/1} + \sqrt{5}\, \cos(\kappa/2)\, \boldsymbol{P}_{4/2}] \end{aligned} \tag{5.27}$$

where $\boldsymbol{H}^3{}_1$, $\boldsymbol{H}^2{}_2$, $\boldsymbol{H}^1{}_3$, and $\boldsymbol{H}^1{}_4$ denote the completely traceless and symmetric tensors of dimension 7, 5, 3, and 3, and $\boldsymbol{H}_i$ the eigentensors of $\overset{(2,2)}{\boldsymbol{C}}$. Further, $\boldsymbol{P}_{4/1,2}$ are the isotropic projectors from the spectral decomposition of isotropic stiffness tetradics with the compression modulus K and shear modulus G

$$\overset{(1,1)}{\boldsymbol{C}} = 3K\, \boldsymbol{P}_{4/1} + 2G\, \boldsymbol{P}_{4/2} \tag{5.28}$$

with the isotropic fourth-order eigenprojectors $\boldsymbol{P}_{4/1} = {}^1\!/_3\ \mathbf{I} \otimes \mathbf{I}$ and $\boldsymbol{P}_{4/2} = (\boldsymbol{I}^S - {}^1\!/_3\, \mathbf{I} \otimes \mathbf{I})$ with the fourth-order symmetrizer $\boldsymbol{I}^S := \frac{1}{2}\, (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i) \otimes (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i)$.

The 7-dimensional eigenspace $\mathcal{H}_1$

A displacement field $\mathbf{u}$ associated with the third-order harmonic tensor $\overset{(2)}{\boldsymbol{U}}$ has the following properties

$$u_{i\,,jj} = 0 \quad \Leftrightarrow \quad \Delta\,\mathbf{u} = div\,grad\,\mathbf{u} = \mathbf{o}$$

(5.29) $$u_{i\,,ik} = 0 \quad \Leftrightarrow \quad grad\,div\,\mathbf{u} = \mathbf{o}$$

$$u_{i\,,jk} = u_{j\,,ik} \Leftrightarrow \quad grad\,rot\,\mathbf{u} = \mathbf{0}\ .$$

Here Δ denotes the LAPLACE operator. Thus, $\mathbf{u}$ is a harmonic function, and the volumetric strain must be homogeneous. The HELMHOLTZ representation theorem tells us that there is a scalar field ϕ and a vector field $\mathbf{a}$ that is divergence free (COULOMB´s gauge) such that

(5.30) $$\mathbf{u} = \nabla\phi + \nabla \times \mathbf{a} \quad \text{and} \quad div\,\mathbf{a} = 0$$

hold. Using $u_{i\,,i} = div\,\mathbf{u} = tr\,\mathbf{H} = \Delta\,\phi$, we find with (5.29.2)

(5.31) $$grad\,\Delta\,\phi = \Delta\,grad\,\phi = \mathbf{o}$$

since for sufficiently smooth fields the LAPLACEan and the gradient operation commute. (5.29.3) is equivalent to

(5.32) $$grad\,rot\,\mathbf{u} = -(\mathbf{u} \times \nabla) \otimes \nabla = \mathbf{0}\ .$$

So the rotational part of $\mathbf{u}$ is also homogeneous. The HELMHOLTZ representation and COULOMB´s condition imply

(5.33) $$(\Delta\,\mathbf{a}) \otimes \nabla = \Delta\,grad\,\mathbf{a} = \mathbf{0}\ .$$

Since the LAPLACEan and the gradient operation commute, $\Delta\,\phi = u_{i\,,i}$ and $\Delta\,\mathbf{a}$ are homogeneous (5.31), (5.33), and $\nabla\phi$ and $\mathbf{a} \otimes \nabla$ are harmonic functions due to (5.31) and (5.33).

In conclusion, the displacement fields that generate the eigenstrain gradients in $\mathscr{H}_1$

- are free from volumetric strain gradients (5.29.2),
- their components have zero mean curvature (5.29.1),
- the gradient of the axial vector $\mathbf{u} \times \nabla$ vanishes (5.33), *i.e.*, the rotational part of the displacement field is homogeneous.

The 5-dimensional eigenspace $\mathscr{H}_2$

For convenience we drop the indices of the components of $\boldsymbol{H}^2{}_1$ in what follows. With respect to an ONB we get

(5.34) $$u_{i\,,jk} = \tfrac{1}{2}\,(\varepsilon_{ijl}H_{lk} + \varepsilon_{ikl}H_{lj})$$

where u_i is the component of the displacement field that produces only strain gradients in the 5-dimensional eigenspace that is isomorphic to $\mathscr{H}_2$.

We cannot directly transfer the traceless and symmetric properties of $\boldsymbol{H}^2{}_1$ to the displacement gradient, since a summation index is involved in $\boldsymbol{H}^2{}_1$ but not in $u_{i\,,jk}$. Taking the two independent traces $u_{i\,,jk}$ gives

(5.35) $$u_{i\,,jj} = \tfrac{1}{2}\,(\varepsilon_{ijl}H_{lj} + \varepsilon_{ijl}H_{lj}) = \varepsilon_{ijl}H_{lj} = 0 \quad \Leftrightarrow \quad axi\,skw\,\boldsymbol{H}^2{}_1 = \mathbf{o}$$

$$u_{j\,,jk} = \tfrac{1}{2}\,(\varepsilon_{jjl}H_{lk} + \varepsilon_{jkl}H_{lj}) = \tfrac{1}{2}\,\varepsilon_{ijl}H_{lj} = 0 \quad \Leftrightarrow \quad axi\,skw\,\boldsymbol{H}^2{}_1 = \mathbf{o}\ .$$

so that both conditions bear the same information.

The skew part of $\boldsymbol{H}^2{}_2$ (and hence the axial vector of it) is zero by definition. Thus we find that the eigenstrain gradients of the 5-dimensional eigenspace belong to harmonic displacement fields without volumetric strain gradient, as in the case before.

Now we consider

$$
\begin{aligned}
& 2\,(\varepsilon_{nij}\,u_{i\,,j})\,,_k \;=\; 2\;\varepsilon_{nij}\,u_{i\,,jk} = \varepsilon_{nij}\,(u_{i\,,jk} - u_{j\,,ik}) \\
& = \varepsilon_{nij}\;\tfrac{1}{2}\,(2\varepsilon_{ijm}\,H_{mk} + \varepsilon_{ikm}\,H_{mj} - \varepsilon_{jkm}\,H_{mi}) \\
& = \tfrac{1}{2}\,(2\varepsilon_{ijn}\,\varepsilon_{ijm}\,H_{mk} + \varepsilon_{ijn}\,\varepsilon_{ikm}\,H_{mj} - \varepsilon_{jni}\,\varepsilon_{jkm}\,H_{mi}) \\
& = 2\,\delta_{nm}\,H_{mk} + \tfrac{1}{2}\,[(\delta_{jk}\,\delta_{nm} - \delta_{jm}\,\delta_{nk})\,H_{mj} - (\delta_{nk}\,\delta_{im} - \delta_{nm}\,\delta_{ik})\,H_{mi}\,] \\
& = 2\,H_{nk} + \tfrac{1}{2}\,(H_{nk} - \delta_{nk}\,H_{mm} - \delta_{nk}\,H_{mm} + H_{nk}) \\
& = 3\,H_{nk}\,.
\end{aligned}
\tag{5.36}
$$

In symbolic notation we thus have

$$
\boldsymbol{H}^2{}_1 \;\propto\; (\mathbf{u} \times \nabla) \otimes \nabla \;=\; (-\Delta\,\mathbf{a}) \otimes \nabla \;=\; -\Delta\,(\mathbf{a} \otimes \nabla)\,. \tag{5.37}
$$

$\boldsymbol{H}^2{}_1$ is symmetric and deviatoric. The latter property is in accordance with COULOMB´s condition on $\mathbf{a}$. The symmetry of $\boldsymbol{H}^2{}_1$ implies another constraint on $\mathbf{a}$

$$
\Delta\,(\mathbf{a} \otimes \nabla) \;=\; \Delta\,(\nabla \otimes \mathbf{a}) \tag{5.38}
$$

$$
\Leftrightarrow \qquad \mathbf{0} \;=\; \Delta\,(\mathbf{a} \otimes \nabla - \nabla \otimes \mathbf{a})
$$

$$
= \Delta\,\boldsymbol{\mathcal{E}} \cdot (\mathbf{a} \times \nabla) \tag{5.39}
$$

$$
\Leftrightarrow \qquad \mathbf{o} \;=\; \Delta\,(\mathbf{a} \times \nabla) \;=\; (\Delta\,\mathbf{a}) \times \nabla\,. \tag{5.40}
$$

The divergence of (5.38) provides by means of the COULOMB condition

$$
\begin{aligned}
& \mathbf{o} \;=\; [\Delta\,(\mathbf{a} \otimes \nabla - \nabla \otimes \mathbf{a})] \cdot \nabla \\
& = \Delta\,[(\mathbf{a} \otimes \nabla) \cdot \nabla - (\nabla \otimes \mathbf{a}) \cdot \nabla] \\
& = \Delta\,(\Delta\,\mathbf{a} - \nabla\,(\mathbf{a} \cdot \nabla) \;=\; \Delta\,\Delta\,\mathbf{a}\,.
\end{aligned}
\tag{5.41}
$$

Therefore, $\mathbf{a}$ must be a biharmonic function. In conclusion, the displacement fields that generate eigenstrain gradients in $\mathscr{H}_2$

- are free from volumetric strain gradients,
- have zero mean curvature,
- and the divergence of the gradient of the axial vector $\mathbf{u} \times \nabla$ vanishes.

