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ORIGINAL ARTICLE

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The thermodynamics of gradient elastoplasticity

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Abstract A thermomechanical framework for the modelling of gradient plasticity is developed within the range of linear strains. Full anisotropy is considered. Special focus is given to the restrictions imposed by the Clausius–Duhem inequality. A rather general example gives a complete anisotropic model and shows its thermodynamic consistency. This is finally particularized for the isotropic case by using isotropic tensor-function representations.

Keywords Gradient plasticity · Thermoplasticity

1 Introduction

In the literature there are several suggestions to enlarge the classical elasticity and plasticity by higher gradients in order to include size effects which are apparent in many situations. For both the experimental motivations and the generalized continuum theories, the reader is referred to the references quoted in the recent books by [29] and [3]. A natural extension of classical plasticity is to include the gradient of the deformation tensor into the list of (observable) independent variables. As a dual concept of this third-order tensor field, one would expect a third-order generalized stress tensor (see [4]).

A large amount of the literature is devoted to the introduction of the gradient of plastic deformation aiming at setting a thermodynamically consistent framework for accommodating Aifantis initial phenomenological ansatz [2]. Fleck and Hutchinson [15] and [25] established balance and constitutive equations for materials incorporating the effect of the gradient of plastic deformation. Related thermodynamical formulations were also discussed recently by [19].

In contrast, less attention has been focused on the elastoplasticity of Toupin and Mindlin's original strain gradient media. Toupin [37] and [30] only envisaged the linear elastic behaviour of the media they had created. The foundations of strain gradient theory were more recently revisited by [38–40] for solids and fluids. These works were followed by [11] who probably developed the first elastoplastic strain gradient theory. Motivations for the development of elastoplastic strain gradient models mainly stem from the mechanics of heterogeneous materials. For instance, [32] and [45] derived effective strain gradient properties for porous materials, whereas [27] considered the effective properties of micro-cracked solids. However, their theories are confined to linear

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strain gradient elasticity with a dependence on pore volume fraction or crack density. An explicit yield function for strain gradient media incorporating both the second- and third-order stress tensors is proposed by [24] derived from a micromechanical analysis. The proposed model is applicable to the simulation of the ductile fracture of metals and alloys including pore size effects and regularization of the plastic strain localization zones, see [12]. Size effects induced by plasticity are also significant in single crystals for which [14] formulated a crystal elastoviscoplasticity theory based on Mindlin's strain gradient continuum. An elastoplastic strain gradient model is also relevant for the modelling of strain localization phenomena in granular media for civil engineering applications as shown by [9,10] and [44]. Finally, it is worth mentioning applications related to shear banding and deformation patterning during metal forming, for instance according to the second-gradient viscoplasticity model by [13]. The formation of strain localization zones in granular media like soils or metal polycrystals provides a clear example of a mechanical situation where the need for a characteristic size arises in relation to the measurable finite width of such bands, generally of the order of 5–10 grain diameters.

All the previously mentioned models are special cases of the thermodynamically consistent formulation of constitutive equations for nonlinear strain gradient media proposed by [17] (see also [33]). This constitutive setting represents an extension of the concepts of continuum thermodynamics involving the definition of free energy functional and dissipation potential (see [23] and [28]). An alternative framework was proposed simultaneously by [34]. More recent contributions to the thermomechanics of elastoviscoplastic strain gradient theories are due to [26] and [35].

In this paper, we adopt a different perspective and generalize the approach of [5] to such gradient materials. Within the frame of plasticity, we decompose the deformation tensor and the gradient of the deformation tensor into its elastic and its plastic parts. The theory is based on the usual assumption of identical thermoelastic behaviour in all elastic ranges, which means that all measurable thermoelastic properties are not affected by plastic deformations. This concept has been introduced by [6,7] in the context of large deformations. In the present work, however, we limit ourselves to small deformations for the sake of simplicity and clarity. Specific features related to the formulation of constitutive equations for strain gradient media at finite deformations can be found in [9,10,17] and [35].

After exploiting this assumption, an example is given, which generalizes the classical J_2-theory to gradient plasticity. Then the restrictions from the second law of thermodynamics are worked out. The example still contains an arbitrary anisotropy. Only in the last section its reduction to the isotropic case is shown.

The intention of this paper is to give a thermodynamical framework for the modelling of gradient effects in elastoplasticity. Even our example is still formulated in a rather general manner, thus leaving enough space for specific models for further research work in the field.

Notations We indicate the order of a material tensor in brackets above like, e.g., $\overset{\langle 4\rangle}{\mathbf{C}}$ for the fourth-order elasticity tensor. For every contraction between tensors, we use one point. The P-fold contraction of a K-fold tensor product with an M-fold one for $K \geq P \leq M$ is defined as the $(K + M - 2P)$-fold tensor product

$$
\begin{aligned}
&(\mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_K) \cdot \ldots \cdot (\mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_M) \\
&\quad := \mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_{K-P} \otimes \mathbf{x}_{P+1} \otimes \ldots \otimes \mathbf{x}_M (\mathbf{v}_{K-P+1} \cdot \mathbf{x}_1)(\mathbf{v}_{K-P+2} \cdot \mathbf{x}_2) \ldots (\mathbf{v}_K \cdot \mathbf{x}_P)
\end{aligned}
$$

with "$\cdot \ldots \cdot$" standing for P product points. A straightforward generalization of this defines the according products between complete higher-order tensors

2 The mechanical theory of gradient plasticity

Before we go into the thermodynamical theory, we want to briefly repeat the basic concepts of a mechanical theory of gradient plasticity which consists of the following ingredients:

1. an *additive decomposition* of the linear strain tensor $\mathbf{E}$ into an elastic and a plastic part

$$\mathbf{E} = \mathbf{E}_e + \mathbf{E}_p \tag{1}$$

and an analogous one of the gradient of the strain tensor.

$$\mathbf{K} := \operatorname{grad} \mathbf{E} = \mathbf{E} \otimes \nabla = \mathbf{K}_e + \mathbf{K}_p \tag{2}$$

with the following symmetry properties: $\mathbf{E} = \mathbf{E}^T$ and $K^{ijk} = K^{jik}$, which also apply to their elastic and plastic parts. It depends on the specific approach whether $\mathbf{E}_p$ and $\mathbf{K}_p$ are considered as independent variables or, alternatively, we set $\mathbf{K}_p = \text{grad } \mathbf{E}_p$ in analogy with (2). Since the first case seems to be more general, we will in the sequel consider this case and only eventually mention the second one.

2. two *elastic laws* taken as linear mappings of the elastic strain tensors into the second- and third-order stress tensors

$$\mathbf{T} = \overset{\langle 4\rangle}{\mathbf{C}}_T \cdot\cdot \mathbf{E}_e + \overset{\langle 5\rangle}{\mathbf{C}}_{TM} \cdot\cdot\cdot \mathbf{K}_e \tag{3}$$

$$\mathbf{M} = \overset{\langle 5\rangle}{\mathbf{C}}_{MT} \cdot\cdot \mathbf{E}_e + \overset{\langle 6\rangle}{\mathbf{C}}_M \cdot\cdot\cdot \mathbf{K}_e \tag{4}$$

(see [30]) by use of a fourth-order elasticity tensor $\overset{\langle 4\rangle}{\mathbf{C}}_T$, two fifth-order elasticity coupling tensors $\overset{\langle 5\rangle}{\mathbf{C}}_{TM}$ and $\overset{\langle 5\rangle}{\mathbf{C}}_{MT}$, and a sixth-order elasticity tensor $\overset{\langle 6\rangle}{\mathbf{C}}_M$. These laws can be isotropic or anisotropic. In the central symmetric case, however, the fifth-order tensors disappear, as we will see below. The underlying assumption of these laws is that the stresses depend only on the elastic variables and are unaffected by plastic deformations.

At any instant these variables have to fulfil the mechanical balance laws (see [30] and [22]), namely

- the balance of linear momentum $\text{div}\,(\mathbf{T} - \text{div}\,\mathbf{M}) + \rho\mathbf{b} = \rho\mathbf{u}^{\bullet\bullet}$
- the balance of moment of momentum $\mathbf{T} = \mathbf{T}^T$

with ρ being the mass density, $\mathbf{b}$ the specific body force, and $\mathbf{u}$ the displacement vector. In the entire text the upper dot stands for the material time derivative.

3. a *yield limit* (yield criterion), which indicates the limit of the current elastic range. The general ansatz for the yield criterion in the strain space is

$$\varphi(\mathbf{E}_e, \mathbf{K}_e, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}) \tag{5}$$

where $\mathbf{H}$ is the vector of additional scalar or tensorial internal variables such as hardening variables. In the sequel we will note it as a second-order tensor without intending to constrain our considerations to this special case. The internal variables introduced in the formulation comprise the classical hardening or damage variables used in material modelling but also new variables related to the strain gradient or the elastic or plastic parts of the strain gradient. An internal variable is characterized by the fact that its evolution law is given by an ODE. An example of such a new internal variables could be the use of the norm of the plastic part of the strain gradient in the evolution of the yield stress, as done for instance in the *mechanism-based strain gradient plasticity* (see [21]) and in the recent models by [42]. The yield limit is the kernel of this function

$$\varphi(\mathbf{E}_e, \mathbf{K}_e, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}) = 0 \tag{6}$$

(*yield condition*), while we assume

$$\varphi(\mathbf{E}_e, \mathbf{K}_e, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}) < 0 \tag{7}$$

(only) in the interior of the elastic range. Necessary and sufficient conditions for the material to yield are the yield condition and the *loading condition*

$$\frac{\partial\varphi}{\partial\mathbf{E}_e} \cdot\cdot \mathbf{E}^\bullet + \frac{\partial\varphi}{\partial\mathbf{K}_e} \cdot\cdot\cdot \mathbf{K}^\bullet > 0 \tag{8}$$

which states a violation of the yield criterion if the total deformation increments would be purely elastic. Our presentation is limited to rate-independent plasticity, but viscoplasticity can be introduced in a straightforward manner in the model, for instance, but not exclusively, by the introduction of a viscoplastic potential from which viscoplastic flow rule and evolution laws for hardening variables are derived (see [17]).

4. *flow rules* which determine the evolution of $\mathbf{E}_p$ and $\mathbf{K}_p$. A general rate-independent ansatz for them would be first-order ODEs depending on practically all variables and the rates of the total deformations

$$\mathbf{E}_p^\bullet = E\big(\mathbf{E}_p, \mathbf{E}_e, \mathbf{K}_p, \mathbf{K}_e, \mathbf{H}, \mathbf{E}^\bullet, \mathbf{K}^\bullet\big) \tag{9}$$

$$\mathbf{K}_p^\bullet = K\big(\mathbf{E}_p, \mathbf{E}_e, \mathbf{K}_p, \mathbf{K}_e, \mathbf{H}, \mathbf{E}^\bullet, \mathbf{K}^\bullet\big). \tag{10}$$

If we, however, would assume $\mathbf{K}_p = \text{grad } \mathbf{E}_p$, then the second flow rule K is not needed since the evolution of $\mathbf{K}_p$ would be given by

$$\mathbf{K}_p^\bullet = \text{grad } E\big(\mathbf{E}_p, \mathbf{E}_e, \mathbf{K}_p, \mathbf{K}_e, \mathbf{H}, \mathbf{E}^\bullet, \mathbf{K}^\bullet\big). \tag{11}$$

In the sequel we will not use Eq. (11) but instead (8) and (9) which we consider as describing the more general case.

