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Mathematical analysis of the primary network effect

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Abstract Communication and similar markets can only be understood thoroughly if network effects are taken into account. For an oligopolistic market without competition, a static mathematical model is derived. This discrete model allows to analyse in detail marginal utility and net gain for a network as well as its critical sizes and limit of growth or saturation. They are described in terms of connection values and price that might depend on the network size. Moreover, a generalisation of Metcalfe's law is shown.

JEL Classification C0, D4, D7, L1, L68, L86, L96

Key words network – externalities – mathematical model – difference equation – Metcalfe's law

1 Introduction

The study of complex networked systems with an origin in sociology, biology, ecology, physics, technology (especially, information technology), and other fields has become topical again in the last decade. In particular, methods of dynamical systems and graph theory as well as probabilistic methods have been employed in order to develop models and to construct tools for a better understanding, prediction or control of the behaviour of such networks, their temporal evolution, and spatial distribution. For a review, we refer to Newman (2003), Newman *et al.* (2006), Strogatz (2001), and the literature cited therein.

Networks also very often arise in economics although the mathematical description seems rather to be in an early stage here. Network effects play, however,

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an important role in the understanding of e.g. communication markets where consumer decisions depend upon the behaviour of other consumers and the availability of (real or fictitious) connections of a certain value. For a discussion of assumptions, implications, and limits of the network effect theory, we refer to Weitzel *et al.* (2000). For earlier results on the analysis of network externalities in economics, we refer to Economides (1996) and the references cited therein.

The effect of networks in markets such as telecommunication and fax, internet, video cassette recorders, electricity, railroads etc. has recently been analysed in detail by some authors.

So, in Park (2004), an oligopolistic market for newly introduced durable products with network externalities is described by means of a dynamic structural model for the consumer's choice and producer's pricing. The dynamic aspect is taken into account by considering characteristic values at different time periods. Markov processes, the Bellmann equation, and the parameter identification from statistic moment equations form the mathematical background. Based upon this model, Park explains the superiority of the VHS format over the Betamax format.

Ohashi (2003) presents a hedonic approach in order to explain the development of the price index for video cassette recorders.

The U.S. fax market has been considered by Shapiro and Varian (1998). They distinguish between three possible network equilibria: the trivial one where no network exists, an equilibrium with a small network size where potential consumers are not willing to pay too much as they expect only a small network, and an equilibrium with a large network size where prices are low as the good only has a small value for new users. The analysis relies upon the comparison of the demand and the supply curve. However, the model is rather simple: the demand is supposed to be a quadratic function of the network's size whereas the supply is constant and thereby independent of the network's size.

A similar approach can be found in Economides and Himmelberg (1995). The authors again distinguish between the zero size, intermediate size, and large size network equilibria within an appropriate range of prices. As the latter one is Pareto optimal, it is expected that the market will choose this large size equilibrium. Moreover, small networks (with network sizes below a critical mass) cannot be observed. The model suggested by the authors relies upon a network externalities function.

Schoder (2000) analyses network effects for telecommunication services, especially diffusion phenomena such as critical mass. The probabilistic model there is based upon the master equation approach and is applied to the question of superiority of one attitude over others.

The U.S. as well as Korean fax market has also been considered in Lim *et al.* (2003), and a new diffusion model describing network externalities in telecommunication services has been proposed. Beside the critical mass, the authors discuss the late-take off and other phenomena.

We may distinguish between real or fictitious networks: In a fictitious network, goods and services are exchanged whereas in a real network also other objects are exchanged such as data packets within a communication network. An essential property of real networks is that they are determined by a specific technological standard. The participation in a communication network for instance requires to possess an apparatus that is compatible with those used by the other participants of the network. Moreover, we find central networks with only a few sources but many sinks as well as local networks in which any node may act as a source and a sink. In this paper, we focus on local real networks.

The static model suggested here is a discrete deterministic one that relies upon difference equations. It takes into account connection diversity (links between different nodes may have different connection values). The connection value and price function, however, which are assumed to be given *a priori*, can result from probabilistic considerations. The model suggested might easily be extended to incorporate dynamical aspects.

It is important to distinguish between direct versus indirect network effects. Whereas direct network effects are demand-side user externalities, indirect effects are supply-side user externalities due to e.g. decreasing unit prices. In our model, direct externalities as well as the consumer's expectation are incorporated by the user's utility and the connection values. Indirect externalities can be described by the price function.