The last restriction is weaker than in the case of $\mathscr{H}_1$ since we have less constraints to exploit in the present case (one zero trace and one symmetry *vs*. two zero traces and one symmetry).

The 3-dimensional eigenspaces

Unfortunately, $\boldsymbol{H}^1{}_3$ and $\boldsymbol{H}^1{}_4$ have no symmetry or zero trace that could be exploited. The third and the fourth eigenmode depend on the angle κ which depends on the coefficients $c_{1,\,2,\,3,\,5}$ through (5.16). Thus we can determine canonical angles by taking one of the $c_{1,\,2,\,3,\,5}$ as

infinite, or consider more general directional limits with fixed rations between $c_{1,\,2,\,3,\,5}$. When doing so, two special cases emerge, namely when c_2 or c_5 are infinite. In both cases, the third eigenvalue λ_3 becomes infinite, and its eigenprojector $\boldsymbol{P}_3$ becomes $^1/_8\,\boldsymbol{B}_2$ or $^1/_3\,\boldsymbol{B}_5$, respectively. The angles κ that belong to these materials can be inferred from (5.16) and one finds the following limits

$$c_2 \to \infty : \; cos\,\kappa \to -\,^2/_3\,, \;\; \boldsymbol{P}_3 = {}^1/_8\,\boldsymbol{B}_2 \;\;, \; \lambda_3 \to \infty$$

$$c_5 \to \infty : \; cos\,\kappa \to 1\,, \;\; \boldsymbol{P}_3 = {}^1/_3\,\boldsymbol{B}_5 \;\;, \; \lambda_3 \to \infty$$

However, we can also adjust κ and the eigenvalues $\lambda_{1,\,2,\,3,\,4}$ independently.

The case $cos\,\kappa = -\,^2/_3$

The eigentensors of the third and fourth eigenvalue are related to the harmonic tensors $\boldsymbol{H}^{\boldsymbol{1}}{}_3$ and $\boldsymbol{H}^{\boldsymbol{1}}{}_4$ through

(5.42)
$$\boldsymbol{H}_3 \;=\; \boldsymbol{H}^{\boldsymbol{3}}{}_1 \cdot \boldsymbol{I}^{\mathsf{S}} \;=\; sym_{23}\,\mathbf{I} \otimes \boldsymbol{H}^{\boldsymbol{1}}{}_3$$

$$\boldsymbol{H}_4 \;=\; \boldsymbol{H}^{\boldsymbol{1}}{}_4 \cdot (\boldsymbol{I}^{\mathsf{S}} - 6\,\boldsymbol{P}_{4/1}) / \sqrt{5}\;.$$

This case is closest to the usual strain decomposition into dilatoric and deviatoric parts. The eigenmodes of the third eigenvalue are gradients of the volumetric strain. Unfortunately, the fourth eigenmode does not correspond to a gradient of a deviatoric strain.

By considering

(5.43)
$$cos\,\kappa = \frac{c_3}{c_r} = -\,^2/_3 \quad \text{and} \quad sin\,\kappa = \frac{c_4}{c_r} = \sqrt{5}\,/_3$$

– remember the restriction of κ to the interval $[0\,,\,\pi)$ – , by eliminating and summarizing, one finds that this case corresponds to

(5.44)
$$4c_1 + 2\,c_3 + c_5 \;=\; 0\,.$$

The case $cos\,\kappa = 1$

The eigentensors of the third and fourth eigenvalue are related to the harmonic tensors $\boldsymbol{H}^{\boldsymbol{1}}{}_3$ and $\boldsymbol{H}^{\boldsymbol{1}}{}_4$ by

(5.45)
$$\boldsymbol{H}_3 \;=\; \boldsymbol{H}^{\boldsymbol{1}}{}_3 \cdot \boldsymbol{P}_{4/1}$$

$$\boldsymbol{H}_4 \;=\; \boldsymbol{H}^{\boldsymbol{1}}{}_4 \cdot \boldsymbol{P}_{4/2}\;.$$

A calculation similar to the symbolic examination of the 7- and 5-dimensional eigenspaces shows that both eigenstrain gradients $\boldsymbol{H}_3$ and $\boldsymbol{H}_4$ result from displacement fields with a biharmonic field ϕ in their HELMHOLTZ representations. In terms of c_i , this case corresponds to

(5.46)
$$3c_1 + 2\,c_2 + 2\,c_3 \;=\; 0\,.$$

Relations to other forms of the strain gradient elasticity

We next summarize the conversion of the parameters between the two forms of strain gradient elasticity, namely those of MINDLIN/ ESHEL (1968), NEFF/ JEONG/ RAMEZANI (2009), and LAZAR/ MAUGIN/ AIFANTIS (2006).

MINDLIN/ ESHEL´s second form of strain gradient plasticity

MINDLIN and ESHEL´s two forms for the isotropic strain gradient energy comprise the following non-classical terms

$$\text{(5.47)} \qquad \tfrac{1}{2}\, \overset{(2)}{\boldsymbol{U}} \cdots \overset{\langle 6\rangle}{\boldsymbol{C}}_U \cdots \overset{(2)}{\boldsymbol{U}} \qquad \text{with } \overset{(2)}{\boldsymbol{U}} := \mathit{grad}\, \mathit{grad}\, \mathbf{u}$$

$$\tfrac{1}{2}\, \boldsymbol{M} \cdots \overset{\langle 6\rangle}{\boldsymbol{C}}_M \cdots \boldsymbol{M} \qquad \text{with } \boldsymbol{M} := \nabla \otimes \mathbf{E}\,.$$

With respect to the basis $\{\boldsymbol{B}_i\}$, the components of the first hexadic expressed by those of the second are related as

$$\begin{aligned} c_{U1} &= {}^{1}\!/_{2}\,(c_{M1} + c_{M2}) \\ c_{U2} &= c_{M1}/2 + c_{M2}/4 + c_{M5}/4 \\ c_{U3} &= 3c_{M3}/4 + c_{M4}/4 \\ c_{U4} &= {}^{1}\!/_{2}\,(c_{M3} + c_{M4}) \\ c_{U5} &= c_{M2}\,. \end{aligned} \qquad \text{(5.48)}$$

Note that MINDLIN uses another basis as we do, so that his components are also different from ours.

The form of NEFF/ JEONG/ RAMEZANI (2009)

The following table gives the conversion of special cases of strain gradient elasticity to the components of MINDLIN´s isotropic stiffness hexadic (5.47).

elastic energy w	$c_{1,2,3,4,5}$	$\lambda_{1,2,3,4}$, κ
$\lvert \mathbf{u} \otimes \nabla \otimes \nabla \rvert^2$	*0, 0, 0, 1, 0*	*2, 2, 2, 2 , arbitrary*
$\lvert \Delta \mathbf{u} \rvert^2$	*0, 0, 0, 0, 2*	*0, 0, 6, 0, 0*
$\lvert sym(\mathbf{u} \otimes \nabla) \otimes \nabla \rvert^2$	*0, 0, 1/4, 1/2, 0*	*2, 1/2, 2, 1/2, arccos(1/9)*
$\lvert dev\, sym(\mathbf{u} \otimes \nabla) \otimes \nabla \rvert^2$	*0, −1/6, 1/4, 1/2, 0*	*2, 1/2, 7/6, 0, arccos(19/21)*
$\lvert skw(\mathbf{u} \times \nabla) \otimes \nabla) \rvert^2$	*−1/2, 1/4, 0, 0, 1*	*0, 0, 3, 0, arccos(−1/9)*
$\lvert (\mathbf{u} \times \nabla) \times \nabla \rvert^2$	*−1, 1/2, 0, 0, 2*	*0, 0, 6, 0, arccos(−1/9)*
$\lvert (\mathbf{u} \cdot \nabla) \nabla \rvert^2$	*0, 1/2, 0, 0, 0*	*0, 0, 4, 0, arccos(−2/3)*
$\lvert (\mathbf{u} \times \nabla) \otimes \nabla \rvert^2$	*0, 0, −1/2, 1, 0*	*0, 3, 3, 0, arccos(−1/9)*
$\lvert dev[(\mathbf{u} \times \nabla) \otimes \nabla] \rvert^2$	*0, 0, −1/2, 1, 0*	*0, 3, 3, 0, arccos(−1/9)*
$\lvert sym[(\mathbf{u} \times \nabla) \otimes \nabla] \rvert^2$	*1/2, −1/4, −1/2, 1, −1*	*0, 3, 0, 0, arbitrary*
$\lvert dev\, sym[(\mathbf{u} \times \nabla) \otimes \nabla] \rvert^2$	*1/2, −1/4, −1/2, 1, −1*	*0, 3, 0, 0, arbitrary*
$\lvert sym[sym(\mathbf{u} \otimes \nabla) \times \nabla] \rvert^2$	*1/8, −1/16, −1/8, 1/4, −1/4*	*0, 3/4, 0, 0, arbitrary*