5. an evolution equation called *hardening rule* for the additional variable(s) **H,** which is assumed to be of the same form as the flow rules above

$$\mathbf{H}^\bullet = H\big(\mathbf{E}_p, \mathbf{E}_e, \mathbf{K}_p, \mathbf{K}_e, \mathbf{H}, \mathbf{E}^\bullet, \mathbf{K}^\bullet\big). \tag{12}$$

These constitutive laws establish a complete mechanical rate-independent format for a gradient elastoplasticity. In the next section we will generalize this format to thermomechanics by introducing the thermodynamical variables and laws.

3 Thermoplasticity—general theory

For this purpose we have to enlarge the set of purely mechanical variables by the thermodynamical ones. These are the following local quantities:

ε	the specific *internal energy* (a time-dependent scalar)
$\mathbf{q}$	the *heat flux* per unit time and area (a time-dependent vector)
θ	the *absolute temperature* (a time-dependent positive scalar)
$\mathbf{g} := \text{grad } \theta$	the *temperature gradient* (a time-dependent vector)
η	the specific *entropy* (a time-dependent scalar).

By *specific* we always mean with respect to unit mass.

We use the *energy balance* (first law of thermodynamics) in the local form

$$\rho\varepsilon^\bullet = \rho Q + \mathbf{T} \cdot\cdot\, \mathbf{E}^\bullet + \mathbf{M} \cdot\cdot\cdot\, \mathbf{K}^\bullet. \tag{13}$$

with the stress power density $\mathbf{T} \cdot\cdot\, \mathbf{E}^\bullet + \mathbf{M} \cdot\cdot\cdot\, \mathbf{K}^\bullet$ and the heat supply per unit mass and time Q, which results from irradiation r and conduction $\mathbf{q}$ in the usual form

$$Q = r - (\text{div } \mathbf{q})/\rho. \tag{14}$$

By the introduction of the *Helmholtz free energy*

$$\psi := \varepsilon - \theta\eta \tag{15}$$

we assume for the second law the *Clausius–Duhem inequality* in the form

$$\psi^\bullet + \eta\theta^\bullet - \frac{\mathbf{T}}{\rho} \cdot\cdot\, \mathbf{E}^\bullet - \frac{\mathbf{M}}{\rho} \cdot\cdot\cdot\, \mathbf{K}^\bullet + \frac{\mathbf{q}\cdot\mathbf{g}}{\rho\theta} \leq 0. \tag{16}$$

Thus, the specific *dissipation*, which consists of the mechanical dissipation

$$\delta_m := \frac{\mathbf{T}}{\rho} \cdot\cdot\, \mathbf{E}^\bullet + \frac{\mathbf{M}}{\rho} \cdot\cdot\cdot\, \mathbf{K}^\bullet - \psi^\bullet - \eta\theta^\bullet = \theta\eta^\bullet - Q \tag{17}$$

by using Eqs. (13) and (15), and the thermal dissipation

$$\delta_{\text{th}} := -\frac{\mathbf{q}\cdot\mathbf{g}}{\rho\,\theta} \tag{18}$$

fulfils the *dissipation inequality*

$$\delta = \delta_m + \delta_{\mathrm{th}} \geq 0. \tag{19}$$

In order to enlarge the mechanical plasticity theory to a thermomechanical one, we add the temperature and the temperature gradient to the list of independent variables called *thermokinematical variables*, and the heat flux, the entropy, and the internal energy or the free energy to the dependent variables called *caloro-dynamical variables*.

Thermoplastic materials can be understood as material models with *internal variables*. The set of the internal variables contains in the case of plasticity the first- and second-order plastic strains and, eventually, hardening variables. For all of these variables we assume as generalizations of Eqs. (9), (10), and (12) rate-independent evolution equations in the general form

$$\mathbf{E}_p^\bullet = E\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}, \mathbf{E}^\bullet, \mathbf{K}^\bullet, \theta^\bullet, \mathbf{g}^\bullet\right) \tag{20}$$

$$\mathbf{K}_p^\bullet = K\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}, \mathbf{E}^\bullet, \mathbf{K}^\bullet, \theta^\bullet, \mathbf{g}^\bullet\right) \tag{21}$$

$$\mathbf{H}^\bullet = H\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}, \mathbf{E}^\bullet, \mathbf{K}^\bullet, \theta^\bullet, \mathbf{g}^\bullet\right). \tag{22}$$

Again, K will only be needed if we consider $\mathbf{K}_p$ as an internal variable independent of $\mathbf{E}_p$. The set of additional constitutive equations for a thermomechanical material with internal variables is assumed to be

$$\begin{aligned}
\mathbf{T} &= T\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\right)\\
\mathbf{M} &= M\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\right)\\
\mathbf{q} &= q\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\right)\\
\eta &= \eta\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\right)\\
\varepsilon &= \varepsilon\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\right)\\
\text{or}\quad \psi &= \psi\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\right).
\end{aligned} \tag{23}$$

Instead of elastic ranges, we will now have to deal with *thermoelastic ranges*. These are specified by a *yield criterion*

$$\varphi\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\right) \tag{24}$$

which induces the *yield condition*

$$\varphi\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\right) = 0 \tag{25}$$

and the *loading condition*

$$\frac{\partial \varphi}{\partial \mathbf{E}_e} \cdot\cdot\, \mathbf{E}^\bullet + \frac{\partial \varphi}{\partial \mathbf{K}_e} \cdot\cdot\cdot\, \mathbf{K}^\bullet + \frac{\partial \varphi}{\partial \theta}\theta^\bullet + \frac{\partial \varphi}{\partial \mathbf{g}} \cdot \mathbf{g}^\bullet > 0. \tag{26}$$

If both conditions are not simultaneously fulfilled, the event is (thermo)*elastic* and, hence, the plastic variables do not evolve

$$\mathbf{E}_p^\bullet \equiv \mathbf{0} \qquad \mathbf{K}_p^\bullet \equiv \mathbf{0} \qquad \mathbf{H}^\bullet \equiv \mathbf{0} \tag{27}$$

which are side conditions of the functions E, K, and H. Otherwise, it is a *plastic event* or an event of *yielding*, in which the plastic variables necessarily have to evolve according to (20)–(22).

We will next investigate the restrictions imposed on the constitutive equations by the Clausius–Duhem inequality. With the free energy (23.6) we obtain for the inequality (16)

$$\begin{aligned}
0 \geq\ & \frac{\partial \psi}{\partial \mathbf{E}_e} \cdot\cdot\, \mathbf{E}_e^\bullet + \frac{\partial \psi}{\partial \mathbf{K}_e} \cdot\cdot\cdot\, \mathbf{K}_e^\bullet + \frac{\partial \psi}{\partial \theta}\theta^\bullet + \frac{\partial \psi}{\partial \mathbf{g}} \cdot \mathbf{g}^\bullet + \frac{\partial \psi}{\partial \mathbf{E}_p} \cdot\cdot\, \mathbf{E}_p^\bullet + \frac{\partial \psi}{\partial \mathbf{K}_p} \cdot\cdot\cdot\, \mathbf{K}_p^\bullet\\
& + \frac{\partial \psi}{\partial \mathbf{H}} \cdot\cdot\, \mathbf{H}^\bullet + \eta\theta^\bullet - \frac{\mathbf{T}}{\rho} \cdot\cdot \left(\mathbf{E}_e^\bullet + \mathbf{E}_p^\bullet\right) - \frac{\mathbf{M}}{\rho} \cdot\cdot\cdot \left(\mathbf{K}_e^\bullet + \mathbf{K}_p^\bullet\right) + \frac{\mathbf{q}\cdot\mathbf{g}}{\rho\theta}\\
=\ & \left(\frac{\partial \psi}{\partial \mathbf{E}_e} - \frac{\mathbf{T}}{\rho}\right) \cdot\cdot\, \mathbf{E}_e^\bullet + \left(\frac{\partial \psi}{\partial \mathbf{K}_e} - \frac{\mathbf{M}}{\rho}\right) \cdot\cdot\cdot\, \mathbf{K}_e^\bullet + \left(\frac{\partial \psi}{\partial \theta} + \eta\right)\theta^\bullet + \frac{\partial \psi}{\partial \mathbf{g}} \cdot \mathbf{g}^\bullet\\
& + \left(\frac{\partial \psi}{\partial \mathbf{E}_p} - \frac{\mathbf{T}}{\rho}\right) \cdot\cdot\, \mathbf{E}_p^\bullet + \left(\frac{\partial \psi}{\partial \mathbf{K}_p} - \frac{\mathbf{M}}{\rho}\right) \cdot\cdot\cdot\, \mathbf{K}_p^\bullet + \frac{\partial \psi}{\partial \mathbf{H}} \cdot\cdot\, \mathbf{H}^\bullet + \frac{\mathbf{q}\cdot\mathbf{g}}{\rho\theta}.
\end{aligned} \tag{28}$$

If we first consider elastic events, then there remains only the inequality

$$0 \geq \left(\frac{\partial\psi}{\partial \mathbf{E}_e} - \frac{\mathbf{T}}{\rho}\right) \cdot\cdot\, \mathbf{E}_e^\bullet + \left(\frac{\partial\psi}{\partial \mathbf{K}_e} - \frac{\mathbf{M}}{\rho}\right) \cdot\cdot\cdot\, \mathbf{K}_e^\bullet + \left(\frac{\partial\psi}{\partial\theta} + \eta\right)\theta^\bullet + \frac{\partial\psi}{\partial \mathbf{g}} \cdot \mathbf{g}^\bullet + \frac{\mathbf{q}\cdot\mathbf{g}}{\rho\theta}. \tag{29}$$

The exploitation of this inequality for states below the yield limit, where $\mathbf{E}_e^\bullet$, $\mathbf{K}_e^\bullet$, and $\theta^\bullet$ are not restricted, leads by standard arguments to the conditions from gradient thermoelasticity, namely the independence of the free energy of the temperature gradient

$$\psi = \psi\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \tag{30}$$

instead of (23.6), and the *thermoelastic potentials*

$$\mathbf{T} = \rho\frac{\partial\psi}{\partial \mathbf{E}_e} = T\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \tag{31}$$

$$\mathbf{M} = \rho\frac{\partial\psi}{\partial \mathbf{K}_e} = M\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \tag{32}$$

$$\eta = \frac{\partial\psi}{\partial\theta} = \eta\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \tag{33}$$

as well as the *heat conduction inequality*

$$-\,\mathbf{q}\cdot\mathbf{g} \geq 0 \tag{34}$$

as necessary and sufficient conditions for the second law to hold during elastic events. Note that the consideration of a dependence of the free energy function on the gradient of the temperature therefore requires an extension of the present theory which has been proposed in the context of linear strain gradient thermoelasticity by [16], see also the formulations by [18] and [42].

