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TWO-STEP BDF TIME DISCRETISATION OF NONLINEAR EVOLUTION PROBLEMS GOVERNED BY MONOTONE OPERATORS WITH STRONGLY CONTINUOUS PERTURBATIONS

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Abstract — The time discretisation of the initial-value problem for a first-order evolution equation by the two-step backward differentiation formula (BDF) on a uniform grid is analysed. The evolution equation is governed by a time-dependent monotone operator that might be perturbed by a time-dependent strongly continuous operator. Well-posedness of the numerical scheme, a priori estimates, convergence of a piecewise polynomial prolongation, stability as well as smooth-data error estimates are provided relying essentially on an algebraic relation that implies the G-stability of the two-step BDF with constant time steps.

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1. Introduction

In this paper, we are concerned with the time discretisation of the initial-value problem for a nonlinear evolution equation,

$$u' + Au = f \quad \text{in } (0,T), \quad u(0) = u_0. \tag{1.1}$$

The operator A is supposed to be the sum of Nemytskii operators A_0 and B corresponding to the families of nonlinear operators $\{A_0(t)\}_{t\in[0,T]}$ and $\{B(t)\}_{t\in[0,T]}$, respectively. The main assumptions are that, uniformly in $t \in [0,T]$, $A(t) + \kappa I : V \to V^*$ (with $V \subseteq H \subseteq V^*$ being a Gelfand triple and I being the identity) is coercive for some $\kappa \geqslant 0$, $A_0(t) + \kappa I : V \to V^*$ is hemicontinuous and monotone, and $B(t) : V \to V^*$ is strongly continuous, has range in H, and fulfills a local Lipschitz-type condition. Moreover, $A_0(t)$ and $B(t)$ are supposed to fulfill a growth condition.

The time discretisation under consideration is a two-step backward differentiation formula (BDF) with constant time steps $\Delta t = T/N$ ($N \in \mathbb{N}$) for the computation of $u^n \approx u(t_n)$ ($n = 2, 3, \ldots, N$, $t_n = n\Delta t$),

$$\frac{1}{\Delta t}\left(\frac{3}{2}u^n - 2u^{n-1} + \frac{1}{2}u^{n-2}\right) + A(t_n)u^n = f^n, \quad n = 2, 3, \ldots, N, \tag{1.2}$$

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with given approximation $\{f^n\}_{n=1}^N$ of the right-hand side f and starting values $u^0 \approx u_0$, $u^1 \approx u(\Delta t)$.

Nonlinear evolution equations of first order are well known for the mathematical description of many time-dependent real-world phenomena, and so there is a vast literature on their analysis as well as approximate solution (cf. the monographs of Barbu [5], Brézis [7], Dautray and Lions [10], Fujita *et al.* [15], Gajewski *et al.* [16], Henry [26], Kačur [29], Lions [33], Lunardi [37], Martin [38], Pazy [42], Rektorys [43], Roubíček [44], Sell and You [45], Showalter [46], Tanabe [50], Temam [53], Zeidler [55], and the references therein). Our framework follows essentially Emmrich [14], Gajewski *et al.* [16], Roubíček [44], and Zeidler [55].

For the time discretisation of linear evolution problems, we refer to the standard text of Thomée [54]. Many authors have analysed the backward Euler method for the time discretisation of particular nonlinear partial differential equations and abstract evolution equations. There is, however, comparatively little knowledge about the stability, convergence, and error estimates for other schemes, especially as applied to rather general classes of nonlinear evolution problems.

The approximation of semilinear evolution equations by means of single-step methods was considered, e.g., by Crouzeix and Thomée [9], Slodička [47, 48], Lubich and Ostermann [36], and by means of linear multistep methods, e.g., by Hill and Süli [27]. Calvo and Palencia [8] studied explicit multistep exponential integrators. Implicit-explicit multistep methods were considered by Akrivis *et al.* [1–3].

The time discretisation for a certain class of quasi-linear evolution problems by explicit multistep schemes was studied by Hass and Kreth [25]. Zlámal [56] has analysed A-stable two- and one-step methods for quasi-linear parabolic problems of second order, and Le Roux [32] analysed A(θ)-stable multistep methods for abstract quasi-linear evolution equations. For the analysis of Runge — Kutta methods applied to quasi-linear evolution equations, we refer to González and Palencia [19] and Lubich and Ostermann [34].