Table. Special cases of strain gradient elasticity translated into the parameter set c_i . *dev* means deviatoric part, *skw* means skew part, *sym* means symmetric part.

The form of LAZAR/ MAUGIN/ AIFANTIS (2006)

These authors recommend the use of

$$\overset{(2,2)}{C}_{ijklnm} = l^2 \overset{(1,1)}{C}_{jkmn} \delta_{il} \tag{5.49}$$

with an internal length parameter l with respect to the form (5.47.1). In the case of anisotropic elasticity, the second order tensor that extends the stiffness tetradic is invariant under the action of the material symmetry transformations. In the case of isotropy and cubic elasticity, this tensor is spherical. This leads to the following relationships with our coefficients

$$\begin{aligned} c_1 &= 0 \\ c_2 &= l^2 (K/4 - G/6) \\ c_3 &= l^2 G/4 \\ c_4 &= l^2 G/2 \\ c_5 &= 0 \end{aligned} \tag{5.50}$$

with compression modulus K and shear modulus G.

This gives for MINDLIN´s second form (5.47.2) only two non-zero parameters

$$c_{1,\,2,\,3} = 0$$
$$c_4 = l^2 G \tag{5.51}$$
$$c_5 = l^2 \lambda$$

with the LAMÉ constant λ.

6. Internal Constraints

The following chapter is based on the article

> Bertram, A.; Glüge, R.: *Gradient Materials with Internal Constraints.* Mathematics and Mechanics of Complex Systems (to appear)

6.1 Mechanical Internal Constraints

The theory of internal constraints as it is described in, *e.g.*, TRUESDELL/ NOLL sect. 30 (1965), is a useful tool for the description of incompressible materials, inextensible composites, and many more material classes. It provides us with a basis upon which both theoretical and practical investigations can be developed. Particularly, it provides a change of the structure of the basic balance equations, which can be helpful for the construction of solutions of the field problem. This way, the only non-homogeneous universal solutions for simple materials are those for constrained materials[40].

For gradient materials, one wants to introduce internal constraints other than the classical ones to again benefit from such extensions. The question arises whether such an extension is possible, or demands substantial alterations of the entire format. It turns out, and will be shown in the sequel, that such an extension is in fact straight-forward once a theory of gradient materials has been constructed, at least within the mechanical context.

Classical Internal Constraints

The classical theory of internal constraints is based on two assumptions.

Assumption 6.1. (internal constraint)
There are restrictions upon the possible deformations of the material, such that a scalar valued function of the motion and its gradient equals zero for all possible deformations

$$\gamma(\boldsymbol{\chi}, \mathit{Grad}\,\boldsymbol{\chi}) = 0 \tag{6.1}$$

Assumption 6.2. *Principle of determinism for simple materials subject to internal constraints*
The stress is determined by the deformation process only to within an additive part $\mathbf{T}_R$ that does no work in any possible motion satisfying the constraint.

If one applies the *Principle of Invariance under Rigid Body Modifications* (Axiom 2.1) to the material function γ, one can show that a function $\gamma_{\text{red}}(\mathbf{C})$ with the right CAUCHY-GREEN tensor $\mathbf{C}$ (0.36) is invariant and, hence, a reduced form.

Such a constraint would be considered as isotropic if it is invariant under arbitrary rotations

$$\gamma_{\text{red}}(\mathbf{C}) = \gamma_{\text{red}}(\mathbf{Q} \cdot \mathbf{C} \cdot \mathbf{Q}^T) \tag{6.2}$$

[40] see ERICKSEN (1955)

for all orthogonal tensors $\mathbf{Q}$. In this sense, incompressibility would be an isotropic constraint, while inextensibility in one direction is not.

By exploiting the second assumption, we start with an additive split of the CAUCHY stresses into an extra part and a reactive part

(6.3) $$\mathbf{T} = \mathbf{T}_E + \mathbf{T}_R$$

so that the specific stress power of the latter vanishes for every compatible process

(6.4) $$1/\rho\, \mathbf{T}_R \cdot\cdot\, \mathbf{D} = 0 .$$

We can bring the constraint equation into a rate form

$$\gamma_{\text{red}}(\mathbf{C})^{\bullet} = \mathit{grad}\, \gamma_{\text{red}}(\mathbf{C}) \cdot\cdot\, \mathbf{C}^{\bullet} = 0$$

and express the stress power in terms of a reactive 2nd PIOLA-KIRCHHOFF stress

(6.5) $$1/\rho_0\, \mathbf{S}_R \cdot\cdot\, \mathbf{C}^{\bullet} = 0 \qquad \text{with} \qquad \mathbf{S}_R := \mathbf{F}^{-1} * J\, \mathbf{T}_R .$$

If we multiply this equation by a LAGRANGEan multiplier and add it to the constraint equation, we find that the 2nd PIOLA-KIRCHHOFF reaction stress $\mathbf{S}_R$ must have the representation

(6.6) $$\mathbf{S}_R = \alpha\, \partial_{\mathbf{C}}\, \gamma_{\text{red}}(\mathbf{C})$$

and the CAUCHY reaction stress

(6.7) $$\mathbf{T}_R = \alpha\, \mathbf{F} \cdot \partial_{\mathbf{C}}\, \gamma(\mathbf{C}) \cdot \mathbf{F}^T \qquad \text{with } \alpha \in \mathscr{R}.$$

As a normalization of the decomposition, we can pose the orthogonality condition

(6.8) $$\mathbf{T}_R \cdot\cdot\, \mathbf{T}_E = 0$$

or

(6.9) $$\mathbf{S}_R \cdot\cdot\, \mathbf{S}_E = 0 .$$

This is, however, not necessary and often not even practical.

If there is more than one internal constraint (say $N \leq 6$), we also have more than one reaction stress, which can be superimposed to the total stress as

(6.10) $$\mathbf{T} = \mathbf{T}_E + \sum_{i=1}^{N} \alpha_i\, \mathbf{F} \cdot \partial_{\mathbf{C}}\, \gamma_i(\mathbf{C}) \cdot \mathbf{F}^T \qquad \text{with } \alpha_i \in \mathscr{R}.$$

TRUESDELL/ NOLL (1965) put these assumptions in an axiomatic way, without giving any substantiation for them other than the plausibility of their consequences in particular applications.

Non-Classical Internal Constraints

The question arises, if one could generalize such a construction to gradient materials. It turns out that such a generalization is straight-forward. We again make the two assumptions.

Assumption 6.3. There are restrictions upon the possible deformations of the material, such that a scalar valued function of the motion, deformation gradient, and the second gradient equals zero for all possible deformations

(6.11) $\gamma(\boldsymbol{\chi}, \mathit{Grad}\,\boldsymbol{\chi}, \mathit{Grad\,Grad}\,\boldsymbol{\chi}) = 0$

If this is understood as a constitutive equations, it must fulfill the EUCLIDean invariance Axiom 2.1. This leads to the following reduced form of the non-classical internal constraint

$$\gamma_{\text{red}} : \mathscr{Conf} \to \mathscr{R}$$

such that

(6.12) $\gamma_{\text{red}}(\mathbf{C}, \boldsymbol{K}) = 0$

holds for all motions. The rate form of the constraint equation is

(6.13) $\partial_{\mathbf{C}}\, \gamma_{\text{red}} \cdot\cdot\, \mathbf{C}^{\bullet} + \partial_{\boldsymbol{K}}\, \gamma_{\text{red}} \cdot\cdot\cdot \boldsymbol{K}^{\bullet} = 0\,.$

Assumption 6.4. *Principle of determinism for simple materials subject to internal constraints* The stresses and the hyperstresses are determined by the deformation process only to within additive parts $\mathbf{T}_R$ and $\boldsymbol{G}_R$ that do no work in any possible motion satisfying the constraint.