Because of continuity, restrictions (30)–(34) must also hold when reaching the yield limit. If yielding occurs, however, then the additional terms of the Clausius–Duhem inequality (28) must fulfil

$$0 \geq \left(\frac{\partial\psi}{\partial \mathbf{E}_p} - \frac{\mathbf{T}}{\rho}\right) \cdot\cdot\, \mathbf{E}_p^\bullet + \left(\frac{\partial\psi}{\partial \mathbf{K}_p} - \frac{\mathbf{M}}{\rho}\right) \cdot\cdot\cdot\, \mathbf{K}_p^\bullet + \frac{\partial\psi}{\partial \mathbf{H}} \cdot\cdot\, \mathbf{H}^\bullet + \frac{\mathbf{q}\cdot\mathbf{g}}{\rho\theta}. \tag{35}$$

This inequality must be valid for any value of the temperature gradient $\mathbf{g}$ and, hence, for an arbitrarily large absolute value of $\mathbf{q}\cdot\mathbf{g}$. Since the other terms of the inequality do not depend on $\mathbf{g}$, we obtain again the heat conduction inequality (34) separately, and the *residual dissipation inequality* in the form

$$\left(\frac{\mathbf{T}}{\rho} - \frac{\partial\psi}{\partial \mathbf{E}_p}\right) \cdot\cdot\, \mathbf{E}_p^\bullet + \left(\frac{\mathbf{M}}{\rho} - \frac{\partial\psi}{\partial \mathbf{K}_p}\right) \cdot\cdot\cdot\, \mathbf{K}_p^\bullet - \frac{\partial\psi}{\partial \mathbf{H}} \cdot\cdot\, \mathbf{H}^\bullet \geq 0 \tag{36}$$

as a restriction of the yield criterion and the flow and hardening rules. The terms $\rho\frac{\partial\psi}{\partial \mathbf{E}_p}$ and $\rho\frac{\partial\psi}{\partial \mathbf{K}_p}$ can be interpreted as *back stresses*, while the terms in brackets contain the *effective stresses*.

4 Identical thermoelastic behaviour

Up to this point, our analysis has been rather general. In order to further specify this setting for gradient elastoplastic materials, we introduce another important assumption, which is the generalization of the assumption previously mentioned of identical elastic behaviour within all elastic ranges.

Assumption The thermoelastic behaviour within all thermoelastic ranges of the elastoplastic material is identical.

This means more precisely that during elastic events all measurable quantities like the stresses, the heat flux, and the heat supply depend only upon the elastic strains and the temperature and its gradient, but not upon the plastic variables $\mathbf{E}_p, \mathbf{K}_p,$ and $\mathbf{H}$. The consequences of this assumption shall be investigated next.

We obtain for elastic events zero mechanical dissipation and hence from Eq. (17)

$$\delta_m = 0 \Rightarrow Q = \theta \eta^{\bullet} \tag{37}$$

with the local heat supply Q, which we consider as a measurable quantity, at least in principle. With Eq. (33) the right-hand side becomes for elastic events with $\mathbf{E} = \mathbf{E}_e$ and $\mathbf{K} = \mathbf{K}_e$

$$\theta \eta^{\bullet} = \theta \left(\frac{\partial \eta}{\partial \mathbf{E}_e} \cdot\cdot \, \mathbf{E}^{\bullet} + \frac{\partial \eta}{\partial \mathbf{K}_e} \cdot\cdot\cdot \, \mathbf{K}^{\bullet} + \frac{\partial \eta}{\partial \theta} \theta^{\bullet} \right) \tag{38}$$

which shall be independent of $\mathbf{E}_p$, $\mathbf{K}_p$, and $\mathbf{H}$. This can only be the case if the following decomposition of the entropy exists

$$\eta\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) = \eta_e(\mathbf{E}_e, \mathbf{K}_e, \theta) + \eta_p\big(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big)\,. \tag{39}$$

After the above assumption, also the stresses shall not depend on the plastic variables. Hence, we have the reduced stress laws

$$\mathbf{T} = T(\mathbf{E}_e, \mathbf{K}_e, \theta) \tag{40}$$

$$\mathbf{M} = M(\mathbf{E}_e, \mathbf{K}_e, \theta) \tag{41}$$

and instead of Eqs. (31) and (32)

$$T(\mathbf{E}_e, \mathbf{K}_e, \theta) = \rho \frac{\partial \psi}{\partial \mathbf{E}_e} \tag{42}$$

$$M(\mathbf{E}_e, \mathbf{K}_e, \theta) = \rho \frac{\partial \psi}{\partial \mathbf{K}_e} \tag{43}$$

Instead of Eqs. (33) we obtain with (39)

$$\eta_e(\mathbf{E}_e, \mathbf{K}_e, \theta) + \eta_p\big(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) = -\frac{\partial \psi}{\partial \theta}, \tag{44}$$

which leads to the split of the free energy

$$\psi = \psi_e(\mathbf{E}_e, \mathbf{K}_e, \theta) - \theta\ \eta_p\big(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) + \varepsilon_p\big(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \tag{45}$$

such that

$$\eta_e(\mathbf{E}_e, \mathbf{K}_e, \theta) = -\frac{\partial \psi_e(\mathbf{E}_e, \mathbf{K}_e, \theta)}{\partial \theta},$$

and with an (up to now) arbitrary function ε_p of the plastic variables $\mathbf{E}_p$, $\mathbf{K}_p$, and $\mathbf{H}$.

The internal energy follows from Eq. (15)

$$\varepsilon = \varepsilon_e(\mathbf{E}_e, \mathbf{K}_e, \theta) + \varepsilon_p\big(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \tag{46}$$

with

$$\varepsilon_e(\mathbf{E}_e, \mathbf{K}_e, \theta) := \psi_e(\mathbf{E}_e, \mathbf{K}_e, \theta) + \theta\ \eta_e(\mathbf{E}_e, \mathbf{K}_e, \theta)\,. \tag{47}$$

After the above assumption, the heat conduction can neither depend on the plastic variables $\mathbf{E}_p$, $\mathbf{K}_p$, and $\mathbf{H}$, which gives rise to the reduced form of Eq. (23.3)

$$\mathbf{q} = q(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}). \tag{48}$$

In all of these (thermoelastic) constitutive laws, no rates of deformations or temperature appear. They are, thus, always valid, i.e., for both elastic and plastic events. We state these findings in the following.

Theorem *The assumption of equal thermoelastic behaviour in all elastic ranges is fulfilled if and only if the following representations hold for the*

entropy	$\eta = -\frac{\partial \psi_e(\mathbf{E}_e, \mathbf{K}_e, \theta)}{\partial \theta} + \eta_p(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H})$
internal energy	$\varepsilon = \psi_e(\mathbf{E}_e, \mathbf{K}_e, \theta) - \theta \frac{\partial \psi_e(\mathbf{E}_e, \mathbf{K}_e, \theta)}{\partial \theta} + \varepsilon_p(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H})$
free energy	$\psi = \psi_e(\mathbf{E}_e, \mathbf{K}_e, \theta) - \theta \eta_p(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H}) + \varepsilon_p(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H})$
heat conduction	$\mathbf{q} = q(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g})$
2nd order stresses	$\mathbf{T} = \rho \frac{\partial \psi_e(\mathbf{E}_e, \mathbf{K}_e, \theta)}{\partial \mathbf{E}_e}$
3rd order stresses	$\mathbf{M} = \rho \frac{\partial \psi_e(\mathbf{E}_e, \mathbf{K}_e, \theta)}{\partial \mathbf{K}_e}$.

Accordingly, the following constitutive equations completely constitute the thermoplastic material model

the elastic free energy	$\psi_e(\mathbf{E}_e, \mathbf{K}_e, \theta)$
the plastic entropy	$\eta_p(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H})$
the plastic internal energy	$\varepsilon_p(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H})$
the heat conduction law	$q(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g})$
the yield criterion	$\varphi(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H})$

together with the flow and hardening rules to be considered later.

Equation (45) gives for the residual dissipation inequality (36)

$$0 \leq \left(\frac{\mathbf{T}}{\rho} - \frac{\partial \varepsilon_p}{\partial \mathbf{E}_p} + \theta \frac{\partial \eta_p}{\partial \mathbf{E}_p}\right) \cdot\cdot\, \mathbf{E}_p^\bullet + \left(\frac{\mathbf{M}}{\rho} - \frac{\partial \varepsilon_p}{\partial \mathbf{K}_p} + \theta \frac{\partial \eta_p}{\partial \mathbf{K}_p}\right) \cdot\cdot\cdot\, \mathbf{K}_p^\bullet + \left(-\frac{\partial \varepsilon_p}{\partial \mathbf{H}} + \theta \frac{\partial \eta_p}{\partial \mathbf{H}}\right) \cdot\cdot\, \mathbf{H}^\bullet$$
$$= \frac{\mathbf{T}}{\rho} \cdot\cdot\, \mathbf{E}_p^\bullet + \frac{\mathbf{M}}{\rho} \cdot\cdot\cdot\, \mathbf{K}_p^\bullet + \theta \eta_p(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H})^\bullet - \varepsilon_p(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H})^\bullet . \tag{49}$$

In order to determine the change of the temperature of the material point under consideration, we use the first law of thermodynamics Eq. (13) together with Eqs. (1) and (2) and (46)

$$Q = \varepsilon^\bullet - \frac{\mathbf{T}}{\rho} \cdot\cdot\, \mathbf{E}^\bullet - \frac{\mathbf{M}}{\rho} \cdot\cdot\cdot\, \mathbf{K}^\bullet = Q_e + Q_p \tag{50}$$

with a split of the heat supply into an elastic part

$$Q_e\left(\mathbf{E}_e, \theta, \mathbf{E}_e^\bullet, \mathbf{K}_e^\bullet, \theta^\bullet\right) := \varepsilon_e(\mathbf{E}_e, \mathbf{K}_e, \theta)^\bullet - \frac{\mathbf{T}}{\rho} \cdot\cdot\, \mathbf{E}_e^\bullet - \frac{\mathbf{M}}{\rho} \cdot\cdot\cdot\, \mathbf{K}_e^\bullet \tag{51}$$

with (37)

$$= \theta \eta_e(\mathbf{E}_e, \mathbf{K}_e, \theta)^\bullet$$
$$= c\, \theta^\bullet - \theta \frac{1}{\rho}\left(\overset{\langle 2\rangle}{\mathsf{R}} \cdot\cdot\, \mathbf{E}_e^\bullet - \overset{\langle 3\rangle}{\mathsf{S}} \cdot\cdot\cdot\, \mathbf{K}_e^\bullet\right)$$

with

the *specific heat*	$c\,(\mathbf{E}_e, \mathbf{K}_e, \theta) := \theta \frac{\partial \eta_e}{\partial \theta}$
the 2*nd order stress-temperature tensor*	$\overset{\langle 2\rangle}{\mathsf{R}}(\mathbf{E}_e, \mathbf{K}_e, \theta) := -\rho \frac{\partial \eta_e}{\partial \mathbf{E}_e}$
the 3*rd order stress-temperature tensor*	$\overset{\langle 3\rangle}{\mathsf{S}}(\mathbf{E}_e, K_e, \theta) := -\rho \frac{\partial \eta_e}{\partial \mathbf{K}_e}$

and a plastic part

$$Q_p := \varepsilon_p(\mathbf{E}_p, \mathbf{K}_p, \mathbf{H})^\bullet - \frac{\mathbf{T}}{\rho} \cdot\cdot\, \mathbf{E}_p^\bullet - \frac{\mathbf{M}}{\rho} \cdot\cdot\cdot\, \mathbf{K}_p^\bullet . \tag{52}$$

Equation (51) can be solved for

$$c\theta^{\bullet} = Q - Q_p + \theta\frac{1}{\rho}\Big(\overset{\langle 2\rangle}{\mathbf{R}} \cdot\cdot\, \mathbf{E}_e^{\bullet} + \overset{\langle 3\rangle}{\mathbf{S}} \cdot\cdot\cdot\, \mathbf{K}_e^{\bullet}\Big). \tag{53}$$

By this equation, we can integrate the temperature along the process and so determine the final temperature after some elastoplastic process. Accordingly, temperature changes are caused by

1. the heat supply Q from the outside
2. the heat $-Q_p$ generated by plastic yielding and hardening, and
3. thermoelastic transformations due to the last term in Eq. (53).