With the static model presented here, we can strictly derive Metcalfe's law and also some generalisation. Moreover, we show that the marginal net gain is the sum of the net gain for a new user and the net gain all old users have when a new user joins a network. The latter one incorporates changes due to size-dependent connection values and unit prices. Critical network sizes are then sizes where these two net gains change their sign.

As an example, we consider connection values that are constant for all connections but are monotonically decreasing with the network's size and a typical hyperbolic prize function. Decreasing connection values can be interpreted as a model for overloading or capacity bounds or diminishing interest in the network. We show, in particular, that the connection values are not allowed to decay faster than 1/size in order to guarantee an expanding network.

2 Mathematical model

We consider a network of $n \in \{2, 3, \dots\}$ users U_i ($i = 1, 2, \dots, n$). The network users take (directly or indirectly) advantage of the connections to other users. We assume that

(H1) a connection takes only place between two users (plane network);
(H2) a connection between two users has a value only for these two connected users.

As the value of the connection between the two users U_i and U_j might be different for both the users, we introduce a_{ij} being the value of the connection for user U_i (and a_{ji} being the value for user U_j):

$$a_{ij} : \boxed{U_i} \longleftrightarrow U_j\,, \quad i, j \in \{1, 2, \dots, n\}\,.$$

Note that[1]

$$a_{ii} = 0 \quad \text{for all } i = 1, 2, \ldots, n\,.$$

We, therefore, can associate with the network the matrix

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & 0 & a_{23} & \ldots & a_{2n} \\ & & \ldots\ldots & & \\ a_{n1} & a_{n2} & a_{n3} & \ldots & 0 \end{pmatrix}.$$

In general, the matrix A is unsymmetric.

If each user can connect to each other user, there are

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

connections in a network of size n. If this is not the case, we may model non-existing connections by setting its value equal zero.

Often the value of a connection depends on the network's size. Due to e.g. an overload, the value might be decreasing with increasing size. So the values a_{ij} and the matrix A are indeed functions of n,

$$a_{ij} = a_{ij}(n)\,, \quad A = A(n)\,.$$

The total utility of the network of size n for user U_i is denoted by $N_{i\to}(n)$ and is given by the i-th row sum of $A(n)$,

$$N_{i\to}(n) = \sum_{j=1}^{n} a_{ij}(n)\,, \quad i \in \{1, 2, \ldots, n\}\,,$$

whereas $N_{\to k}(n)$ denotes the total value of all connections with user U_k for all other users of the network of size n, which is the k-th column sum of $A(n)$,

$$N_{\to k}(n) = \sum_{j=1}^{n} a_{jk}(n)\,, \quad k \in \{1, 2, \ldots, n\}\,.$$

So $i \to$ denotes all possible connections *from* user U_i whereas $\to k$ denotes all possible connections *to* user U_k. The total utility $N(n)$ of the whole network is thus given by

$$N(n) = \sum_{i,j=1}^{n} a_{ij}(n) = \sum_{i=1}^{n} N_{i\to}(n) = \sum_{k=1}^{n} N_{\to k}(n)\,.$$

[1] This assumption is only to hold the presentation as simple as possible. If not only a derivative utility but also an original utility occurs, this could be modelled by assigning it to the value a_{ii} for a connection with oneself, see also (H3).

Example 1 We consider a network of n users, in which all connections have the same value $a(n)/2$. Then

$$a_{ij}(n) = \begin{cases} 0 & \text{for } i = j\,, \\ \dfrac{a(n)}{2} & \text{for } i \neq j\,, \end{cases} \qquad i, j \in \{1, 2, \ldots, n\}\,.$$

It follows

$$N(n) = \frac{a(n)}{2}\, n(n-1)\,, \tag{1}$$

which coincides with Metcalfe's law if $a = const$. □

Example 2 Often the connections differ in their values. As a simplified model we may assume that the connection value $a_{ij}(n)$ within a fixed network of size n depends in some sense on the distance between the users U_i and U_j.