The decomposition of the stress tensors is therefore

$$\mathbf{T} = \mathbf{T}_E + \mathbf{T}_R \qquad \text{and} \qquad \boldsymbol{G} = \boldsymbol{G}_E + \boldsymbol{G}_R$$

or of the material stresses

$$\mathbf{S} = \mathbf{S}_E + \mathbf{S}_R \qquad \text{and} \qquad \boldsymbol{H} = \boldsymbol{H}_E + \boldsymbol{H}_R\,.$$

It has been shown in (2.24) that the specific stress power for a second-gradient material can be brought into the following EULERean and LAGRANGEan forms. After the second assumption we have

(6.14) $$0 = 1/\rho\,(\mathbf{T}_R \cdot\cdot\, \mathit{grad}\,\mathbf{v} + \boldsymbol{G}_R \cdot\cdot\cdot \mathit{grad\,grad}\,\mathbf{v})$$
$$= 1/\rho_0\,(\tfrac{1}{2}\,\mathbf{S}_R \cdot\cdot\, \mathbf{C}^{\bullet} + \boldsymbol{H}_R \cdot\cdot\cdot \boldsymbol{K}^{\bullet})\,.$$

By subtracting an α-fold of the constraint equation in the rate form, we obtain

(6.15) $$0 = (\frac{1}{2\rho_0}\,\mathbf{S}_R - \alpha\,\partial_{\mathbf{C}}\,\gamma_{\text{red}}) \cdot\cdot\, \mathbf{C}^{\bullet} + (\frac{1}{\rho_0}\,\boldsymbol{H}_R - \alpha\,\partial_{\boldsymbol{K}}\,\gamma_{\text{red}}) \cdot\cdot\cdot \boldsymbol{K}^{\bullet}$$

so that the following equations must hold

(6.16) $$\mathbf{S}_R = \alpha\,\partial_{\mathbf{C}}\,\gamma_{\text{red}}(\mathbf{C}, \boldsymbol{K})$$
$$\boldsymbol{H}_R = \alpha\,\partial_{\boldsymbol{K}}\,\gamma_{\text{red}}(\mathbf{C}, \boldsymbol{K})$$

with a joint LAGRANGEan parameter α which couples the two reactive stresses.

The constraint equation allows for the following choice: *Certain parts of the deformation gradient cannot vary in space.*

For $\mathbf{F}$ having nine independent components, and the space having three linear independent directions, 9 x 3 = 27 such constraints on *Grad* $\mathbf{F}$ are possible. This, however, reduces to 18 independent constraints because of SCHWARZ´ commutation law since *Grad* $\mathbf{F}$ has the right subsymmetry

(6.17) $$F_{ij\,,\,k} = \chi_{i\,,\,jk} = \chi_{i\,,\,kj} = F_{ik\,,\,j}\,.$$

This subsymmetry also applies to the configuration tensor $\boldsymbol{K}$.

In SEPPECHER/ ALIBERT/ DELL'ISOLA (2011) one finds examples of materials with microstructures with such properties.

By imposing 18 independent constraints of this kind, the deformation gradient can only be constant in space. Bodies with this property have been investigated in the past under the label *homogeneous strains*, see SLAWIANOWSKI (1974, 1975), or *pseudo-rigidity*, see COHEN (1981), COHEN/ MUNCASTER (1984), COHEN/ MAC SITHIGH (1989), ANTMANN/ MARLOW (1991), CASEY (2004, 2006, 2007). However, these approaches are completely different from the present one, since there the homogeneity of strains is imposed on the body as a global constraint, while in the present approach we still assume local constraints as an extension of classical constraints to gradient materials.

6.2 Thermomechanical Constraints

Not only in mechanics, but also more general in thermo-mechanics, the introduction of internal constraints can be advantageous. In the literature, several suggestions have been made to generalize the mechanical concepts of constraints to thermodynamics, see GREEN/ NAGHDI/ TRAPP (1970), TRAPP (1971), ANDREUSSI/ PODIO-GUIDUGLI (1973), GURTIN/ PODIO-GUIDUGLI (1973), CASEY / KRISHNASWAMY (1998), BERTRAM (2005), CASEY (2011).

In what follows, we extend the concept made in TRAPP (1971) and BERTRAM (2005), where one also finds examples for thermomechanical constraints like temperature-dependent incompressibility or inextensibility.

Definition 6.1. A **thermo-mechanical internal constraint** consists of material functions

$$\begin{aligned}
&\boldsymbol{J} \;:\; \mathscr{C}onf \times \mathscr{R}^{+} \;\to\; \mathscr{T}riad \;\big|\; (\mathbf{C}, \boldsymbol{K}, \theta) \;\mapsto\; \boldsymbol{J}(\mathbf{C}, \boldsymbol{K}, \theta) \\
&\mathbf{J} \;:\; \mathscr{C}onf \times \mathscr{R}^{+} \;\to\; \mathscr{S}ym \;\big|\; (\mathbf{C}, \boldsymbol{K}, \theta) \;\mapsto\; \mathbf{J}(\mathbf{C}, \boldsymbol{K}, \theta) \qquad (6.18) \\
&\mathbf{j} \;:\; \mathscr{P}sym \times \mathscr{R}^{+} \;\to\; \mathscr{V}^{3} \;\big|\; (\mathbf{C}, \boldsymbol{K}, \theta) \;\mapsto\; \mathbf{j}(\mathbf{C}, \boldsymbol{K}, \theta) \\
&j \;:\; \mathscr{P}sym \times \mathscr{R}^{+} \;\to\; \mathscr{R} \;\big|\; (\mathbf{C}, \boldsymbol{K}, \theta) \mapsto j(\mathbf{C}, \boldsymbol{K}, \theta)
\end{aligned}$$

such that for all admissible thermo-kinematical processes the constraint equation

$$\boldsymbol{J}(\mathbf{C}, \boldsymbol{K}, \theta) \cdots \boldsymbol{K}^{\bullet} + \mathbf{J}(\mathbf{C}, \boldsymbol{K}, \theta) \cdot\cdot \mathbf{C}^{\bullet} + \mathbf{j}(\mathbf{C}, \boldsymbol{K}, \theta) \cdot \mathbf{g}_0 + j(\mathbf{C}, \boldsymbol{K}, \theta)\, \theta^{\bullet} = 0 \qquad (6.19)$$

holds at each instant.

Since we made use of material variables, this constraint is already in a reduced form. The first two terms, to which the equation is reduced in the isothermal case, corresponds to the mechanical constraint in its rate form (6.13).

Once again, we have to modify the *Principle of Determinism*.

Assumption 6.5. **Principle of determinism for materials with thermo-mechanical internal constraints**
The current values of the hyperstress, stress, heat flux, internal energy, and entropy are determined by the thermo-kinematical process only up to additive parts that are not dissipative during all admissible processes that satisfy the constraint equation (6.19).

Thus, we have the decompositions of the dependent variables into reactive parts and extra parts:

$$\begin{aligned}
&\text{hyperstress} && \boldsymbol{H} = \boldsymbol{H}_E + \boldsymbol{H}_R \\
&\text{PIOLA-KIRCHHOFF stress} && \mathbf{S} = \mathbf{S}_E + \mathbf{S}_R \\
&\text{heat flux} && \mathbf{q}_0 = \mathbf{q}_{0E} + \mathbf{q}_{0R} \qquad (6.20) \\
&\text{internal energy} && \varepsilon = \varepsilon_E + \varepsilon_R
\end{aligned}$$

entropy $\eta = \eta_E + \eta_R$

and, consequently, also for the

free energy $\psi = \varepsilon_E + \varepsilon_R - \theta\,\eta_E - \theta\,\eta_R =: \psi_E + \psi_R$

where only the extra-terms depend on the thermo-kinematical process. The reaction parts do not dissipate in the sense of the CLAUSIUS-DUHEM inequality

$$(6.21)\qquad \frac{1}{\rho_0}\,(\tfrac{1}{2}\,\mathbf{S}_R \cdot\cdot\, \mathbf{C}^{\bullet} + \boldsymbol{H}_R \cdot\cdot\cdot \boldsymbol{K}^{\bullet}) - \frac{1}{\theta\,\rho_0}\,\mathbf{q}_{0R}\cdot\mathbf{g}_0 - \psi_R^{\bullet} - \eta_R\,\theta^{\bullet} = 0$$

for all admissible thermo-kinematical processes. If we subtract from this equation an α-fold of the constraint equation we get