We now specify the ansatz for the rate-independent evolution Eqs. (20)–(22) for the plastic variables, namely the flow and the hardening rules in the following form

$$\mathbf{E}_p^{\bullet} = \lambda E\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \tag{54}$$
$$\mathbf{K}_p^{\bullet} = \lambda K\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \tag{55}$$
$$\mathbf{H}^{\bullet} = \lambda H\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \tag{56}$$

with a joint *plastic consistency parameter* λ and three functions E, K, and H of the listed arguments. λ is zero if and only if no yielding occurs, i.e., during thermoelastic events. Note that the case of multiple yield functions and corresponding plastic multipliers as proposed by [17] is not envisaged in the present work for the sake of conciseness. During yielding, however, λ is positive. In all cases the Kuhn–Tucker conditions hold in the form

$$\lambda\varphi = 0 \quad \text{with} \quad \lambda \geq 0 \quad \text{and} \quad \varphi \leq 0. \tag{57}$$

The plastic parameter can be determined during yielding by the *consistency condition* as a consequence of the yield condition Eq. (25)

$$\begin{aligned} 0 = {} & \frac{\partial\varphi}{\partial\mathbf{E}_e} \cdot\cdot\, \mathbf{E}_e^{\bullet} + \frac{\partial\varphi}{\partial\mathbf{K}_e} \cdot\cdot\cdot\, \mathbf{K}_e^{\bullet} + \frac{\partial\varphi}{\partial\theta}\theta^{\bullet} + \frac{\partial\varphi}{\partial\mathbf{g}} \cdot \mathbf{g}^{\bullet} \\ & + \frac{\partial\varphi}{\partial\mathbf{E}_p} \cdot\cdot\, \lambda\, E\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \\ & + \frac{\partial\varphi}{\partial\mathbf{K}_p} \cdot\cdot\cdot\, \lambda\, K\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \\ & + \frac{\partial\varphi}{\partial\mathbf{H}} \cdot\cdot\, \lambda\, H\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \end{aligned} \tag{58}$$

which can be solved for the plastic parameter

$$\lambda = \alpha^{-1}\left(\frac{\partial\varphi}{\partial\mathbf{E}_e} \cdot\cdot\, \mathbf{E}_e^{\bullet} + \frac{\partial\varphi}{\partial\mathbf{K}_e} \cdot\cdot\, \mathbf{K}_e^{\bullet} + \frac{\partial\varphi}{\partial\theta}\theta^{\bullet} + \frac{\partial\varphi}{\partial\mathbf{g}} \cdot \mathbf{g}^{\bullet}\right) \tag{59}$$

with a scalar denominator

$$\begin{aligned} \alpha\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) := {} & -\frac{\partial\varphi}{\partial\mathbf{E}_p} \cdot\cdot\, E\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) - \frac{\partial\varphi}{\partial\mathbf{K}_p} \cdot\cdot\cdot\, K\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\big) \\ & - \frac{\partial\varphi}{\partial\mathbf{H}} \cdot\cdot\, H(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}). \end{aligned} \tag{60}$$

Due to the loading condition Eq. (26), $\lambda\alpha$ must be positive during plastic events. After (57), λ alone is positive, and so α must also be positive. After Eq. (60) this is a restriction to the functions E, K, H, and the yield criterion.

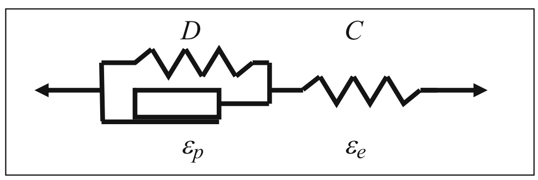
Another restriction to these functions is obtained by the second law. We substitute Eqs. (54)–(56) into the residual dissipation inequality (49)

$$\left(\mathbf{T} - \rho\frac{\partial \varepsilon_p}{\partial \mathbf{E}_p} + \rho\theta\frac{\partial \eta_p}{\partial \mathbf{E}_p}\right)\cdot\cdot\, E\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\right)$$
$$+\left(\mathbf{M} - \rho\frac{\partial \varepsilon_p}{\partial \mathbf{K}_p} + \rho\theta\frac{\partial \eta_p}{\partial \mathbf{K}_p}\right)\cdots K\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\right)$$
$$+\left(-\rho\frac{\partial \varepsilon_p}{\partial \mathbf{H}} + \rho\,\theta\frac{\partial \eta_p}{\partial \mathbf{H}}\right)\cdot\cdot\, H\left(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, \mathbf{H}\right) \geq 0. \tag{61}$$

This restriction to the functions E, K, H, and the yield criterion will be considered in the following example.

5 Example

In the sequel we will discuss the foregoing framework for gradient plasticity by means of a simple example, which is a generalization of the one in [5]. It is based on a one-dimensional Prager model with two springs and a St.Venant element which stands for dry friction. This model performs an elastoplastic behaviour with linear kinematical hardening induced by the spring D while the stresses can be determined by the strains in the spring C. We will generalize this model into three dimensions with full anisotropy first and finally particularize it to the isotropic case.



Within the format of a linear theory, the elastic part of the free energy has a quadratic form familiar from linear thermoelasticity

$$\rho\psi_e(\mathbf{E}_e, \mathbf{K}_e, \theta) = \frac{1}{2}\,\mathbf{E}_e\cdot\cdot\overset{\langle 4\rangle}{\mathbf{C}}_T\cdot\cdot\,\mathbf{E}_e + \frac{1}{2}\,\mathbf{K}_e\cdots\overset{\langle 6\rangle}{\mathbf{C}}_M\cdots\mathbf{K}_e + \mathbf{E}_e\cdot\cdot\overset{\langle 5\rangle}{\mathbf{C}}_{TM}\cdots\mathbf{K}_e$$
$$+c\,\rho\left(\Delta\theta - \theta ln\frac{\theta}{\theta_0}\right) + \Delta\theta\left(\overset{\langle 2\rangle}{\mathbf{R}}\cdot\cdot\,\mathbf{E}_e + \overset{\langle 3\rangle}{\mathbf{S}}\cdots\mathbf{K}_e\right) \tag{62}$$

with higher-order elasticity tensors $\overset{\langle 4\rangle}{\mathbf{C}}_T, \overset{\langle 6\rangle}{\mathbf{C}}_M, \overset{\langle 5\rangle}{\mathbf{C}}_{TM}$, two constant stress-temperature tensors $\overset{\langle 2\rangle}{\mathbf{R}}$ and $\overset{\langle 3\rangle}{\mathbf{S}}$, a constant specific heat c, a reference temperature θ_0, and the deviation from it $\Delta\theta := \theta - \theta_0$.

Since all material parameters are taken as constant, this model applies only to moderate temperature changes and small deformations.

We obtain with Eqs. (42), (43) the thermoelastic laws for the stresses

$$\mathbf{T} = \overset{\langle 4\rangle}{\mathbf{C}}_T\cdot\cdot\,\mathbf{E}_e + \overset{\langle 5\rangle}{\mathbf{C}}_{TM}\cdots\mathbf{K}_e + \Delta\theta\,\overset{\langle 2\rangle}{\mathbf{R}} \tag{63}$$
$$\mathbf{M} = \overset{\langle 5\rangle}{\mathbf{C}}_{TM}{}^{*}\cdot\cdot\,\mathbf{E}_e + \overset{\langle 6\rangle}{\mathbf{C}}_M\cdots\mathbf{K}_e + \Delta\theta\,\overset{\langle 3\rangle}{\mathbf{S}} \tag{64}$$

in analogy with Eqs. (3), (4), where the superimposed asterisk indicates that particular transposition of a fifth-order tensor which fulfils

$$\mathbf{M}\cdots\overset{\langle 5\rangle}{\mathbf{C}}{}^{*}\cdot\cdot\,\mathbf{T} = \mathbf{T}\cdot\cdot\overset{\langle 5\rangle}{\mathbf{C}}\cdots\mathbf{M}$$

for every second-order tensor $\mathbf{T}$ and every third-order tensor $\mathbf{M}$.

With Eq. (44) we get for the elastic entropy

$$\rho\ \eta_e(\mathbf{E}_e,\mathbf{K}_e,\theta)\ = -\overset{\langle 2\rangle}{\mathbf{R}}\cdot\cdot\,\mathbf{E}_e - \overset{\langle 3\rangle}{\mathbf{S}}\cdot\cdot\cdot\,\mathbf{K}_e + \rho\ c\ ln\frac{\theta}{\theta_0} \tag{65}$$

and after Eq. (47) for the elastic internal energy

$$\begin{aligned}&\rho\varepsilon_e(\mathbf{E}_e,\mathbf{K}_e,\theta)\\ &= \frac{1}{2}\mathbf{E}_e\cdot\cdot\overset{\langle 4\rangle}{\mathbf{C}}_T\cdot\cdot\,\mathbf{E}_e + \frac{1}{2}\mathbf{K}_e\cdot\cdot\cdot\overset{\langle 6\rangle}{\mathbf{C}}_M\cdot\cdot\cdot\,\mathbf{K}_e\\ &+\mathbf{E}_e\cdot\cdot\overset{\langle 5\rangle}{\mathbf{C}}_{TM}\cdot\cdot\cdot\,\mathbf{K}_e + c\ \rho\ \Delta\ \theta - \theta_0\Big(\overset{\langle 2\rangle}{\mathbf{R}}\cdot\cdot\,\mathbf{E}_e + \overset{\langle 3\rangle}{\mathbf{S}}\cdot\cdot\cdot\,\mathbf{K}_e\Big).\end{aligned} \tag{66}$$

For the heat flux we choose a Fourier-type law

$$\mathbf{q} = -\overset{\langle 2\rangle}{\mathbf{F}}\cdot\mathbf{g} \tag{67}$$

with a second-order positive semi-definite tensor $\overset{\langle 2\rangle}{\mathbf{F}}$ so that the heat conduction inequality is always fulfilled.

In accordance with the Prager model, we write for the back stresses two elastic laws in the plastic strains

$$\mathbf{T}_B = \overset{\langle 4\rangle}{\mathbf{D}}_E\ \cdot\cdot\,\mathbf{E}_p + \overset{\langle 5\rangle}{\mathbf{D}}_{KE}\ \cdot\cdot\cdot\ \mathbf{K}_p \tag{68}$$

$$\mathbf{M}_B = \overset{\langle 6\rangle}{\mathbf{D}}_K\cdot\cdot\cdot\,\mathbf{K}_p + \overset{\langle 5\rangle}{\mathbf{D}}_{KE}{}^{*}\cdot\cdot\,\mathbf{E}_p \tag{69}$$

with material tensors $\overset{\langle 4\rangle}{\mathbf{D}}_{E,}\overset{\langle 6\rangle}{\mathbf{D}}_{K,}\overset{\langle 5\rangle}{\mathbf{D}}_{KE}$ with the usual symmetry properties.