Let us, for instance, assume that for each user U_i there are m ($0 < m < n$) other users such that the connections from U_i to these users have the value $a(n)/2$ and all other connections from U_i are valued with $b(n)/2$. It then follows

$$N(n) = n\left(m\,\frac{a(n)}{2} + (n-m-1)\,\frac{b(n)}{2}\right),$$

which generalises Metcalfe's law. □

In a recent discussion (see Odlyzko and Tilly (2005), Briscoe *et al.* (2006)), the authors claim that Metcalfe's law is wrong and instead of $N(n) \sim n^2$ the relation $N(n) \sim n \log n$ is proposed. However, even the latter relation can be derived from (1) by assuming $a(n) \sim (\log n)/n$ which models a connection value function that is decreasing for n larger than some critical size.

Let us consider now the case that a new user joins a network of n users. So the new network consists of $n+1$ users. More precisely, we compare a network of n users *ceteris paribus* with a network of $n+1$ users. For simplicity, we firstly consider networks with connection values that are independent of the network's size and afterwards the more general case of networks with changing connection values.

2.1 Networks with connection values that are independent of the network's size

If the connection values are independent of the network's size, we find for the associated matrix of connection values

$$A(n+1) = \begin{pmatrix} & & & & a_{1n+1} \\ & & & & a_{2n+1} \\ & & A(n) & & \vdots \\ & & & & a_{nn+1} \\ a_{n+11} & a_{n+12} & \ldots & a_{n+1n} & 0 \end{pmatrix}.$$

Moreover, we obtain

$$N(n+1) = N(n) + N_{n+1\to}(n+1) + N_{\to n+1}(n+1)\,,$$

i.e. the total utility of the network with $n+1$ users equals the total utility of the network with n users *plus* the total value for the new $(n+1)$-th user *plus* the total value all connections to the new user have for all old users. The marginal utility is thus given by[2]

$$\Delta N(n) := N(n+1) - N(n) = N_{n+1\to}(n+1) + N_{\to n+1}(n+1) \qquad (2)$$

and is the sum of the total value $N_{n+1\to}(n+1)$ for the new user as he or she can connect to all old users plus the total value $N_{\to n+1}(n+1)$ all old users have as they can now connect to the new user.

Example 3 We assume $a_{ij}(n) \equiv a/2$ for $i \neq j$ with $a > 0$ and $a_{ii}(n) \equiv 0$. It follows

$$N_{n+1\to}(n+1) = N_{\to n+1}(n+1) = \frac{an}{2}$$

as there are n connections to and from the new user U_{n+1}. Hence, the marginal utility is given by

$$\Delta N(n) = an\,.$$

□

Let p be the price of a good for one user. In general, p is a (monotonically decreasing) function of n. The expenses for a network of size n are then given by $K(n) = np(n)$ and the total net gain of the network is given by

$$G(n) = N(n) - K(n) = N(n) - np(n)\,.$$

Here we assume that

(H3) the good is of *no* value without the network,

although it would easily be possible to incorporate a positive value of the good in the absence of the network.

For the marginal net gain, it follows

$$\Delta G(n) = \Delta N(n) - \Delta\big(np(n)\big)\,,$$

where

$$\Delta\big(np(n)\big) := (n+1)p(n+1) - np(n) = p(n+1) + n\Delta p(n)\,. \qquad (3)$$

To be precise, the marginal net gain above is the difference between the net gain of a network of size $n+1$ and the net gain of another network of size n *ceteris paribus*.

[2] Here and in the following let $\Delta y(n) := y(n+1) - y(n)$ for any quantity y that depends on the network size n.

From (2) and (3), we conclude

$$\begin{aligned}\Delta G(n) &= N_{n+1\to}(n+1) - p(n+1) + N_{\to n+1}(n+1) - n\Delta p(n) \\ &=: G_{n+1\to}(n+1) + G_{\to n+1}(n+1)\,. \end{aligned} \tag{4}$$

Here, $G_{n+1\to}(n+1) = N_{n+1\to}(n+1) - p(n+1)$ denotes the net gain for the new user U_{n+1}, whereas $G_{\to n+1}(n+1) = N_{\to n+1}(n+1) - n\Delta p(n)$ denotes the net gain for all old users $U_1, \ldots, U_n$ when U_{n+1} joins the network.

The new user will only be willing to enter the network if

$$G_{n+1\to}(n+1) > 0\,, \tag{5}$$

which is a condition for the size of the network and determines the break-even point. On the other hand, the old network will be willing to accept a new user only if

$$G_{\to n+1}(n+1) \geq 0\,, \tag{6}$$

which is in particular true if $p = p(n)$ is constant or monotonically decreasing such that $\Delta p(n) \leq 0$.