$$(6.22)\qquad (\frac{1}{2\rho_0}\,\mathbf{S}_R - \alpha\,\mathbf{J}) \cdot\cdot\, \mathbf{C}^{\bullet} + (\frac{1}{\rho_0}\,\boldsymbol{H}_R - \alpha\,\boldsymbol{J}) \cdot\cdot\cdot \boldsymbol{K}^{\bullet} - (\frac{1}{\theta\,\rho_0}\,\mathbf{q}_{0R} + \alpha\,\mathbf{j})\cdot\mathbf{g}_0$$
$$- \psi_R^{\bullet} - (\eta_R + \alpha j)\,\theta^{\bullet} = 0$$

for any real α. Because of the arbitrariness of $\mathbf{C}^{\bullet}$, $\boldsymbol{K}^{\bullet}$, $\mathbf{g}_0$, and $\theta^{\bullet}$, this is solved for all constrained materials only by

$$(6.23)\qquad \begin{aligned} \boldsymbol{H}_R &= \alpha\,\rho_0\,\boldsymbol{J}(\mathbf{C}, \boldsymbol{K}, \theta) \\ \mathbf{S}_R &= 2\,\alpha\,\rho_0\,\mathbf{J}(\mathbf{C}, \boldsymbol{K}, \theta) \\ \mathbf{q}_{0R} &= -\,\alpha\,\rho_0\,\theta\,\mathbf{j}(\mathbf{C}, \boldsymbol{K}, \theta) \\ \psi_R^{\bullet} &= 0 \\ \eta_R &= -\,\alpha j(\mathbf{C}, \boldsymbol{K}, \theta) \end{aligned}$$

or spatially

$$(6.24)\qquad \begin{aligned} \boldsymbol{G}_R &= \alpha\,\rho\,\mathbf{F} \circ \boldsymbol{J}(\mathbf{C}, \boldsymbol{K}, \theta) && \text{with } \boldsymbol{G}_R = \mathbf{F} \circ J^{-1}\,\boldsymbol{H}_R \\ \mathbf{T}_R &= 2\alpha\,\rho\,\mathbf{F} * \mathbf{J}(\mathbf{C}, \boldsymbol{K}, \theta) && \text{with } \mathbf{T}_R = \mathbf{F} * J^{-1}\,\mathbf{S}_R \\ \mathbf{q}_R &= -\,\alpha\,\rho\,\theta\,\mathbf{F} * \mathbf{j}(\mathbf{C}, \boldsymbol{K}, \theta) && \text{with } \mathbf{q}_R = \mathbf{F} * J^{-1}\,\mathbf{q}_{0R}\,. \end{aligned}$$

With this form, for no $\alpha \in \mathscr{R}$ can a contradiction to the CLAUSIUS-DUHEM inequality occur if the extra terms already fulfill it alone.

Since the free energy is only determined up to a constant, we can principally assume $\psi_R = 0$.

As a normalization of the decomposition, one can pose the orthogonality condition

$$(6.25)\qquad \boldsymbol{H}_R \cdot\cdot\cdot \boldsymbol{H}_E + \tfrac{1}{4}\,\mathbf{S}_R \cdot\cdot\, \mathbf{S}_E + \mathbf{q}_{0R}\cdot\mathbf{q}_{0E}/\theta^2 + \rho_0^2\,\eta_R\cdot\eta_E = 0\,.$$

This is, however, not compulsory and perhaps not even practical.

If more than one constraint is active, then the reactive parts are simply additive superpositions of those resulting from each constraint alone.

Introduction of internal constraints in a natural way

In BERTRAM (1980, 1982) another approach to establish a theory of internal constraints has been suggested, claiming to be *in a natural way*. Here only solids are considered for which the stresses are (at least partly) caused by elastic deformations. The idea there is, roughly speaking, to consider constraint material behavior as a limit of hyperelastic behavior with increasing stiffness for certain deformation modes.

If one starts with hyperelastic behavior of a simple material, one can consider a tangential stiffness tensor with 6 (not necessary different) eigenvalues called principal stiffnesses in the case of classical (non-gradient) materials. If one produces a series of such materials by incrementing one of these eigenvalues to infinity and keeping all others finite, one produces in the limit a material behavior that is constrained in such a way that the deformation mode belonging to this eigenvalue tends to zero if only finite stresses are applied. It has been shown there that for an isotropic or anisotropic hyperelastic material, this construction exactly leads to the above two Assumptions 4.1 and 4.2.

The method to produce internal constraints in a natural way can also be applied to gradient hyperelastic materials. We can (locally) linearise the elastic laws (2.34) and (2.35) by taking their incremental forms

$$\mathrm{d}\mathbf{S} = \overset{\langle 4\rangle}{\boldsymbol{E}} \cdot\cdot\, \mathrm{d}\mathbf{C} + \overset{\langle 5\rangle}{\boldsymbol{E}} \cdot\cdot\cdot\, \mathrm{d}\boldsymbol{K} \tag{6.26}$$

$$\mathrm{d}\boldsymbol{G} = \mathrm{d}\mathbf{C} \cdot\cdot\, \overset{\langle 5\rangle}{\boldsymbol{E}} + \overset{\langle 6\rangle}{\boldsymbol{E}} \cdot\cdot\cdot\, \mathrm{d}\boldsymbol{K}$$

with a

- fourth-order symmetric stiffness tensor (tetradic) $\overset{\langle 4\rangle}{\boldsymbol{E}} := 2\rho_0\, \partial_{\mathbf{CC}}\, w(\mathbf{C}, \boldsymbol{K})$
- sixth-order symmetric stiffness tensor (hexadic) $\overset{\langle 6\rangle}{\boldsymbol{E}} := \rho_0\, \partial_{\boldsymbol{KK}}\, w(\mathbf{C}, \boldsymbol{K})$
- fifth-order stiffness tensor $\overset{\langle 5\rangle}{\boldsymbol{E}} := 2\rho_0\, \partial_{\mathbf{C}\boldsymbol{K}}\, w(\mathbf{C}, \boldsymbol{K})$.

Interesting for us is the stiffness hexadic $\overset{\langle 6\rangle}{\boldsymbol{E}}$ since it does not exist for classical materials. We can bring the hexadic into a spectral form

$$\overset{\langle 6\rangle}{\boldsymbol{E}} = \sum_{i=1}^{18} \lambda_i\, \overset{\langle 6\rangle}{\boldsymbol{P}}_i \tag{6.27}$$

with 18 (not necessarily distinct) eigenvalues λ_i and the same number of eigenspace projectors of sixth-order $\overset{\langle 6\rangle}{\boldsymbol{P}}_i$. These are related to the third-order normalized and orthogonal eigentensors $\boldsymbol{E}_i$ of the stiffness hexadic by the sum

$$\overset{\langle 6\rangle}{\boldsymbol{P}}_i = \sum_{j=1}^{M_i} \boldsymbol{E}_j \otimes \boldsymbol{E}_j \tag{6.28}$$

over the multiplicity M_i of the particular eigenvalue. The construction of internal constraints in a natural way consists of taking for all of these eigenvalues finite values except for one, say λ_1 .

Let us first consider an eigenvalue of multiplicity one. In the limit, one would not be able to deform the material in the corresponding mode by applying finite stresses. Thus, we obtain the constraint equation (in this case independent of $\mathbf{C}$)

$$\gamma_{\text{red}}(\boldsymbol{K}) := \boldsymbol{E}_1 \cdots \boldsymbol{K} = 0 \tag{6.29}$$

and expect the reaction hyperstresses after (6.16.2) as

$$\boldsymbol{H}_R = \alpha\, \partial_{\boldsymbol{K}} \gamma_{\text{red}}(\boldsymbol{K}) = \alpha\, \boldsymbol{E}_1 \tag{6.30}$$

with some scalar field α .

Such a constraint would be considered as isotropic if it is invariant under arbitrary rotations

$$\gamma_{\text{red}}(\boldsymbol{K}) = \gamma_{\text{red}}(\mathbf{Q} * \boldsymbol{K}) \tag{6.31}$$

for all orthogonal tensors $\mathbf{Q}$. Clearly, this is the case if and only if

$$\boldsymbol{E}_1 = \mathbf{Q} * \boldsymbol{E}_1 \tag{6.32}$$

i.e., for isotropic tensors. But this would be a rather drastic restriction, which should not be made in general.

We can also superimpose the M constraints of a multiple eigenvalue in one equation

$$\gamma_{\text{red}}(\boldsymbol{K}) := \boldsymbol{K} \cdots \overset{\langle 6 \rangle}{\boldsymbol{P}}_{\boldsymbol{1}} \cdots \boldsymbol{K} = 0 \,. \tag{6.33}$$

An alternative approach to create internal constraints was suggested by CASEY (1995) and BAESU/ CASEY (2000) in a mechanical, and CASEY/ KRISHNASWAMY (1998) and CASEY (2011) in a thermomechanical setting where the constrained material is identified as an equivalence class of unconstrained ones.