We define the specific *plastic work* in some time interval $[t_{0,}t_1]$ as the work of the effective stresses upon the plastic deformations

$$\begin{aligned}w_p &:= \int_{t_0}^{t_1}\Big(\frac{\mathbf{T}-\mathbf{T}_B}{\rho}\cdot\cdot\,\mathbf{E}_p^{\bullet} + \frac{\mathbf{M}-\mathbf{M}_B}{\rho}\cdot\cdot\cdot\,\mathbf{K}_p^{\bullet}\Big)dt\\ \Rightarrow\ w_p^{\bullet} &:= \frac{\mathbf{T}-\mathbf{T}_B}{\rho}\cdot\cdot\,\mathbf{E}_p^{\bullet} + \frac{\mathbf{M}-\mathbf{M}_B}{\rho}\cdot\cdot\cdot\,\mathbf{K}_p^{\bullet}\end{aligned} \tag{70}$$

as the only hardening variable in our example.

The plastic part of the entropy is set

$$\eta_p\big(\mathbf{E}_p,\mathbf{K}_p,\mathbf{H}\big) = 0. \tag{71}$$

We will later show under which restrictions this ansatz will satisfy the residual dissipation inequality and also that other non-trivial choices for the plastic entropy will work as well.

We also assume a quadratic form for the plastic part of the internal energy

$$\begin{aligned}\rho\ \varepsilon_p\big(\mathbf{E}_p,\mathbf{K}_p,\mathbf{H}\big) &= \frac{1}{2}\ \mathbf{E}_p\cdot\cdot\overset{\langle 4\rangle}{\mathbf{D}}_E\cdot\cdot\,\mathbf{E}_p + \frac{1}{2}\mathbf{K}_p\cdot\cdot\cdot\overset{\langle 6\rangle}{\mathbf{D}}_K\cdot\cdot\cdot\,\mathbf{K}_p\\ &+\mathbf{E}_p\cdot\cdot\overset{\langle 5\rangle}{\mathbf{D}}_{KE}\cdot\cdot\cdot\,\mathbf{K}_p + \rho\mu w_p\end{aligned} \tag{72}$$

with a scalar coefficient μ.

In accordance with Eqs. (36), (45), and (71), we can in fact verify the ansatz (68) and (69) for the back stresses.

We use a generalization of the anisotropic v. Mises yield criterion [31] in the stress space

$$\begin{aligned}\varphi = (\mathbf{T}-\mathbf{T}_B)\cdot\cdot\overset{\langle 4\rangle}{\mathbf{G}}\cdot\cdot(\mathbf{T}-\mathbf{T}_B) + (\mathbf{M}-\mathbf{M}_B)\cdot\cdot\cdot\overset{\langle 6\rangle}{\mathbf{G}}\cdot\cdot\cdot(\mathbf{M}-\mathbf{M}_B)\\ +(\mathbf{T}-\mathbf{T}_B)\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}}\cdot\cdot\cdot(\mathbf{M}-\mathbf{M}_B) - \sigma_Y(w_p,\theta,\mathbf{g})^2\end{aligned} \tag{73}$$

with three material tensors $\overset{\langle 4\rangle}{\mathsf{G}}$, $\overset{\langle 5\rangle}{\mathsf{G}}$, and $\overset{\langle 6\rangle}{\mathsf{G}}$ reflecting the symmetry of the material, and a scalar yield stress σ_Y depending on the plastic work, the temperature, and the temperature gradient. To the best of our knowledge, a direct dependence of the yield function on the temperature gradient has not yet been proposed in the literature. However, strongly coupled gradient thermomechanical models have been recently developed for the viscoplasticity of laser material processing techniques involving strong strain and temperature gradients (see [43]).

Uniqueness and regularity of the model are ensured if the yield function is convex with respect to all its arguments, including the newly introduced higher-order strain (or stress). In the proposed quadratic potential (73), convexity of the yield surface is ensured if the linear elastic mapping is positive definite.

If the stresses and the back stresses are substituted by Eqs. (63)–(64), (68)–(69), the assumed form of the yield criterion can in fact be transformed into the strain space

$$
\begin{aligned}
&\varphi\big(\mathbf{E}_e, \mathbf{K}_e, \theta, \mathbf{g}, \mathbf{E}_p, \mathbf{K}_p, w_p\big)\\
&= \Big(\overset{\langle 4\rangle}{\mathsf{C}}_T \cdot\cdot\, \mathbf{E}_e + \overset{\langle 5\rangle}{\mathsf{C}}_{TM} \cdot\cdot\cdot\, \mathbf{K}_e + \Delta\theta\, \overset{\langle 2\rangle}{\mathsf{R}} - \overset{\langle 4\rangle}{\mathsf{D}}_E \cdot\cdot\, \mathbf{E}_p - \overset{\langle 5\rangle}{\mathsf{D}}_{KE} \cdot\cdot\cdot\, \mathbf{K}_p\Big) \cdot\cdot\, \overset{\langle 4\rangle}{\mathsf{G}}\\
&\quad \cdot\cdot \Big(\overset{\langle 4\rangle}{\mathsf{C}}_T \cdot\cdot\, \mathbf{E}_e + \overset{\langle 5\rangle}{\mathsf{C}}_{TM} \cdot\cdot\cdot\, \mathbf{K}_e + \Delta\theta\, \overset{\langle 2\rangle}{\mathsf{R}} - \overset{\langle 4\rangle}{\mathsf{D}}_E \cdot\cdot\, \mathbf{E}_p - \overset{\langle 5\rangle}{\mathsf{D}}_{KE} \cdot\cdot\cdot\, \mathbf{K}_p\Big)\\
&\quad + \Big(\mathbf{E}_e \cdot\cdot\, \overset{\langle 5\rangle}{\mathsf{C}}_{TM} + \overset{\langle 6\rangle}{\mathsf{C}}_M \cdot\cdot\cdot\, \mathbf{K}_e + \Delta\theta\, \overset{\langle 3\rangle}{\mathsf{S}} - \overset{\langle 6\rangle}{\mathsf{D}}_K \cdot\cdot\cdot\, \mathbf{K}_p - \mathbf{E}_p \cdot\cdot\, \overset{\langle 5\rangle}{\mathsf{D}}_{KE}\Big) \cdot\cdot\cdot\, \overset{\langle 6\rangle}{\mathsf{G}}\\
&\quad \cdot\cdot\cdot \Big(\mathbf{E}_e \cdot\cdot\, \overset{\langle 5\rangle}{\mathsf{C}}_{TM} + \overset{\langle 6\rangle}{\mathsf{C}}_M \cdot\cdot\cdot\, \mathbf{K}_e + \Delta\,\theta\, \overset{\langle 3\rangle}{\mathsf{S}} - \overset{\langle 6\rangle}{\mathsf{D}}_K \cdot\cdot\cdot\, \mathbf{K}_p - \mathbf{E}_p \cdot\cdot\, \overset{\langle 5\rangle}{\mathsf{D}}_{KE}\Big)\\
&\quad + \Big(\overset{\langle 4\rangle}{\mathsf{C}}_T \cdot\cdot\, \mathbf{E}_e + \overset{\langle 5\rangle}{\mathsf{C}}_{TM} \cdot\cdot\cdot\, \mathbf{K}_e + \Delta\,\theta\, \overset{\langle 2\rangle}{\mathsf{R}} - \overset{\langle 4\rangle}{\mathsf{D}}_E \cdot\cdot\, \mathbf{E}_p - \overset{\langle 5\rangle}{\mathsf{D}}_{KE} \cdot\cdot\cdot\, \mathbf{K}_p\Big) \cdot\cdot\, \overset{\langle 5\rangle}{\mathsf{G}}\\
&\quad \cdot\cdot\cdot \Big(\mathbf{E}_e \cdot\cdot\, \overset{\langle 5\rangle}{\mathsf{C}}_{TM} + \overset{\langle 6\rangle}{\mathsf{C}}_M \cdot\cdot\cdot\, \mathbf{K}_e + \Delta\,\theta\, \overset{\langle 3\rangle}{\mathsf{S}} - \overset{\langle 6\rangle}{\mathsf{D}}_K \cdot\cdot\cdot\, \mathbf{K}_p - \overset{\langle 5\rangle}{\mathsf{D}}_{KE}{}^{*} \cdot\cdot\, \mathbf{E}_p\Big)\\
&\quad \cdot\cdot\cdot \Big(\mathbf{E}_e \cdot\cdot\, \overset{\langle 5\rangle}{\mathsf{C}}_{TM} + \overset{\langle 6\rangle}{\mathsf{C}}_M \cdot\cdot\cdot\, \mathbf{K}_e + \Delta\,\theta\, \overset{\langle 3\rangle}{\mathsf{S}} - \overset{\langle 6\rangle}{\mathsf{D}}_K \cdot\cdot\cdot\, \mathbf{K}_p - \overset{\langle 5\rangle}{\mathsf{D}}_{KE}{}^{*} \cdot\cdot\, \mathbf{E}_p\Big)\\
&\quad - \sigma_Y(w_p, \theta, \mathbf{g})^2
\end{aligned}
\tag{74}
$$

such that the loading condition (26) becomes

$$
\begin{aligned}
0 < &\left[2\,(\mathbf{T} - \mathbf{T}_B) \cdot\cdot\, \overset{\langle 4\rangle}{\mathsf{G}} + (\mathbf{M} - \mathbf{M}_B) \cdot\cdot\cdot\, \overset{\langle 5\rangle}{\mathsf{G}}{}^{*}\right] \cdot\cdot \left(\overset{\langle 4\rangle}{\mathsf{C}}_T \cdot\cdot\, \mathbf{E}^\bullet + \overset{\langle 5\rangle}{\mathsf{C}}_{TM} \cdot\cdot\cdot\, \mathbf{K}^\bullet + \theta^\bullet \overset{\langle 2\rangle}{\mathsf{R}}\right)\\
&+ \left[2\,(\mathbf{M} - \mathbf{M}_B) \cdot\cdot\cdot\, \overset{\langle 6\rangle}{\mathsf{G}} + (\mathbf{T} - \mathbf{T}_B) \cdot\cdot\, \overset{\langle 5\rangle}{\mathsf{G}}\right] \cdot\cdot\cdot \left(\overset{\langle 5\rangle}{\mathsf{C}}_{TM}{}^{*} \cdot\cdot\, \mathbf{E}^\bullet + \overset{\langle 6\rangle}{\mathsf{C}}_M \cdot\cdot\cdot\, \mathbf{K}^\bullet + \theta^\bullet \overset{\langle 3\rangle}{\mathsf{S}}\right)\\
&- 2\sigma_Y \left(\frac{\partial \sigma_Y}{\partial \theta} \theta^\bullet + \frac{\partial \sigma_Y}{\partial \mathbf{g}} \cdot \mathbf{g}^\bullet\right).
\end{aligned}
\tag{75}
$$