Linearisation was used by Lubich and Ostermann [35] in order to prove stability and error estimates for linearly implicit one-step methods applied to nonlinear evolution equations posed in a Gelfand triple. The backward Euler and strongly $A(\theta)$-stable Runge — Kutta discretisations of fully nonlinear problems, which are governed by a densely defined nonlinear mapping in a Banach space whose first Fréchet derivative is sectorial, have been dealt with, again by linearisation, Ostermann and Thalhammer [40] (see also [18] for a similar approach). Within the same analytical framework, stability of linear multistep methods has been studied in [41].

Axelsson and Gololobov [4] have derived stability and error estimates for the θ-scheme applied to an evolution equation governed by a strongly monotone operator. Evolution equations governed by maximal monotone operators and their approximation by Runge — Kutta as well as linear multistep methods (including the two-step BDF) have been studied by Hansen [22–24].

Among the abundance of methods, the backward differentiation formulae (BDF) seem to be of particular interest as they are favourable for the integration of stiff problems. For analysis of the BDF applied to ordinary differential equations, we refer in particular to the monographs of Hairer et al. [20, 21] and Stuart and Humphnes [49].

Error estimates for the two-step BDF with constant time steps applied to the semilinear incompressible Navier — Stokes problem have been considered, e.g., by Girault and Raviart [17], Le Roux [32], Hill and Süli [28], and Emmrich [12], whereas the convergence of a piecewise polynomial prolongation of the discrete solution has been shown in [13]. In [39], singly implicit Runge — Kutta methods and also BDF have been applied to a class of quasi-linear parabolic problems. Emmrich [11] has studied the two-step BDF with variable time steps for a class of mildly semilinear evolution equations. Kreth [30, 31] has studied the two-step BDF with constant time steps applied to a nonlinear evolution problem governed by a family of time-dependent, continuous, strongly monotone operators mapping a Hilbert space into itself.

In this paper, we prove, for a large class of nonlinear evolution problems, the convergence of piecewise polynomial prolongations of the discrete numerical solution towards a weak solution to the nonlinear initial-value problem (1.1). *A priori* estimates for the numerical solution are the essential prerequisite for the convergence. Moreover, we show stability of the numerical solution with respect to the data. Stability estimates uniform in Δt then allow to derive *a priori* error estimates for sufficiently regular solutions.

All results rely upon the theory of monotone operators and compactness arguments together with algebraic relations describing the properties of the temporal discretisation. Possible extensions and restrictions of the results are exemplified for the incompressible Navier — Stokes problem for which A_0 is linear.

It should be noted that results analogous to those obtained here for the two-step BDF can similarly, although somewhat more easily, be derived for the implicit Euler method (cf. also corresponding results in [44, Ch. 8.2]). For evolution equations governed by a monotone potential operator, Roubíček [44, Rem. 8.20] postulates convergence results for the two-step BDF relying upon an algebraic relation that is different from those employed here.

Unlike other work, we allow explicitly time depending operators and perturbations of the monotone main part. Moreover, our results do not rely upon linearisation and, therefore, do not require differentiability of the nonlinear operator A. Finally, the operator A is not supposed to be a potential operator. Note, however, that our assumptions imply global well-posedness of the original problem (1.1) which is different from the approach in, e.g., [40], [18], and [41].

The paper is organized as follows: In Section 2, we describe the analytical framework and assumptions on the nonlinear operator A. Moreover, we collect some results concerning the well-posedness of the original problem (1.1). The discretisation and its main properties are discussed in Section 3. *A priori* estimates and the main convergence result are then proven in Section 4. Finally, in Section 5, stability and error estimates are derived.

2. Time continuous problem

Let $V \subseteq H \subseteq V^*$ be a Gelfand triple with a reflexive, separable, real Banach space $(V, \|\cdot\|)$ which is dense and continuously embedded in the Hilbert space $(H, (\cdot,\cdot), |\cdot|)$. The dual V^* of V is equipped with the usual norm $\|f\|_* := \sup_{v\in V\setminus\{0\}} \langle f, v\rangle/\|v\|$, where $\langle\cdot,\cdot\rangle$ denotes the dual pairing. Sometimes we emphasise the spaces by a subscript as in $\langle\cdot,\cdot\rangle_{V^*\times V}$.