References

Adams, R. A.: *Sobolev Spaces*. Academic Press, New York (1975)

Aifantis, E.C.: *The physics of plastic deformation*. Int. J. Plasticity **3**, 211-247 (1987)

Anand, L.; Aslan, O.; Chester, S. A.: *A large-deformation gradient theory for elastic–plastic materials: Strain softening and regularization of shear bands*. Int. J. Plasticity **30–31**, 116–143 (2012)

Andreussi, F.; Podio Guidugli, P.: *Thermomechanical constraints in simple materials*. Bull. Acad. Pol. Sci. Ser. Sci. Techn. **11**,4, 199-205 (1973)

Antman, S.S.; Marlow, R.S.: *Material constraints, Lagrange multipliers, and compatibility. Applications to rod and shell theories*. Arch. Rational Mech. Anal. **116**, 257-299 (1991)

Auffray, N.; Le Quang, H.; He, Q. C. : *Matrix representations for 3D strain-gradient elasticity.* J. Mechanics Physics Solids **61**, 1202-1223 (2013)

Bammann, D. J.: *A model of crystal plasticity containing a natural length scale*. Materials Sc. Engin. A309–310, 406–410 (2001)

Baesu, E.; Casey, J.: *A treatment of internally constrained elastic-plastic materials.* Int. J. Engng. Sci. **38**, 1677-1698 (2000)

Bertram, A.: *An introduction of internal constraints in a natural way.* Z. ang. Math. Mech. **60**, T100-T101 (1980)

Bertram, A.: *Material Systems - a framework for the description of material behavior.* Arch. Rat. Mech. Anal., **80**,2, 99-133 (1982)

Bertram, A.: *An Axiomatic Approach to Classical Dynamics Based on an Objective Power Functional*. Forschungsbericht Nr. 1 des 2. Instituts für Mechanik, TU Berlin (1983)

Bertram, A.: *Axiomatische Einführung in die Kontinuumsmechanik.* BI-Wissenschaftsverlag, Mannheim, Wien, Zürich (1989)

Bertram, A.: *An alternative approach to finite plasticity based on material isomorphisms.* Int. J. Plasticity **52**, 353-374 (1998)

Bertram, A.; Svendsen, B.: *On material objectivity and reduced constitutive equations*, Archives of Mechanics **53**,6, 653-675 (2001)

Bertram, A.: *Elasticity and Plasticity of Large Deformations - an Introduction.* Springer-Verlag (2005, 2008, 2012)

Bertram, A.; Forest, S.: *Mechanics based on an Objective Power Functional*. Techn. Mechanik **27**,1, 1-17 (2007)

Bertram, A.; Krawietz, K.: *On the introduction of thermoplasticity*. Acta Mechanica **223**,10, 2257-2268 (2012)

Bertram, A.: *The Mechanics and Thermodynamics of Finite Gradient Elasticity and Plasticity*. Preprint Otto-von-Guericke Universität Magdeburg http://www.uni-magdeburg.de/ifme/l-festigkeit/pdf/1/Preprint_ Gradientenplasti_finite_16.10.12.pdf (2013)

Bertram, A.; Forest, S.: *The thermodynamics of gradient elastoplasticity*. Continuum Mech. Thermodyn. **26**,269–286 (2014)

Bertram, A.: *On the History of Material Theory - A Critical Review.* In: *The History of Theoretical, Material and Computational Mechanics - Mathematics Meets Mechanics and Engineering*, Lecture Notes in Applied Mathematics and Mechanics Volume 1, Editor: Erwin Stein, Springer, 119-131 (2014)

Bertram, A.: *Finite gradient elasticity and plasticity: a constitutive mechanical framework*. Continuum Mech. Thermodyn., **27**,6, 1039-1058 (2015)

Bertram, A.: *Finite gradient elasticity and plasticity: a constitutive thermodynamical framework*, Continuum Mech. Thermodyn, DOI 10.1007/s00161-015-0417-6 (in press)

Bertram, A.; Glüge, R.: *Gradient Materials with Internal Constraints.* Mathematics and Mechanics of Complex Systems (to appear)

Bleustein, J. L.: *A note on the boundary conditions of Toupin´s strain-gradient theory*. Int. J. Solids Structures **3**, 1053-1057 (1967)

Böhlke, T.; Bertram, A.: *The evolution of Hooke's law due to texture development in polycrystals.* Int. J. Sol. Struct., **38**,52, 9437-9459 (2001)

Brand, L.: *Vector and Tensor Analysis.* John Wiley & Sons, New York, London (1947)

Cardona, J.-M.; Forest, S.; Sievert, R.: *Towards a theory of second grade thermoelasticity.* Extracta Mathematicae **14**, 127-140 (1999)

Casey, J.: *A treatment of internally constrained materials.* Trans. ASME **62**, 542-544 (1995)

Casey, J.; Krishnaswamy, S.: *A characterization of internally constrained thermoelastic materials*. Math. Mech. Solids **3**, 71-89 (1998)

Casey, J.: *Pseudo-Rigid Continua: Basic Theory and a Geometrical Derivation of Lagrange's Equations*. Proceedings of the Royal Society, **A 460**, 2021-2049 (2004)

Casey, J.: *The ideal pseudo-rigid continuum*. Proceedings of the Royal Society A **462**, 3185-3195, (2006)

Casey, J.: *A New Definition of a Pseudo-Rigid Continuum*. Note di Matematica, **27**, 43-53 (2007)

Casey, J.: *Nonlinear thermoelastic materials with viscosity, and subject to internal constraints: a classical continuum thermodynamics approach.* J. Elast. **104**, 91-104 (2011)

Cauchy, A.-L.: Recherches sur l´équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques. Bull. Soc. Philomath. Paris (1823)

Cauchy, A.-L.: Sur les équations qui expriment les conditions d´équilibre ou les lois du mouvement intérieur d´un corps solide, élastique ou non élastique. Ex. de Math. **3**, 160-187 (1828)

Chambon, R.; Caillerie, D.; Tamagnini, C.: *A finite deformation second gradient theory of plasticity*. Comptes Rendus de l'Académie des Sciences Paris - Series IIb - Mechanics **329**, 797-802 (2001)

Ciarletta, P.; Maugin, G. A.: *Elements of a finite strain-gradient thermomechanical theory for material growth and remodeling.* I. J. Nonlinear Mechanics **46**, 1341-1346 (2011)

Clayton, J. D.; McDowell, D. L.; Bammann, D. J.: *Modeling dislocations and disclinations with finite micropolar elstoplasticity.* Int. J. Plasticity 22, 210-256 (2006)

Cleja-Ţigoiu, S.: *Couple stresses and non-Riemannian plastic connection in finite elasto-plasticity*. Z. angew. Math. Phys. 53, 996-1013 (2002)

Cleja-Ţigoiu, S.: *Elasto-plastic materials with lattice defects modeled by second-order deformations with non-zero curvature.* Int. J. Fract. **166**, 61-75 (2010)

Cleja-Ţigoiu, S.: *Non-local elasto-viscoplastic models with dislocations in finite elasto-plasticity. Part I: Constitutive framework.* Mathematics Mechanics Solids **18**,4, 349-372 (2013)

Cleja-Ţigoiu, S.; Paşcan, R.: *Non-local elasto-viscoplastic models with dislocations in finite elasto-plasticity. Part II: Influence of dislocations in crystal plasticity.* Mathematics Mechanics Solids **18**,4, 373-396 (2012)

Cohen, H.: *Pseudo-rigid bodies*. Utilitas Matematica **20**, 221-247 (1981)

Cohen, H.; Muncaster, R. G.: *The dynamics of pseudo-rigid bodies: general structure and exact solutions*. J. Elasticity **14**, 127-154 (1984)

Cohen, H.; Mac Sithigh, G. P.: *Plane motions of elastic pseudo-rigid bodies*. J. Elast. **21**, 193-226 (1989)

Cordero, N. M.; Forest, S.; Busso, E. P.: *Second strain gradient elasticity of nano-objects*. To appear in J. Mech. Phys. Solids.