The associated flow rules are

$$
\mathbf{E}_p^\bullet = \lambda \frac{\partial \varphi}{\partial \mathbf{T}} = \lambda \Big\{2 \overset{\langle 4\rangle}{\mathsf{G}} \cdot\cdot\, (\mathbf{T} - \mathbf{T}_B) + \overset{\langle 5\rangle}{\mathsf{G}} \cdot\cdot\cdot\, (\mathbf{M} - \mathbf{M}_B)\Big\}
\tag{76}
$$

$$
\mathbf{K}_p^\bullet = \lambda \frac{\partial \varphi}{\partial \mathbf{M}} = \lambda \Big\{2 \overset{\langle 6\rangle}{\mathsf{G}} \cdot\cdot\, (\mathbf{M} - \mathbf{M}_B) + \overset{\langle 5\rangle}{\mathsf{G}}{}^{*} \cdot\cdot\cdot\, (\mathbf{T} - \mathbf{T}_B)\Big\}
\tag{77}
$$

with a joint plastic consistency parameter λ. The consistency condition Eq. (58) requires with Eqs. (1), (2), (63), (64), (68), (69), (74), (76), (77)

$$
\begin{aligned}
0 = \varphi^\bullet = &\ 2\,(\mathbf{T} - \mathbf{T}_B) \cdot\cdot\, \overset{\langle 4\rangle}{\mathsf{G}} \cdot\cdot\, (\mathbf{T} - \mathbf{T}_B)^\bullet + 2\,(\mathbf{M} - \mathbf{M}_B) \cdot\cdot\cdot\, \overset{\langle 6\rangle}{\mathsf{G}} \cdot\cdot\cdot\, (\mathbf{M} - \mathbf{M}_B)^\bullet\\
&+ (\mathbf{T} - \mathbf{T}_B)^\bullet \cdot\cdot\, \overset{\langle 5\rangle}{\mathsf{G}} \cdot\cdot\cdot\, (\mathbf{M} - \mathbf{M}_B) + (\mathbf{T} - \mathbf{T}_B) \cdot\cdot\, \overset{\langle 5\rangle}{\mathsf{G}} \cdot\cdot\cdot\, (\mathbf{M} - \mathbf{M}_B)^\bullet
\end{aligned}
$$

$$
\begin{aligned}
&-2\sigma_Y \left[\frac{\partial \sigma_Y}{\partial w_p} \left(\frac{\mathbf{T}-\mathbf{T}_B}{\rho} \cdot\cdot\, \mathbf{E}_p^\bullet + \frac{\mathbf{M}-\mathbf{M}_B}{\rho} \cdot\cdot\cdot\, \mathbf{K}_p^\bullet \right) + \frac{\partial \sigma_Y}{\partial \theta}\theta^\bullet + \frac{\partial \sigma_Y}{\partial \mathbf{g}} \cdot \mathbf{g}^\bullet \right] \\
= &\left[2\,(\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 4\rangle}{\mathbf{G}} + (\mathbf{M}-\mathbf{M}_B) \cdot\cdot\cdot \overset{\langle 5\rangle}{\mathbf{G}}{}^* \right] \\
&\cdot\cdot \left[\overset{\langle 4\rangle}{\mathbf{C}}_T \cdot\cdot\, (\mathbf{E}-\mathbf{E}_p)^\bullet + \overset{\langle 5\rangle}{\mathbf{C}}_{TM} \cdot\cdot\cdot\, (\mathbf{K}-\mathbf{K}_p)^\bullet + \Delta\theta^\bullet \overset{\langle 2\rangle}{\mathbf{R}} - \overset{\langle 4\rangle}{\mathbf{D}}_E \cdot\cdot\, \mathbf{E}_p^\bullet - \overset{\langle 5\rangle}{\mathbf{D}}_{KE} \cdot\cdot\cdot\, \mathbf{K}_p^\bullet \right] \\
&+\left[2\,(\mathbf{M}-\mathbf{M}_B) \cdot\cdot\cdot \overset{\langle 6\rangle}{\mathbf{G}} + (\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 5\rangle}{\mathbf{G}} \right] \\
&\cdot\cdot\cdot \left[\overset{\langle 5\rangle}{\mathbf{C}}_{TM}{}^* \cdot\cdot\, (\mathbf{E}-\mathbf{E}_p)^\bullet + \overset{\langle 6\rangle}{\mathbf{C}}_M \cdot\cdot\cdot\, (\mathbf{K}-\mathbf{K}_p)^\bullet + \Delta\theta^\bullet \overset{\langle 3\rangle}{\mathbf{S}} - \overset{\langle 6\rangle}{\mathbf{D}}_K \cdot\cdot\cdot\, \mathbf{K}_p^\bullet - \overset{\langle 5\rangle}{\mathbf{D}}_{KE}{}^* \cdot\cdot\, \mathbf{E}_p^\bullet \right] \\
&-2\sigma_Y \left[\frac{\partial \sigma_Y}{\partial w_p} \left(\frac{\mathbf{T}-\mathbf{T}_B}{\rho} \cdot\cdot\, \mathbf{E}_p^\bullet + \frac{\mathbf{M}-\mathbf{M}_B}{\rho} \cdot\cdot\cdot\, \mathbf{K}_p^\bullet \right) + \frac{\partial \sigma_Y}{\partial \theta}\theta^\bullet + \frac{\partial \sigma_Y}{\partial \mathbf{g}} \cdot \mathbf{g}^\bullet \right] \\
= &\left[2\,(\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 4\rangle}{\mathbf{G}} + (\mathbf{M}-\mathbf{M}_B) \cdot\cdot\cdot \overset{\langle 5\rangle}{\mathbf{G}}{}^* \right] \\
&\cdot\cdot \Bigg[\overset{\langle 4\rangle}{\mathbf{C}}_T \cdot\cdot \left(\mathbf{E}^\bullet - \lambda \left\{ 2\overset{\langle 4\rangle}{\mathbf{G}} \cdot\cdot\, (\mathbf{T}-\mathbf{T}_B) + \overset{\langle 5\rangle}{\mathbf{G}} \cdot\cdot\cdot\, (\mathbf{M}-\mathbf{M}_B) \right\} \right) \\
&+\overset{\langle 5\rangle}{\mathbf{C}}_{TM} \cdot\cdot\cdot \left(\mathbf{K}^\bullet - \lambda \left\{ 2\overset{\langle 6\rangle}{\mathbf{G}} \cdot\cdot\cdot\, (\mathbf{M}-\mathbf{M}_B) + \overset{\langle 5\rangle}{\mathbf{G}}{}^* \cdot\cdot\, (\mathbf{T}-\mathbf{T}_B) \right\} \right) + \Delta\theta^\bullet \overset{\langle 2\rangle}{\mathbf{R}} \\
&-\overset{\langle 4\rangle}{\mathbf{D}}_E \cdot\cdot\, \lambda \left\{ 2\overset{\langle 4\rangle}{\mathbf{G}} \cdot\cdot\, (\mathbf{T}-\mathbf{T}_B) + \overset{\langle 5\rangle}{\mathbf{G}} \cdot\cdot\cdot\, (\mathbf{M}-\mathbf{M}_B) \right\} \\
&-\overset{\langle 5\rangle}{\mathbf{D}}_{KE} \cdot\cdot\cdot\, \lambda \left\{ 2\overset{\langle 6\rangle}{\mathbf{G}} \cdot\cdot\cdot\, (\mathbf{M}-\mathbf{M}_B) + \overset{\langle 5\rangle}{\mathbf{G}}{}^* \cdot\cdot\, (\mathbf{T}-\mathbf{T}_B) \right\} \Bigg] \\
&+\left[2\,(\mathbf{M}-\mathbf{M}_B) \cdot\cdot\cdot \overset{\langle 6\rangle}{\mathbf{G}} + (\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 5\rangle}{\mathbf{G}} \right] \\
&\cdot\cdot\cdot \Bigg[\overset{\langle 5\rangle}{\mathbf{C}}_{TM}{}^* \cdot\cdot \left(\mathbf{E}^\bullet - \lambda \left\{ 2\overset{\langle 4\rangle}{\mathbf{G}} \cdot\cdot\, (\mathbf{T}-\mathbf{T}_B) + \overset{\langle 5\rangle}{\mathbf{G}} \cdot\cdot\cdot\, (\mathbf{M}-\mathbf{M}_B) \right\} \right) \\
&+\overset{\langle 6\rangle}{\mathbf{C}}_M \cdot\cdot\cdot \left(\mathbf{K}^\bullet - \lambda \left\{ 2\overset{\langle 6\rangle}{\mathbf{G}} \cdot\cdot\cdot\, (\mathbf{M}-\mathbf{M}_B) + \overset{\langle 5\rangle}{\mathbf{G}}{}^* \cdot\cdot\, (\mathbf{T}-\mathbf{T}_B) \right\} \right) + \Delta\theta^\bullet \overset{\langle 3\rangle}{\mathbf{S}} \\
&-\overset{\langle 6\rangle}{\mathbf{D}}_K \cdot\cdot\cdot\, \lambda \left\{ 2\overset{\langle 6\rangle}{\mathbf{G}} \cdot\cdot\cdot\, (\mathbf{M}-\mathbf{M}_B) + \overset{\langle 5\rangle}{\mathbf{G}}{}^* \cdot\cdot\, (\mathbf{T}-\mathbf{T}_B) \right\} \\
&-\overset{\langle 5\rangle}{\mathbf{D}}_{KE}{}^* \cdot\cdot\, \lambda \left\{ 2\overset{\langle 4\rangle}{\mathbf{G}} \cdot\cdot\, (\mathbf{T}-\mathbf{T}_B) + \overset{\langle 5\rangle}{\mathbf{G}} \cdot\cdot\cdot\, (\mathbf{M}-\mathbf{M}_B) \right\} \Bigg] \\
&-2\sigma_Y \Bigg[\frac{\partial \sigma_Y}{\partial w_p} \Bigg(\frac{\mathbf{T}-\mathbf{T}_B}{\rho} \cdot\cdot\, \lambda \left\{ 2\overset{\langle 4\rangle}{\mathbf{G}} \cdot\cdot\, (\mathbf{T}-\mathbf{T}_B) + \overset{\langle 5\rangle}{\mathbf{G}} \cdot\cdot\cdot\, (\mathbf{M}-\mathbf{M}_B) \right\} \\
&+\frac{\mathbf{M}-\mathbf{M}_B}{\rho} \cdot\cdot\cdot\, \lambda \left\{ 2\overset{\langle 6\rangle}{\mathbf{G}} \cdot\cdot\cdot\, (\mathbf{M}-\mathbf{M}_B) + \overset{\langle 5\rangle}{\mathbf{G}}{}^* \cdot\cdot\, (\mathbf{T}-\mathbf{T}_B) \right\} \Bigg) + \frac{\partial \sigma_Y}{\partial \theta}\theta^\bullet + \frac{\partial \sigma_Y}{\partial \mathbf{g}} \cdot \mathbf{g}^\bullet \Bigg]
\end{aligned}
\tag{78}
$$