For a Banach space X and the time interval $[0, T]$, let $L^r(0, T; X)$ ($r \in [1, \infty]$) be the usual space of Bochner integrable (for $r = \infty$ Bochner measurable and essentially bounded) abstract functions. The discrete counterpart for functions defined on a time grid will be denoted by $l^r(0, T; X)$. In the following, let $p \in (1, \infty)$ and let $q = p/(p-1)$ be the conjugated exponent. The dual pairing between $L^p(0, T; V)$ and $L^q(0, T; V^*) = (L^p(0, T; V))^*$ is given by

$$\langle f, v\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)} = \int_0^T \langle f(t), v(t)\rangle_{V^*\times V} dt.$$

Similarly we have $(L^1(0,T;H))^* = L^\infty(0,T;H)$ and

$$\langle f, v\rangle_{L^\infty(0,T;H)\times L^1(0,T;H)} = \int_0^T (f(t), v(t)) dt.$$

The inner product in $L^2(0,T;H)$ is denoted by $(\cdot,\cdot)_{L^2(0,T;H)}$.

The space

$$\mathcal{X} := L^p(0,T;V) \cap L^2(0,T;H), \quad \|v\|_{\mathcal{X}} := \|v\|_{L^p(0,T;V)} + \|v\|_{L^2(0,T;H)},$$

is a reflexive, separable Banach space. The dual space $\mathcal{X}^*$ can be identified with $L^q(0,T;V^*)+L^2(0,T;H)$, equipped with the norm

$$\|f\|_{\mathcal{X}^*} := \inf_{\substack{f_1\in L^q(0,T;V^*),\, f_2\in L^2(0,T;H)\\ f=f_1+f_2}} \max\left(\|f_1\|_{L^q(0,T;V^*)}, \|f_2\|_{L^2(0,T;H)}\right).$$

If f possesses the representation $f = f_1 + f_2$ with $f_1 \in L^q(0,T;V^*)$, $f_2 \in L^2(0,T;H)$ then the dual pairing between $f \in \mathcal{X}^*$ and $v \in \mathcal{X}$ is given by

$$\langle f, v\rangle_{\mathcal{X}^*\times\mathcal{X}} = \int_0^T \left(\langle f_1(t), v(t)\rangle_{V^*\times V} + (f_2(t), v(t))\right) dt = \int_0^T \langle f(t), v(t)\rangle_{V^*\times V} dt$$

(see also [16] for more details). Note that $\mathcal{X} \subseteq L^2(0,T;H) \subseteq \mathcal{X}^*$ forms a Gelfand triple. If $p \geqslant 2$, then $\mathcal{X} = L^p(0,T;V)$, $\mathcal{X}^* = L^q(0,T;V^*)$, and we work with the standard norms $\|v\|_{\mathcal{X}} := \|v\|_{L^p(0,T;V)}$, $\|f\|_{\mathcal{X}^*} := \|f\|_{L^q(0,T;V^*)}$.

By v', we denote the time derivative of v in the distributional sense. We remember that for $p \in (1,\infty)$ the Banach space

$$\mathcal{W} := \{v \in \mathcal{X} : v' \in \mathcal{X}^*\}, \quad \|v\|_{\mathcal{W}} := \|v\|_{\mathcal{X}} + \|v'\|_{\mathcal{X}^*},$$

is continuously embedded in $\mathcal{C}([0,T];H)$, the space of uniformly continuous functions with values in H. Due to the compactness theorem by Aubin and Lions (see Lions [33, Thm. 5.2 in Ch. 1]) and the continuous embeddings $\mathcal{X} \hookrightarrow L^p(0,T;V)$, $\mathcal{X}^* \hookrightarrow L^{\min(q,2)}(0,T;V^*)$, we also have $\mathcal{W} \overset{c}{\hookrightarrow} L^p(0,T;H)$ if $V \overset{c}{\hookrightarrow} H$. A sequence that is bounded in $\mathcal{W}$ thus possesses a subsequence that is strongly convergent in $L^p(0,T;H)$. Because of the boundedness in $L^\infty(0,T;H)$ (remember that $\mathcal{W} \hookrightarrow \mathcal{C}([0,T];H)$), there is only one in $L^r(0,T;H)$ for any $r \in [1,\infty)$ follows. Therefore, $\mathcal{W} \overset{c}{\hookrightarrow} L^r(0,T;H)$ for all $r \in [1,\infty)$ if $V \overset{c}{\hookrightarrow} H$.