Obviously, the smallest integer for which both the conditions (5) and (6) are fulfilled is a minimum size of the network.

Example 4 (Example 3 continued)

(a) Let $p(n) \equiv p$ be independent of n. This might be the case for a well-established network. Then

$$\Delta G(n) = an - p\,, \quad G_{n+1\to}(n+1) = \frac{an}{2} - p\,, \quad G_{\to n+1}(n+1) = \frac{an}{2}\,.$$

A new user will be willing to join the network if

$$n > \frac{2p}{a}\,.$$

There will be *no restriction* on the network's size if $p \leq a/2$ as then $n > 1 \geq 2p/a$. However, the value $a/2$ for one connection is, in general, below the price for a good that allows to connect to each member of the network.

(b) Let now

$$p(n) = p_{\min} + \frac{p_{\max} - p_{\min}}{n}\,, \quad 0 \leq p_{\min} < p_{\max}\,. \tag{7}$$

This is a typical price function that decreases with the number of consumers (cf. Figure 1) following the principle of economies of scale. It follows

$$G_{n+1\to}(n+1) = \frac{an}{2} - \left(p_{\min} + \frac{p_{\max} - p_{\min}}{n+1}\right)$$

as well as

$$G_{\to n+1}(n+1) = \frac{an}{2} - n\Delta p(n) = \frac{an}{2} + \frac{p_{\max} - p_{\min}}{n+1}\,.$$

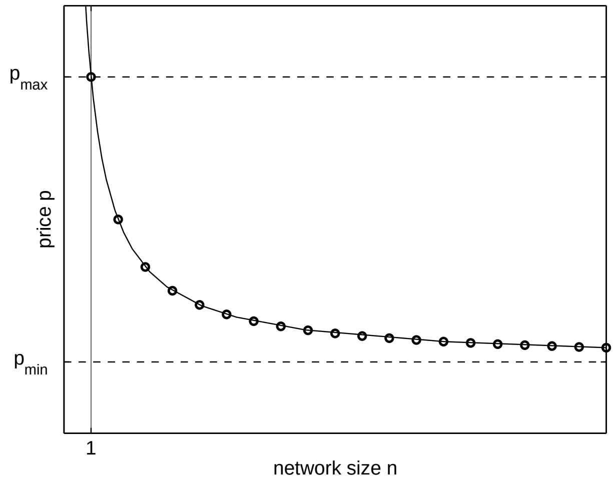

Fig. 1 Price function in Example 3

For large networks, we thus have asymptotically

$$G_{n+1\to}(n+1) \sim \frac{an}{2} - p_{\min}\,, \; G_{\to n+1}(n+1) \sim \frac{an}{2}\,, \; \Delta G(n) \sim an - p_{\min}\,.$$

Note that the total net gain of all old users due to the joining of the new user is nonnegative: $G_{\to n+1}(n+1) \geq 0$. This remains true for arbitrary monotonically decreasing price functions as then $\Delta p(n) \leq 0$.

However, the new user will only be willing to join the network if the net gain $G_{n+1\to}(n+1)$ is positive, i.e. if

$$\frac{an}{2} > p_{\min} + \frac{p_{\max} - p_{\min}}{n+1} = \frac{np_{\min} + p_{\max}}{n+1}\,.$$

This is fulfilled if and only if the network is large enough such that

$$n > \frac{p_{\min}}{a} - \frac{1}{2} + \sqrt{\left(\frac{p_{\min}}{a} - \frac{1}{2}\right)^2 + \frac{2p_{\max}}{a}} =: n^*\,.$$

If e.g. $p_{\min} = a/2$ then $n > \sqrt{p_{\max}/p_{\min}}$ has to be required. There is *no restriction* on the size n if

$$p_{\min} < \frac{3a}{2} \quad \text{and} \quad p_{\max} + p_{\min} \leq a \tag{8}$$

as then $n^* \leq 1$. Again, the value $a/2$ for one connection is, in general, small compared to the price of the good and (8) will be violated.

In Figure 2, the total net gain $G_{n+1\to}(n+1)$ for the new user is illustrated for the rather realistic case that $p_{\min} > a/2$. The critical size is then the integer part of $n^* + 1$.