Cosserat, E.; Cosserat, F.: *Théorie des Corps deformables*. Paris: A, Hermann et Fils (1909)

Cross, J. J.: *Mixtures of fluids and isotropic solids*. Archives Mechanics **25**,6, 1025-1039 (1973)

Dell'Isola, F.; Seppecher, P.: *Edge contact forces and quasi-balanced power*. Meccanica, **32**,1, 33-52 (1997)

Dell´Isola, F.; Sciarra, G.; Vidoli, S.: *Generalized Hooke´s law for isotropic second gradient materials*. Proc. R. Soc. A. **465**, 2177-2196 (2009)

Dell´Isola, F.; Seppecher, P.: *Hypertractions and hyperstresses convey the same mechanical information.* Continuum Mech. Thermodyn. (2010) 22:163-176" *by Prof. Podio Guidugli and Prof. Vianello and some related papers on higher gradient theories*. Continuum Mech. Thermodyn. **23**, 473-478 (2011)

Del Piero, G.: *On the method of virtual power in continuum mechanics*. Journal of Mechanics of Materials and Structures **4**, 281-292 (2009)

Dillon, O.W.; Kratochvil, J.: *A strain gradient theory of plasticity*. Int. J. Solids Structures **6**, 1513-1533 (1970)

Elzanowski, M.; Epstein, M.: *The symmetry group of second-grade materials*. Int. J. Non-Linear Mechanics **27**,4, 635-638 (1992)

Ekh, M.; Grymer, M.; Runesson, K.; Svedberg, T.: *Gradient crystal plasticity as part of the computational modelling of polycrystals.* Int. J. Numer. Meth. Engng 72, 197-220 (2007)

Enakoutsa, K.; Leblond, J.B.: *Numerical implementation and assessment of the GLPD micromorphic model of ductile rupture*. European Journal of Mechanics A/solids, **28**, 445-460 (2009)

Ericksen J. L.: *Deformations possible in every compressible, isotropic, perfectly elastic material.* J. Math. Phys. **34**, 126-128 (1955)

Eringen, A. C.: *Microcontinuum Field Theories. I: Foundations and Solids*. Springer (1999)

Faciu, C.; Molinari, A.: *A non-local rate-type viscoplastic approach to patterning of deformation*. Acta Mech. **126**, 71-99 (1998)

Fleck, N. A.; Muller, G. M.; Ashby, M. F.; Hutchinson, J. W.: *Strain gradient plasticity: theory and experiments*. Acta Metallurgica Materialia **42**,2, 475-487 (1994)

Fleck, N.A.; Hutchinson, J.W.: *Strain gradient plasticity*. Adv. Appl. Mech. **33**, 295-361 (1997)

Fleck, N.A.; Hutchinson, J.W.: *A reformulation of strain gradient plasticity*. Journal of the Mechanics and Physics of Solids **49**, 2245-2271 (2001)

Forest, S.; Cardona, J.M.; Sievert, R., *Thermoelasticity of Second-Grade Media*, Continuum Thermomechanics, The Art and Science of Modelling Material Behaviour, Paul Germain's Anniversary Volume, G.A. Maugin and R. Drouot and F. Sidoroff, Kluwer Academic Publishers, 163-176 (2000)

Forest, S.; Sievert, R.: *Elastoviscoplastic constitutive frameworks for generalized continua*. Acta Mechanica **160**, 71-111 (2003)

Forest S.; Sievert, R.: *Nonlinear microstrain theories*. Int. J. Solids and Structures **43**, 7224-7245 (2006)

Forest, S.; Amestoy, M.: *Hypertemperature in thermoelastic solids*. Comptes Rendus Mécanique **336**, 347-353 (2008)

Forest, S.; Aifantis, E.C.: *Some links between recent gradient thermoelastoplasticity theories and the thermomechanics of generalized continua*. Int. J. Solids Structures **47**, 3367-3376 (2010)

Forest, S.; Trinh, D.K.: *Generalized continua and non-homogeneous boundary conditions in homogenisation methods.* Z. angew. Math. Mech. **91**, 90-109 (2011)

Germain, P.*: Sur l´application de la méthode des puissances virtuelles en mécanique des milieux continus*. C.R. Acad. Sci. Paris Sér. A **274**,2, 1051-1055 (1972)

Germain, P.: *La méthode des puissances virtuelles en mécanique des milieux continus, première partie : théorie du second gradient*. J. Mécanique **12**, 235–274 (1973a)

Germain, P.: *The method of virtual power in continuum mechanics. Part 2: Microstructure*. J. Appl. Math. **25**, 556-575 (1973b)

Glüge, R.; Kalisch, J.; Bertram, A.: *The eigenmodes in isotropic strain gradient elasticity*. To appear in *Generalized Continua as Models for Classical and Advanced Materials.* Edts. H. Altenbach, S. Forest. Springer Verlag (in press)

Golubitsky, M.; Stewart, I.; Schaeffer, D.: *Singularities and Groups in Bifurcation Theory*. Volume II. Springer, New York (1988)

Green, A. E.; Rivlin, R. S.: *On Cauchy´s equations of motion*. ZAMM **15**, 290-292 (1964a)

Green, A. E.; Rivlin, R. S.: *Simple force and stress multipoles*. Archive Rat. Mech. Anal. **16**,5, 325-353 (1964b)

Green, A. E.; Naghdi, P. M.; Trapp, J. A.: *Thermodynamics of a continuum with internal constraints*. Int. J. Engn. Sci. **8**, 891-908 (1970)

Gudmundson, P.: *A unified treatment of strain gradient plasticity.* J. Mech. Phys. Solids **52**, 1379-1406 (2004)

Gurtin, M. E.: *Thermodynamics and the possibility of spacial interaction in elastic materials.* Archive Rat. Mech. Anal. **19**,5, 339-352 (1965)

Gurtin, M. E.; Podio Guidugli, P.: *The thermodynamics of constrained materials*. Arch. Rat. Mech. Anal. **51**,3, 192-208 (1973)

Gurtin, M. E.; Anand, L.: *A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. Part II: Finite deformations.* Int. J. Plasticity **21**, 2297-2318 (2005)

Gurtin, M. E.; Fried, E.; Anand, L.: *The Mechanics and Thermodynamics of Continua.* Cambridge University Press (2009)

Gurtin, M. E.; Anand, L.: *Thermodynamics applied to gradient theories involving the accumulated plastic strain: The theories of Aifantis and Fleck and Hutchinson and their generalization.* J. Mech. Phys. Solids **57**, 405-412 (2009)

Gurtin, M. E.: *A finite-deformation, gradient theory of single-crystal plasticity with free energy dependent on the accumulation of geometrically necessary dislocations*. Int. J. Plasticity **26**, 1073-1096 (2010)

Hirschberger, C.B.; Steinmann, P.: *Classification of Concepts in Thermodynamically Consistent Generalized Plasticity*. ASCE Journal of Engineering Mechanics **135**, 156-170 (2009)

Hwang, K. C.; Jiang, H.; Huang, Y.; Gao, H.; Hu, N.: *A finite deformation theory of strain gradient plasticity*. J. Mech. Phys. Solids **50**, 81-99 (2002)

Ieşan, D.; Quintanilla, R.: *On a strain gradient theory of thermoviscoelasticity*. Mech. Res. Comm. *48*, 52-58 (2013)

Javili, A.; dell'Isola, F.; Steinmann, P.: *Geometrically nonlinear higher-gradient elasticity with energetic boundaries*. Journal of the Mechanics and Physics of Solids, **61**,12, 2381-2401 (2013)

Kalisch, J. and R. Glüge. *Analytical homogenization of linear elasticity based on the interface orientation distribution – a complement to the self-consistent approach*. Composite Structures **126**, 398-416 (2015)

Koiter, W. T.: *Couple-stresses in the theory of elasticity I.* Proc. K. Ned. Akad. Wet. **B 67**, 17-44 (1964)

Korteweg, D. J.: *Sur la forme que presennent les équations du mouvement des fluides si l´on tient compte des forces capillaires causées par de variations de densité considérables mais continues et sur la théorie de la capillarité dans l´hypothése d´une variation continue de la densité.* Archives Néerlandaises Sci. exactes naturelles. Ser. II,6, 1-24 (1901)

Krawietz, A.: Zur Elimination der Verschiebungen bei großen Verformungen. In: Beiträge zu Mechanik, 133-147, Edts. C. Alexandru, G. Gödert, W. Görn, R. Parchem, J. Villwock, TU Berlin (1993)

Lagrange, J.-L.: *Mécanique analytique.* Paris (1788)

Laplace, P. S.: *Traite de mécanique céleste.* Paris (1799)

Lazar, M.; Maugin, G. A.: *Nonsingular stress and strain fields of dislocations and disclinations in first strain gradient elasticity*. International Journal of Engineering Science **43**,13, 1157–1184 (2005)

Lazar, M.; Maugin, G. A.; Aifantis, E.C.: *On a theory of nonlocal elasticity of bi-Helmholtz type and some applications*. Int. J. Solids Structures **43**,6, 1404-1421 (2006)