Let $\{A_0(t)\}_{t\in[0,T]}$ and $\{B(t)\}_{t\in[0,T]}$ be two families of operators $A_0(t) : V \to V^*$ and $B(t) : V \to V^*$. We set $A(t) := A_0(t) + B(t)$ $(t \in [0,T])$ and associate the Nemytskii operators A_0, B, A acting on abstract functions via $(A_0v)(t) := A_0(t)v(t)$, $(Bv)(t) = B(t)v(t)$, $(Av)(t) = A(t)v(t)$ $(t \in [0,T])$ for the function $v : [0,T] \to V$. If $B \neq 0$, then V is assumed to be compactly embedded in H. Each result in this paper relies upon one or more of the following structural assumptions, where $\kappa \geqslant 0$ and $p \in (1,\infty)$ are suitable numbers:

(H1) The mappings $t \mapsto A_0(t)$ and $t \mapsto B(t)$ are weakly measurable on $(0,T)$, i.e., for all $v, w \in V$, $t \mapsto \langle A_0(t)v, w\rangle$ and $t \mapsto \langle B(t)v, w\rangle$ are Lebesgue measurable on $(0,T)$.

(H2) The operators $A_0(t) + \kappa I : V \to V^*$ ($t \in [0,T]$), where I denotes the identity, are hemicontinuous and monotone.

(H3) The operators $B(t)$ ($t \in [0,T]$) map V into H. There is some $\delta \in (0,p]$ and for any $R > 0$ there is some $\beta = \beta(R) > 0$ such that for all $t \in [0,T]$ and $v, w \in V$ with $\max(|v|, |w|) \leqslant R$

$$|B(t)v - B(t)w| \leqslant \beta(R)\left(1 + \|v\|^{p-\delta} + \|w\|^{p-\delta}\right)|v - w|.$$

(H4) For any $R > 0$ there is some $\alpha = \alpha(R) > 0$ such that for all $t \in [0,T]$ and $v \in V$ with $|v| \leqslant R$

$$\|A(t)v\|_* \leqslant \|A_0(t)v\|_* + \|B(t)v\|_* \leqslant \alpha(R)\left(1 + \|v\|^{p-1}\right).$$

(H5) There are constants $\mu > 0$ and $\lambda \geqslant 0$ such that for all $t \in [0,T]$ and $v \in V$

$$\langle (A(t) + \kappa I)v, v\rangle \geqslant \mu\|v\|^p - \lambda.$$

For $B(t)$ ($t \in [0,T]$), the growth condition (H4) already follows from (H3) if $\delta \geqslant 1$ and $t \mapsto |B(t)0|$ is bounded. Besides (H1), we sometimes need the following stronger assumption

(H1′) For all $v \in V$, the mappings $t \mapsto A_0(t)v$ and $t \mapsto B(t)v$ with values in V^* are continuous a.e. in $(0,T)$.

Instead of (H2), we also work with the assumption

(H2′) The operators $A_0(t) + \kappa I : V \to V^*$ ($t \in [0,T]$) are hemicontinuous and there is a constant $\mu_0 > 0$ such that for all $t \in [0,T]$ and $v, w \in V$

$$\langle (A_0(t) + \kappa I)v - (A_0(t) + \kappa I)w, v - w\rangle \geqslant \mu_0\|v - w\|^p.$$

With (H2′), we have to suppose that $p \geqslant 2$ because there is no monotone operator fulfilling the property above with $p \in [1,2)$. Note that (H2′) is stronger than (H2) and implies uniform monotonicity and thus coercivity of $A_0(t) + \kappa I$.

In order to obtain stability and error estimates, we may rely upon the following assumption instead of (H3) if (H2′) holds true:

(H3′) The operators $B(t) : V \to V^*$ ($t \in [0,T]$) are strongly continuous.