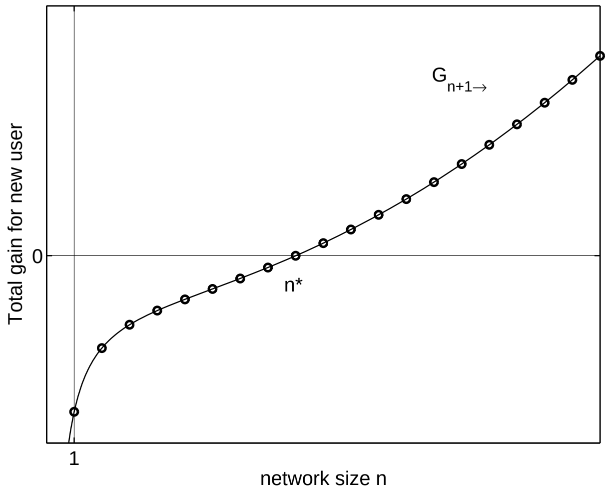

Fig. 2 Total net gain for the new user in Example 3 (b)

(c) We now consider the case of a monotonically increasing price. A first observation is that the net gain $G_{n+1\to}(n+1) = an/2 - p(n+1)$ for the new user will be decreasing with n if $\Delta p(n) > a/2$. So it will be the less attractive to join the network the larger it becomes. In view of

$$G_{\to n+1}(n+1) = \frac{an}{2} - n\Delta p(n) < 0\,,$$

also the old network has no interest in new members if $\Delta p(n) > a/2$. The interest in new members diminishes with increasing n even for $\Delta p(n) \leq a/2$ as long as $p = p(n)$ is such that

$$\Delta\big(n\Delta p(n)\big) > 0\,,$$

which is a condition on the curvature of the price function. Vice versa, a network can –by means of a suitable pricing– remain its size. Such a behaviour can be observed for exclusive networks. □

2.2 Networks with connection values that changes with the network's size

If the connection values a_{ij} are not independent of the network's size n, we obtain because of $a_{n+1n+1} = 0$ from

$$N(n+1) = \sum_{j=1}^{n} a_{n+1j}(n+1) + \sum_{i=1}^{n} a_{in+1}(n+1) + \sum_{i,j=1}^{n} a_{ij}(n+1)$$

$$= N_{n+1\to}(n+1) + N_{\to n+1}(n+1) + \sum_{i,j=1}^{n} a_{ij}(n+1)$$

with $N(n) = \sum_{i,j=1}^{n} a_{ij}(n)$ the relation

$$\Delta N(n) = N_{n+1\rightarrow}(n+1) + N_{\rightarrow n+1}(n+1) + \sum_{i,j=1}^{n} \Delta a_{ij}(n)$$

instead of (2). The additional term $\sum_{i,j=1}^{n} \Delta a_{ij}(n)$ reflects the difference of the utility between a network of size $n+1$ and a network of size n due to the different connection values: Especially for large n, the connection value $a_{ij}(n+1)$ may be less than the corresponding value $a_{ij}(n)$.

Similarly to (4), we find with (3)

$$\Delta G(n) = N_{n+1\rightarrow}(n+1) - p(n+1) + N_{\rightarrow n+1}(n+1) + \sum_{i,j=1}^{n} \Delta a_{ij}(n) - n\Delta p(n)\,.$$

This shows that the marginal net gain is again the sum of the net gain $G_{n+1\rightarrow}(n+1) = N_{n+1\rightarrow}(n+1) - p(n+1)$ of the new user and the net gain

$$G_{\rightarrow n+1}(n+1) = N_{\rightarrow n+1}(n+1) + \sum_{i,j=1}^{n} \Delta a_{ij}(n) - n\Delta p(n)$$

for the old network when U_{n+1} enters it. What is different to the situation with connection values independent of n is the term $\sum_{i,j=1}^{n} \Delta a_{ij}(n)$. As the connection values will often be monotonically decreasing with n (Gossen's law), this term leads to a *reduction of the net gain for the old network if a new user joins it*.

Again the new user will be willing to join the network only if $G_{n+1\rightarrow}(n+1) > 0$. On the other hand, the old network will be willing to accept a new user only if $G_{\rightarrow n+1}(n+1) \geq 0$.