or

$$
\begin{aligned}
&\left[2\,(\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 4\rangle}{\mathbf{G}} + (\mathbf{M}-\mathbf{M}_B) \cdot\cdot\cdot \overset{\langle 5\rangle}{\mathbf{G}}{}^* \right] \cdot\cdot \left[\overset{\langle 4\rangle}{\mathbf{C}}_T \cdot\cdot\, \mathbf{E}^\bullet + \overset{\langle 5\rangle}{\mathbf{C}}_{TM} \cdot\cdot\cdot\, \mathbf{K}^\bullet + \Delta\theta^\bullet \overset{\langle 2\rangle}{\mathbf{R}} \right] \\
&\quad+\left[2\,(\mathbf{M}-\mathbf{M}_B) \cdot\cdot\cdot \overset{\langle 6\rangle}{\mathbf{G}} + (\mathbf{T}-\mathbf{T}_B) \cdot\cdot \overset{\langle 5\rangle}{\mathbf{G}} \right] \cdot\cdot\cdot \left[\overset{\langle 5\rangle}{\mathbf{C}}_{TM}{}^* \cdot\cdot\, \mathbf{E}^\bullet + \overset{\langle 6\rangle}{\mathbf{C}}_M \cdot\cdot\cdot\, \mathbf{K}^\bullet + \Delta\theta^\bullet \overset{\langle 3\rangle}{\mathbf{S}} \right] \\
&\quad-2\sigma_Y \left(\frac{\partial \sigma_Y}{\partial \theta}\theta^\bullet + \frac{\partial \sigma_Y}{\partial \mathbf{g}} \cdot \mathbf{g}^\bullet \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda \Bigg[(\mathbf{T}-\mathbf{T}_B)\cdot\cdot \Bigg\{ 2\overset{\langle 4\rangle}{\mathbf{G}}\cdot\cdot\Big[\overset{\langle 4\rangle}{\mathbf{C}}_T\cdot\cdot 2\overset{\langle 4\rangle}{\mathbf{G}} + \overset{\langle 5\rangle}{\mathbf{C}}_{TM}\cdot\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}}{}^{*} + \overset{\langle 4\rangle}{\mathbf{D}}_E\cdot\cdot 2\overset{\langle 4\rangle}{\mathbf{G}} + \overset{\langle 5\rangle}{\mathbf{D}}_{KE}\cdot\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}}{}^{*}\Big] \\
&\quad + \overset{\langle 5\rangle}{\mathbf{G}}\cdot\cdot\cdot\Big[\overset{\langle 5\rangle}{\mathbf{C}}_{TM}{}^{*}\cdot\cdot 2\overset{\langle 4\rangle}{\mathbf{G}} + \overset{\langle 6\rangle}{\mathbf{C}}_M\cdot\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}}{}^{*} + \overset{\langle 6\rangle}{\mathbf{D}}_K\cdot\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}}{}^{*} + \overset{\langle 5\rangle}{\mathbf{D}}_{KE}{}^{*}\cdot\cdot 2\overset{\langle 4\rangle}{\mathbf{G}}\Big] + 2\sigma_Y\frac{1}{\rho}\frac{\partial\sigma_Y}{\partial w_p}2\overset{\langle 4\rangle}{\mathbf{G}}\Bigg\}\cdot\cdot(\mathbf{T}-\mathbf{T}_B) \\
&\quad + (\mathbf{M}-\mathbf{M}_B)\cdot\cdot\cdot\Bigg\{\overset{\langle 5\rangle}{\mathbf{G}}{}^{*}\cdot\cdot\Big[\overset{\langle 4\rangle}{\mathbf{C}}_T\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}} + \overset{\langle 5\rangle}{\mathbf{C}}_{TM}\cdot\cdot\cdot 2\overset{\langle 6\rangle}{\mathbf{G}} + \overset{\langle 4\rangle}{\mathbf{D}}_E\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}} + \overset{\langle 5\rangle}{\mathbf{D}}_{KE}\cdot\cdot\cdot 2\overset{\langle 6\rangle}{\mathbf{G}}\Big] \\
&\quad + 2\overset{\langle 6\rangle}{\mathbf{G}}\cdot\cdot\Big[\overset{\langle 5\rangle}{\mathbf{C}}_{TM}{}^{*}\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}} + \overset{\langle 6\rangle}{\mathbf{C}}_M\cdot\cdot\cdot 2\overset{\langle 6\rangle}{\mathbf{G}} + \overset{\langle 6\rangle}{\mathbf{D}}_K\cdot\cdot\cdot 2\overset{\langle 6\rangle}{\mathbf{G}} + \overset{\langle 5\rangle}{\mathbf{D}}_{KE}{}^{*}\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}}\Big] \\
&\quad + 2\sigma_Y\frac{1}{\rho}\frac{\partial\sigma_Y}{\partial w_p}2\overset{\langle 6\rangle}{\mathbf{G}}\Bigg\}\cdot\cdot\cdot(\mathbf{M}-\mathbf{M}_B) \\
&\quad + (\mathbf{T}-\mathbf{T}_B)\cdot\cdot\Bigg\{2\overset{\langle 4\rangle}{\mathbf{G}}\cdot\cdot\Big[\overset{\langle 4\rangle}{\mathbf{C}}_T\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}} + \overset{\langle 5\rangle}{\mathbf{C}}_{TM}\cdot\cdot\cdot 2\overset{\langle 6\rangle}{\mathbf{G}} + \overset{\langle 4\rangle}{\mathbf{D}}_E\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}} + \overset{\langle 5\rangle}{\mathbf{D}}_{KE}\cdot\cdot\cdot 2\overset{\langle 6\rangle}{\mathbf{G}}\Big] \\
&\quad + \overset{\langle 5\rangle}{\mathbf{G}}\cdot\cdot\cdot\Big[\overset{\langle 5\rangle}{\mathbf{C}}_{TM}{}^{*}\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}} + \overset{\langle 6\rangle}{\mathbf{C}}_M\cdot\cdot 2\overset{\langle 6\rangle}{\mathbf{G}} + \overset{\langle 6\rangle}{\mathbf{D}}_K\cdot\cdot\cdot 2\overset{\langle 6\rangle}{\mathbf{G}} + \overset{\langle 5\rangle}{\mathbf{D}}_{KE}{}^{*}\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}}\Big] \\
&\quad + 2\sigma_Y\frac{1}{\rho}\frac{\partial\sigma_Y}{\partial w_p}\overset{\langle 5\rangle}{\mathbf{G}}\Bigg\}\cdot\cdot\cdot(\mathbf{M}-\mathbf{M}_B) \\
&\quad + (\mathbf{M}-\mathbf{M}_B)\cdot\cdot\cdot\Bigg\{\overset{\langle 5\rangle}{\mathbf{G}}{}^{*}\cdot\cdot\Big[\overset{\langle 4\rangle}{\mathbf{C}}_T\cdot\cdot 2\overset{\langle 4\rangle}{\mathbf{G}} + \overset{\langle 5\rangle}{\mathbf{C}}_{TM}\cdot\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}}{}^{*} + \overset{\langle 4\rangle}{\mathbf{D}}_E\cdot\cdot 2\overset{\langle 4\rangle}{\mathbf{G}} + \overset{\langle 5\rangle}{\mathbf{D}}_{KE}\cdot\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}}{}^{*}\Big] \\
&\quad + 2\overset{\langle 6\rangle}{\mathbf{G}}\cdot\cdot\cdot\Big[\overset{\langle 5\rangle}{\mathbf{C}}_{TM}{}^{*}\cdot\cdot 2\overset{\langle 4\rangle}{\mathbf{G}} + \overset{\langle 6\rangle}{\mathbf{C}}_M\cdot\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}}{}^{*} + \overset{\langle 6\rangle}{\mathbf{D}}_K\cdot\cdot\cdot\overset{\langle 5\rangle}{\mathbf{G}}{}^{*} + \overset{\langle 5\rangle}{\mathbf{D}}_{KE}{}^{*}\cdot\cdot 2\overset{\langle 4\rangle}{\mathbf{G}}\Big] \\
&\quad + 2\sigma_Y\frac{1}{\rho}\frac{\partial\sigma_Y}{\partial w_p}\overset{\langle 5\rangle}{\mathbf{G}}{}^{*}\Bigg\}\cdot\cdot(\mathbf{T}-\mathbf{T}_B)\Bigg]
\end{aligned}
\tag{79}
$$

This linear equation can be uniquely solved for the consistency parameter λ. The left-hand side of this equation is positive due to the loading condition (75). If the elasticities have the usual positive definiteness properties, and if hardening occurs with $\frac{\partial\sigma_Y}{\partial w_p} > 0$, then the terms in { }-brackets are always positive, so that λ will in fact be positive during yielding, as it should be. If we substitute λ into the flow rules (76), (77), we obtain the *consistent flow rules*, which is straightforward but not done here for brevity.

If we substitute (76), (77) into the definition of the plastic work (70), we obtain

$$
\begin{aligned}
w_p^{\bullet} = \frac{\lambda}{\rho}2\Big\{&(\mathbf{T}-\mathbf{T}_B)\cdot\cdot\Big[\overset{\langle 4\rangle}{\mathbf{G}}\cdot\cdot(\mathbf{T}-\mathbf{T}_B) + \overset{\langle 5\rangle}{\mathbf{G}}\cdot\cdot\cdot(\mathbf{M}-\mathbf{M}_B)\Big] \\
&+ (\mathbf{M}-\mathbf{M}_B)\cdot\cdot\cdot\overset{\langle 6\rangle}{\mathbf{G}}\cdot\cdot\cdot(\mathbf{M}-\mathbf{M}_B)\Big\}
\end{aligned}
\tag{80}
$$

By an appropriate choice of $\overset{\langle 4\rangle}{\mathbf{G}}$, $\overset{\langle 5\rangle}{\mathbf{G}}$, and $\overset{\langle 6\rangle}{\mathbf{G}}$ (positive semi-definiteness), we can assure that the plastic work is non-negative.

The heat generated by yielding -Q_p is then according to Eqs. (72), (68), (69) determined by

$$
\begin{aligned}
-Q_p = -\frac{1}{\rho}\Big(&\mathbf{E}_p\cdot\cdot\overset{\langle 4\rangle}{\mathbf{D}}_E\cdot\cdot\mathbf{E}_p^{\bullet} + \mathbf{K}_p\cdot\cdot\cdot\overset{\langle 6\rangle}{\mathbf{D}}_K\cdot\cdot\cdot\mathbf{K}_p^{\bullet} + \mathbf{E}_p^{\bullet}\cdot\cdot\overset{\langle 5\rangle}{\mathbf{D}}_{KE}\cdot\cdot\cdot\mathbf{K}_p \\
&+ \mathbf{E}_p\cdot\cdot\overset{\langle 5\rangle}{\mathbf{D}}_{KE}\cdot\cdot\cdot\mathbf{K}_p^{\bullet}\Big) - \mu w_p^{\bullet} + \frac{\mathbf{T}}{\rho}\cdot\cdot\mathbf{E}_p^{\bullet} + \frac{\mathbf{M}}{\rho}\cdot\cdot\cdot\mathbf{K}_p^{\bullet}
\end{aligned}
$$

$$= \left(\frac{\mathbf{T}-\mathbf{T}_B}{\rho}\cdot\cdot\,\mathbf{E}_p^\bullet - \theta\frac{\partial\eta_p}{\partial\mathbf{E}_p}\right)\cdot\cdot\,\mathbf{E}_p^\bullet + \left(\frac{\mathbf{M}-\mathbf{M}_B}{\rho} - \rho\,\theta\frac{\partial\eta_p}{\partial\mathbf{K}_p}\right)\cdot\cdot\cdot\,\mathbf{K}_p^\bullet - \mu w_p^\bullet$$
$$= (1-\mu)w_p^\bullet - \theta\left(\frac{\partial\eta_p}{\partial\mathbf{E}_p}\cdot\cdot\,\mathbf{E}_p^\bullet + \frac{\partial\eta_p}{\partial\mathbf{K}_p}\cdot\cdot\cdot\,\mathbf{K}_p^\bullet\right). \tag{81}$$

$(1-\mu)$ can be interpreted as a Taylor–Quinney factor.The residual dissipation inequality- (49) becomes with Eqs. (52), (71), and (81)

$$0 \le -Q_p + \theta\eta_p^\bullet = (1-\mu)w_p^\bullet. \tag{82}$$

This can always be satisfied by choosing $0 \le \mu \le 1$. We see that a non-trivial ansatz for the plastic part of the entropy instead of (71)

$$\eta_p \equiv \gamma w_p \tag{83}$$

with any non-negative real constant γ would also satisfy the dissipation inequality. So the plastic part of the entropy is only weakly restricted by the second law and by no means unique.