Leon, M. de; Epstein, M.: *The geometry of uniformity in second-grade elasticity.* Acta Mechanica **114**, 217-224 (1996)

Leroy, Y. M.; Molinari, A.: *Spatial patterns and size effects in shear zones: A hyperelastic model with higher-order gradients.* J. Mech. Phys. Sol. **41**,4, 631-663 (1993)

Lidström, P.: *On the principle of virtual power in continuum mechanics*. Math. Mech. Solids **17**,5, 16-40 (2012)

Luscher, D. J.; McDowell, D. L.; Bronkhorst, C. A.: *A second gradient theoretical framework for hierarchical multiscale modeling of materials.* Int. J. Plasticity **26**, 1248-1275 (2010)

Maugin, G.: *The method of virtual power in continuum mechanics: Application to coupled fields*. Acta Mech. **35**, 1-70 (1980)

Miehe, C.: *Variational gradient plasticity at finite strains. Part I: Mixed potentials for the evolution and update problems of gradient-extended dissipative solids*. Comput. Methods Appl. Mech. Engrg. **268**, 677-703 (2014)

Mindlin, R.D: *Micro-structure in linear elasticity*. Archive for Rational Mechanics and Analysis **16**,1, 51-78 (1964)

Mindlin, R. D.: *Second gradient of strain and surface-tension in linear elasticity.* Int. J. Solids Structures **1**, 417-438 (1965)

Mindlin, R. D.; Eshel, N. N.: *On first strain-gradient theories in linear elasticity*. Int. J. Solids Structures **4**, 109-124 (1968)

Mises, R. v.: *Mechanik der plastischen Formänderung von Kristallen*. Z. Angew. Math. Mech. **8**,3, 161-185 (1928)

Murdoch, A. I.: *Symmetry considerations for materials of second grade.* J. Elasticity **9**,1, 43-50 (1979)

Neff., P.: *Remarks on invariant modelling in finite strain gradient plasticity.* Technische Mechanik **28**,1, 13-21 (2008)

Neff, P.; Chełmiński, K.; Alber, H.-D.: *Notes on strain gradient plasticity: finite strain covariant modelling and global existence in the infinitesimal rate-independent case.* Math. Models Methods Appl. Sci. **19**,2, 307-346 (2009)

Neff, P. Jeong, J.; Ramézani, H.: *Subgrid interaction and micro-randomness – Novel invariance requirements in infinitesimal gradient elasticity*. Int. J. Solids Structures **46**,25–26, 4261-4276 (2009)

Noll, W.: *La mécanique classique, basée sur un axiome d´objectivité.* (Colloque Int., Paris, 1959) Gauthier-Villars (1963), quoted after Noll, W.: *The Foundations of Mechanics and Thermodynamics, Selected Papers*, Springer-Verlag, Berlin (1974)

Noll, W.: *Materially uniform simple bodies with inhomogeneities.* Arch. Rat. Mech. Anal. **27**, 1-32 (1967)

Olive, M.; Auffray, N.: *Symmetry classes for odd-order tensors*. Zeitschrift für Angewandte Mathematik und Mechanik **94**,5, 421-447 (2014)

Papenfuss, C.; Forest, S.: *Thermodynamical frameworks for higher grade material theories with internal variables or additional degrees of freedom*. Journal of Non-Equilibrium Thermodynamics **31**,4, 319-353 (2006)

Podio-Guidugli, P.; Vianello, M.: *On a stress-power-based characterization of second-gradient elastic fluids*. Continuum Mech. Thermodyn. **25**, 399-421 (2013)

Podio-Guidugli, P.: *A virtual power format for thermodynamics.* Continuum Mech. Thermodyn. **20**, 479-487 (2009)

Polizzotto, C.; Borino, G.: *A thermodynamics-based formulation of gradient-dependent plasticity*. Eur. J. Mech. A/Solids **17**, 741-761(1998)

Polizzotto, C.: *Unified thermodynamic framework-for nonlocal/gradient continuum theories*. Eur. J. Mech., A/Solids **22**, 651-668 (2003)

Polizzotto, C.: *A nonlocal strain gradient plasticity theroy for finite deformations.* Int. J. Plast. **25**, 1280-1300 (2009)

Polizzotto, C.: *A unified residual-based thermodynamic framework for strain gradient theories of plasticity*. Int. J. Plasticity **27**, 388-413 (2011)

Polizzotto, C.: *A second strain gradient elasticity theory with second velocity gradient inertia - Part I: Constitutive equations and quasi-static behavior*. Int. J. Solids Structures **50**,24, 3749-3765 (2013)

Segev, R.: *Forces and the existence of stresses in invariant continuum mechanics*. J. Math. Phys. **27**,1, 163-170 (1986)

Segev, R.; G. De Botton: *On the consistency conditions for force systems*. Int. J. Non-Linear Mechanics **26**,1, 47-59 (1991)

Seppecher, P.; Alibert, J.-J. ; Dell'Isola, F.: *Linear elastic trusses leading to continua with exotic mechanical interactions*. Journal of Physics: Conference Series, **319**,1, 13 pages (2011)

Sievert, R.: *A geometrically nonlinear elasto-viscoplasticity theory of second grade*. Technische Mechanik, **31**, 83-111 (2011)

Sławianowski, J.J.: *Analytical mechanics of finite homogeneous strains*. Arch. Mech. **26**,4, 569-587 (1974)

Sławianowski, J.J.: *Newtonian dynamics of homogeneous strains*. Arch. Mech. **27**,1, 93-102 (1975)

Steinmann, P.: *Geometrical Foundations of Continuum Mechanics*. Springer (2015)

Suiker, A. S. J.; Chang, C. S.: *Application of higher-order tensor theory for formulating enhanced continuum models.* Acta Mech. **142**, 223-234 (2000)

Svendsen, B.; Neff, P.; Menzel, A.: *On constitutive and configurational aspects of models for gradient continua with microstructure*. Z. ang. Math. Mech. **89**,8, 687-697 (2009)

Testa, V.; Vianello, M.: *The symmetry group of gradient sensitive fluids*. Int. J. Non-linear Mechanics **40**, 621-631 (2005)

Toupin, R.A.: *Elastic materials with couple stresses*. Arch. Rat. Mech. Anal. **11**, 385-414 (1962)

Toupin, R.A.: *Theory of elasticity with couple-stress*. Arch. Rat. Mech. Anal. **17**, 85-112 (1964)

Trapp, J.A.: *Reinforced materials with thermo-mechanical constraints*. Int. J. Engng. Sci. **9**,8, 757-773 (1971)

Triantafyllidids, N.; Aifantis, E. C.: *A gradient approach to localization of deformation. I. Hyperelastic materials*. J. Elasticity **16**, 225-237 (1986)

Trostel, R.: *Gedanken zur Konstruktion mechanischer Theorien*. In: *Beiträge zu den Ingenieurwissenschaften*, R. Trostel (Ed.), Univ.-Bibl. Techn. Univ. Berlin, 96-134 (1985)

Trostel, R.: *Mathematische Grundlagen der Technischen Mechanik I - Vektor- und Tensor-Algebra.* Vieweg, Wiesbaden (1993)

Trostel, R.: *Mathematische Grundlagen der Technischen Mechanik II - Vektor- und Tensor-Analysis*. Vieweg, Wiesbaden (1997)

Truesdell, C.; Toupin, R. A.: *The Classical Field Theories*. Handbuch der Physik. Vol. III/1. Ed.: S. Flügge, Springer, Berlin (1960)

Truesdell, C. A.; Noll, W.: *The Non-linear Field Theories of Mechanics*. In: *Handbuch der Physik*. Vol. III/3. Ed.: S. Flügge, Springer, Berlin (1965) , 2nd ed. (1992), 3rd ed. by S. Antman (2004)

Ván, P.; Berezovski, A.; Papenfuss, C.: *Thermodynamic approach to generalized continua.* Continuum Mech. Thermodyn. **26**,3, 403-420 (2014)

Voyiadjis, G. Z.; Faghihi, D.: *Thermo-mechanical strain gradient plasticity with energetic and dissipative length scales*. International Journal of Plasticity **30-31**, 218-247 (2012)

Voyiadjis, G. Z.; Faghihi, D.: *Gradient plasticity for thermo-mechanical processes in metals with length and time scales*. Philosophical Magazine **93**,9, 1013-1053 (2013)

Wang, C.-C.: *Inhomogeneities in second-grade fluid bodies and istotropic solid bodies.* Archives of Mechanics **25**,5, 765-780 (1973)

Zheng, Q.-S.; Zou, W.-N.: *Irreducible decompositions of physical tensors of high orders*. Journal of Engineering Mathematics **37**,1-3, 273-288 (2000)