Example 5 If all connections possess the value $a(n)/2$ (see also Example 1) then

$$N_{n+1\rightarrow}(n+1) = N_{\rightarrow n+1}(n+1) = a(n+1)\,\frac{n}{2}\,,$$
$$\Delta N(n) = a(n+1)\,n + \frac{n(n-1)}{2}\,\Delta a(n)\,.$$

It follows

$$G_{n+1\rightarrow}(n+1) = a(n+1)\,\frac{n}{2} - p(n+1)\,, \tag{9}$$

$$G_{\rightarrow n+1}(n+1) = a(n+1)\,\frac{n}{2} + \frac{n(n-1)}{2}\,\Delta a(n) - n\Delta p(n)\,. \tag{10}$$

In view of an overload or capacity bound, the connection values are monotonically decreasing with the network's size. In a simple case, this behaviour might be modelled by the function

$$a(n) = \begin{cases} a_{\max} & \text{for } n \leq n_1\,, \\ a_{\max} - \dfrac{n-n_1}{n_2-n_1}\,(a_{\max} - a_{\min}) & \text{for } n_1 \leq n \leq n_2\,, \\ a_{\min} & \text{for } n_2 \leq n\,, \end{cases} \tag{11}$$

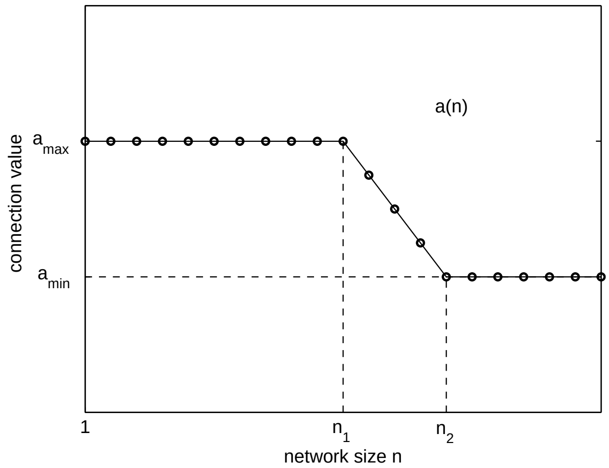

Fig. 3 Connection values in Example 5

where n_1 and n_2 are some specific network sizes and $0 \leq a_{\min} < a_{\max}$ (cf. Figure 3).

(a) Let us now consider a price p that is independent of n which again corresponds to a situation of saturation. The net gain $G_{n+1\to}(n+1)$ of the new user is positive if n is sufficiently large. Indeed, we have to require

$$a(n+1)\,\frac{n}{2} > p\,.$$

This is shown in Figure 4. As we can see, there might be more than one critical

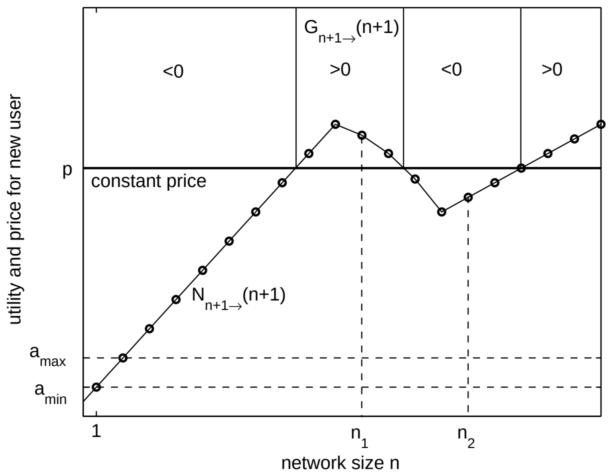

Fig. 4 Total utility and constant price for the new user in Example 5

size of a network. It is a simple but tedious task to calculate these characteristic sizes from (9) and (11). For large networks, we obtain asymptotically

$$G_{n+1\to}(n+1) \sim a_{\min}\,\frac{n}{2} - p\,,\ G_{\to n+1}(n+1) \sim a_{\min}\,\frac{n}{2}\,,\ \Delta G(n) = a_{\min} n - p\,.$$