With Eq. (45) we finally obtain for the free energy for our model with (62), (72), and (83)

$$\psi = \frac{1}{2\rho}\mathbf{E}_e\cdot\cdot\overset{\langle 4\rangle}{\mathbf{C}}_T\cdot\cdot\mathbf{E}_e + \frac{1}{2\rho}\mathbf{K}_e\cdot\cdot\overset{\langle 6\rangle}{\mathbf{C}}_M\cdot\cdot\cdot\mathbf{K}_e + \frac{1}{\rho}\mathbf{E}_e\cdot\cdot\overset{\langle 5\rangle}{\mathbf{C}}_{TM}\cdot\cdot\cdot\mathbf{K}_e$$
$$+c\rho\left(\Delta\theta - \theta ln\frac{\theta}{\theta_0}\right) + \Delta\theta\frac{1}{\rho}\left(\overset{\langle 2\rangle}{\mathbf{R}}\cdot\cdot\,\mathbf{E}_e + \overset{\langle 3\rangle}{\mathbf{S}}\cdot\cdot\cdot\,\mathbf{K}_e\right)$$
$$+\frac{1}{2\rho}\mathbf{E}_p\cdot\cdot\overset{\langle 4\rangle}{\mathbf{D}}_E\cdot\cdot\,\mathbf{E}_p + \frac{1}{2\rho}\mathbf{K}_p\cdot\cdot\cdot\overset{\langle 6\rangle}{\mathbf{D}}_K\cdot\cdot\cdot\,\mathbf{K}_p$$
$$+\frac{1}{\rho}\mathbf{E}_p\cdot\cdot\overset{\langle 5\rangle}{\mathbf{D}}_{KE}\cdot\cdot\cdot\,\mathbf{K}_p + (1-\mu)w_p \tag{84}$$

which completes the model equations for the general anisotropic example.

6 Isotropic material

For isotropic materials we use the isotropic representations by Cauchy (1823) and Mindlin/Eshel [30] for the constitutive equations. For the free energy we obtain after Eq. (84)

$$\rho\psi = a_1(\mathbf{E}_e\cdot\cdot\,\mathbf{I})^2 + a_2\mathbf{E}_e\cdot\cdot\,\mathbf{E}_e + a_3\mathbf{I}\cdot\cdot\,\mathbf{K}_e\cdot\mathbf{K}_e\cdot\cdot\,\mathbf{I} + a_4(\mathbf{K}_e\cdot\cdot\,\mathbf{I})\cdot(\mathbf{K}_e\cdot\cdot\,\mathbf{I})$$
$$+a_5(\mathbf{I}\cdot\cdot\,\mathbf{K}_e)\cdot(\mathbf{I}\cdot\cdot\,\mathbf{K}_e) + a_6\mathbf{K}_e\cdot\cdot\cdot\,\mathbf{K}_e + a_7\mathbf{K}_e\cdot\cdot\cdot\,\mathbf{K}_e^t + c\rho\left(\Delta\theta - \theta ln\frac{\theta}{\theta_0}\right)$$
$$+\Delta\theta a_8\mathbf{I}\cdot\cdot\,\mathbf{E}_e + \rho(\mu-\theta\gamma)w_p$$
$$+b_1\left(\mathbf{E}_p\cdot\cdot\,\mathbf{I}\right)^2 + b_2\mathbf{E}_p\cdot\cdot\,\mathbf{E}_p + b_3\mathbf{I}\cdot\cdot\,\mathbf{K}_p\cdot\mathbf{K}_p\cdot\cdot\,\mathbf{I} + b_4(\mathbf{K}_p\cdot\cdot\,\mathbf{I})\cdot(\mathbf{K}_p\cdot\cdot\,\mathbf{I})$$
$$+b_5(\mathbf{I}\cdot\cdot\,\mathbf{K}_p)\cdot(\mathbf{I}\cdot\cdot\,\mathbf{K}_p) + b_6\mathbf{K}_p\cdot\cdot\cdot\,\mathbf{K}_p + b_7\mathbf{K}_p\cdot\cdot\cdot\,\mathbf{K}_p^t \tag{85}$$

where the following particular transposition for a third-order tensor $[\mathbf{K}^t]_{ijk} := K_{kji}$ is used. a_i and b_i are scalar material constants.

This gives the following stresses after (63) and (64)

$$\mathbf{T} = 2a_1\,(\mathbf{E}_e\cdot\cdot\,\mathbf{I})\,\mathbf{I} + 2a_2\,\mathbf{E}_e + \Delta\theta a_8\,\mathbf{I} \tag{86}$$
$$\mathbf{M} = 2a_3\,\mathbf{I}\otimes\mathbf{K}_e\cdot\cdot\,\mathbf{I} + 2a_4(\mathbf{K}_e\cdot\cdot\,\mathbf{I})\otimes\mathbf{I} + 2a_5\,\mathbf{I}\otimes\mathbf{I}\cdot\cdot\,\mathbf{K}_e + 2a_6\,\mathbf{K}_e + 2a_7\,\mathbf{K}_e^t. \tag{87}$$

The back stresses of Eqs. (68) and (69) reduce in the isotropic case to

$$\mathbf{T}_B = 2d_1\left(\mathbf{E}_p\cdot\cdot\,\mathbf{I}\right)\mathbf{I} + 2d_2\,\mathbf{E}_p \tag{88}$$
$$\mathbf{M}_B = 2d_3\,\mathbf{I}\otimes\mathbf{K}_p\cdot\cdot\,\mathbf{I} + 2d_4(\mathbf{K}_p\cdot\cdot\,\mathbf{I})\otimes\mathbf{I} + 2d_5\,\mathbf{I}\otimes\mathbf{I}\cdot\cdot\,\mathbf{K}_p + 2d_6\,\mathbf{K}_p + 2d_7\,\mathbf{K}_p^t \tag{89}$$

with scalar material constants d_i.

The Fourier law for the heat flux (67) obtains the usual form

$$\mathbf{q} = -\kappa \mathbf{g} \tag{90}$$

with a non-negative coefficient κ.

For the entropy after (65) we obtain with (83)

$$\eta = -\frac{1}{\rho} a_8 \, \mathbf{I} \cdot\cdot \, \mathbf{E}_e + c \, ln \frac{\theta}{\theta_0} + \gamma w_p. \tag{91}$$

The von-Mises-type yield criterion (73) becomes in the isotropic case

$$\begin{aligned}\varphi = \; & g_1[(\mathbf{T} - \mathbf{T}_B) \cdot\cdot \, \mathbf{I}]^2 + g_2 \, (\mathbf{T} - \mathbf{T}_B) \cdot\cdot \, (\mathbf{T} - \mathbf{T}_B) \\ & + g_3 \mathbf{I} \cdot\cdot \, (\mathbf{M} - \mathbf{M}_B) \cdot (\mathbf{M} - \mathbf{M}_B) \cdot\cdot \, \mathbf{I} + g_4[(\mathbf{M} - \mathbf{M}_B) \cdot\cdot \, \mathbf{I}] \cdot [(\mathbf{M} - \mathbf{M}_B) \cdot\cdot \, \mathbf{I}] \\ & + g_5 \, [\mathbf{I} \cdot\cdot \, (\mathbf{M} - \mathbf{M}_B)] \cdot [\mathbf{I} \cdot\cdot \, (\mathbf{M} - \mathbf{M}_B)] + g_6 \, (\mathbf{M} - \mathbf{M}_B) \cdot\cdot\cdot \, (\mathbf{M} - \mathbf{M}_B) \\ & + g_7 \, (\mathbf{M} - \mathbf{M}_B) \cdot\cdot\cdot \, (\mathbf{M} - \mathbf{M}_B)^t - \sigma_Y^2 \left(w_p, \theta, |\mathbf{g}| \right) \end{aligned} \tag{92}$$

with scalar material constants g_i one of which can be generally normalized to one. The associated flow rules (76), (77) are accordingly

$$\mathbf{E}_p^\bullet = \lambda \, \{ g_1[(\mathbf{T} - \mathbf{T}_B) \cdot\cdot \, \mathbf{I}]\mathbf{I} + g_2 \, (\mathbf{T} - \mathbf{T}_B) \} \tag{93}$$

$$\begin{aligned}\mathbf{K}_p^\bullet = \lambda \, \{ & g_3 \, \mathbf{I} \otimes (\mathbf{M} - \mathbf{M}_B) \cdot\cdot \, \mathbf{I} + g_4[(\mathbf{M} - \mathbf{M}_B) \cdot\cdot \, \mathbf{I}] \otimes \mathbf{I} + g_5 \, \mathbf{I} \otimes \mathbf{I} \cdot\cdot \, (\mathbf{M} - \mathbf{M}_B) \\ & + g_6 \, \mathbf{K}_e + g_7 \left(\mathbf{M}^t - \mathbf{M}_B^t \right) \} \end{aligned} \tag{94}$$

where the factor 2 has been drawn into the plastic parameter.

If the flow criterion is density-insensitive, then $g_1 = g_2/3$ and $(\mathbf{T} - \mathbf{T}_B)$ is deviatoric, the same as $\mathbf{E}_p$.

Resume

A thermodynamic framework for material models within gradient plasticity as a generalization of [5] is presented. Here it is worked out for a second-gradient material. However, the same for a higher-gradient material would be straightforward, but still much longer.

An example is given for an anisotropic model which is oriented along the lines of a classical J_2-theory of plasticity, including strain gradients and the first temperature gradient. The thermodynamical restrictions are worked out. The particularization for the isotropic case is then straightforward.

The proposed example already introduces a long list of material parameters related to the strain gradient effects, which must be identified by experimental or computational data. Strain gradient effects often originate from the interaction of different phases in heterogeneous materials. It is thought that homogenization theories can be used to derive higher-order material parameters. Multiscale asymptotic expansion methods have been proposed by [8] to determine higher-order elasticity moduli in composites. Extensions to the case of plasticity have been recently proposed by [20].

Strain gradient plasticity models involving the gradient of plastic strains or of plastic slip as done in [25] and [1] for instance are special cases of the proposed theory when the effect of the gradient of the elastic strain is neglected. This approach is surely well suited to handle generalized crystal plasticity models. However, recent works on continuum theories of crystals suggest that full strain gradient effects might be of interest to model some features of dislocation behaviour like core or grain boundary effects (see for instance the discussion in [41]).

First finite element simulations based on elastoplastic strain gradient constitutive equations, although going back to the work by [36], still represent challenges in computational mechanics. In particular, the use of closely related micromorphic models with internal constraints can ease the numerical implementation of strain gradient formulations, especially regarding boundary conditions, as shown by [12]. It is thought that introducing a rationale in the formulation of such constitutive equations, as done in the present work, can help setting clear and efficient computational frameworks for future large-scale simulations of complex thermomechanical processes based on generalized continua.

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