For an arbitrary function $a = a(n)$ (not given by (11)) and constant price, we infer from (10) the condition

$$a(n+1)\,\frac{n}{2} + \frac{n(n-1)}{2}\,\Delta a(n) = \frac{n}{2}\,(n\Delta a(n) + a(n)) \geq 0$$

in order to have $G_{\to n+1}(n+1) \geq 0$. This is equivalent to the inequality

$$\varepsilon_a(n) := \frac{n\Delta a(n)}{a(n)} \geq -1$$

for the (discrete) elasticity of a that is in particular fulfilled if a is monotonically decreasing and inelastic. An example would be $a(n) = 1/n$ with $\varepsilon_a(n) = -1 + 1/(n+1)$ and a counterexample (for $n \geq 2$) is $a(n) = 1/n^2$ with $\varepsilon_a(n) = -2 + (3n+2)/(n+1)^2$. Indeed, if

$$a(n) = \frac{1}{n^{1+\alpha}} \quad \text{for some } \alpha > 0$$

then for sufficiently large n

$$\varepsilon_a(n) = n^{2+\alpha}\left((n+1)^{-1-\alpha} - n^{-1-\alpha}\right) = n\left(\left(\frac{n}{n+1}\right)^{1+\alpha} - 1\right) < -1\,.$$

Note that $\varepsilon_a(n) \to -(1+\alpha)$ as $n \to \infty$.

Moreover, for $a(n) = a(\log n)/n$ with a constant $a > 0$ (which leads to the suggestion $N(n) \sim n \log n$ as in Briscoe *et al.* (2006), Odlyzko and Tilly (2005)), we find $\varepsilon_a(n) > -1$ and thus $G_{\to n+1}(n+1) \geq 0$.

(b) If the price itself is changing with the network's size, the situation becomes more complicated. In Figure 5, the price is given by the function (7). Nevertheless, it is possible to calculate the critical network sizes from given data. In turn, one can also calculate the price that leads to a positive net gain for the new user if the network is of a given size.

It remains to analyse for which sizes the net gain $G_{\to n+1}(n+1)$ is nonnegative, i.e

$$a(n+1)\,\frac{n}{2} - n\Delta p(n) \geq -\frac{n(n-1)}{2}\,\Delta a(n)\,.$$

Supposing a monotonically decreasing price and a monotonically decreasing connection value, such that $\Delta p(n) \leq 0$ and $\Delta a(n) \leq 0$, the net gain $G_{\to n+1}(n+1)$ is nonnegative if again the network has reached a sufficiently large size as in the previous case (a).

(c) The case of an increasing price follows, although somewhat more involved here, the same lines as in Example 4. In particular, potential new members are the less interested in joining the network the larger $\Delta p(n)$ is. This can be seen from (4). □

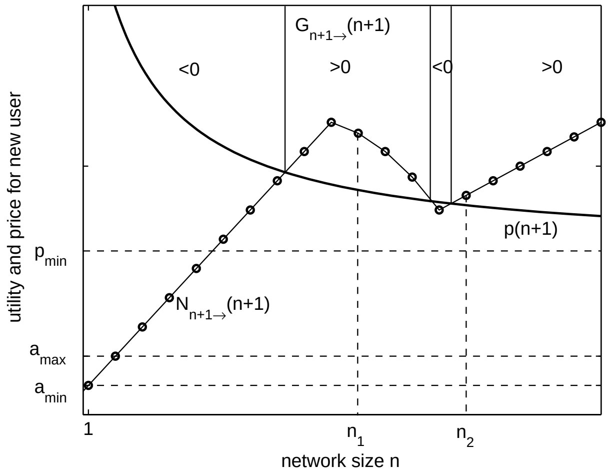

Fig. 5 Total utility and changing price for the new user in Example 5

3 Conclusions

We have shown that the primary network effect given by the marginal net gain of the network can be divided into the net gain $G_{n+1\to}$ the new user has from joining the network plus the net gain $G_{\to n+1}$ the old users have if a new user joins the network. A new user will join the network only if $G_{n+1\to} > 0$ and the network will allow the joining only if $G_{\to n+1} \geq 0$. Both conditions can only be fulfilled if the network's size ranges between critical sizes.

If the value of a connection varies with the size of the network, as is the case when modelling overload, saturation or diminishing interest in the network, then $G_{\to n+1}$ depends on the change in the connection values. In particular, the elasticity of the function $a = a(n)$ of the value for each connection depending on the network size n has to be greater or equal -1 (if, for simplicity, the price is assumed to be constant) in order to have $G_{\to n+1} \geq 0$. This can be interpreted as a condition for the connection values but also for the size of the network. So, a limit of growth can be derived for networks in which connection values decrease with the size of the network.

The model established here allows to calculate critical sizes for a network if the connection values and price function are known *a priori*. It also allows to model exclusive networks that are of limited size.

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