Let $p = 2$. There is some $\delta \in (0,2]$ and $s \in (0,1]$, and for any $R > 0$ there exists $\beta = \beta(R) > 0$ such that for all $t \in [0,T]$ and $v, w \in V$ with $\max(|v|, |w|) \leqslant R$

$$\langle B(t)v - B(t)w, v - w\rangle \geqslant -\beta(R)\left(1 + \|v\|^{2-\delta} + \|w\|^{2-\delta}\right)^s |v - w|^s\|v - w\|^{2-s}.$$

Let $p > 2$. There is some $\delta \in (0,p]$ and for any $R > 0$ there exists $\beta = \beta(R) > 0$ such that for all $t \in [0,T]$ and $v, w \in V$ with $\max(|v|, |w|) \leqslant R$

$$\langle B(t)v - B(t)w, v - w\rangle \geqslant -\beta(R)\left(1 + \|v\|^{p-\delta} + \|w\|^{p-\delta}\right)|v - w|^2. \tag{2.1}$$

Obviously, there is a gap between the two cases $p = 2$ and $p > 2$ as the condition for $p = 2$ is much weaker than that for $p > 2$. However, we were not able to prove reasonable stability or error estimates in the case $p > 2$ with an assumption that is weaker than condition (2.1) and allows a potency of $\|v - w\|$.

Assumption (H3$'$) for $p > 2$ immediately follows from (H3). Moreover, (H3$'$) for $p = 2$ follows from either of the following two continuity assumptions (the first one leads to $s = 1$):

(H3$''_{p=2}$) The operators $B(t) : V \to V^*$ ($t \in [0,T]$) are strongly continuous and their range is in H. There is some $\delta \in (0,2]$ and for any $R > 0$ there exists $\beta = \beta(R) > 0$ such that for all $t \in [0,T]$ and $v, w \in V$ with $\max(|v|, |w|) \leqslant R$

$$|B(t)v - B(t)w| \leqslant \beta(R) \left(1 + \|v\|^{2-\delta} + \|w\|^{2-\delta}\right) \|v - w\|.$$

(H3$'''_{p=2}$) There is some $\delta \in (0,2]$ and $s \in (0,1]$, and for any $R > 0$ there exists $\beta = \beta(R) > 0$ such that for all $t \in [0,T]$ and $v, w \in V$ with $\max(|v|, |w|) \leqslant R$

$$\|B(t)v - B(t)w\|_* \leqslant \beta(R) \left(1 + \|v\|^{2-\delta} + \|w\|^{2-\delta}\right)^s |v - w|^s \|v - w\|^{1-s}.$$

In applications, (H3$'''_{p=2}$) will be weaker than (H3$''_{p=2}$) and s will be small. The growth condition (H4) follows for $B(t)$ ($t \in [0,T]$) already from (H3$'''_{p=2}$) if $\delta \geqslant 1$ and $t \mapsto \|B(t)0\|_*$ is bounded. If $V \overset{c}{\hookrightarrow} H$, then (H3), as well as (H3$'''_{p=2}$), this is a fixed term of $B(t) : V \to V^*$ ($t \in [0,T]$). We remark that, for obtaining convergence and error estimates, the continuity assumptions on $B(t)$ ($t \in [0,T]$) can be further relaxed as we will discuss later.

Finally, the coercivity assumption (H5) can, at least for some of the results, be replaced by the semicoercivity of $A(t) + \kappa I$ ($t \in [0,T]$): Let $|\!|\!| \cdot |\!|\!|$ be a seminorm on V and let there be a constant $c > 0$ such that for all $v \in V$

$$\|v\| \leqslant c\,(|\!|\!|v|\!|\!| + |v|)\,.$$

Then $A(t) + \kappa I$ ($t \in [0,T]$) is said to be semicoercive (see Roubiček [44, p. 202]) if for all $v \in V$

$$\langle (A(t) + \kappa I)v, v\rangle \geqslant \mu |\!|\!|v|\!|\!|^p - \lambda.$$

Indeed, with

$$(a + b)^r \leqslant 2^{r-1}(a^r + b^r), \quad a, b \geqslant 0,\ r \geqslant 1, \tag{2.2}$$

it can easily be shown that then

$$\langle (A(t) + (\kappa + \mu)I)v, v\rangle \geqslant \mu \left(|\!|\!|v|\!|\!|^p + |v|^2\right) - \lambda \geqslant \mu \left(|\!|\!|v|\!|\!|^{\min(p,2)} + |v|^{\min(p,2)} - 2\right) - \lambda \geqslant$$

$$2^{1-\min(p,2)}\mu\,(|\!|\!|v|\!|\!| + |v|)^{\min(p,2)} - 2\mu - \lambda \geqslant 2^{1-\min(p,2)} c^{-\min(p,2)} \mu \|v\|^{\min(p,2)} - 2\mu - \lambda,$$

and so (H5) follows with $p := \min(p,2)$, $\kappa := \kappa + \mu$, $\lambda := 2\mu + \lambda$, $\mu := 2^{1-\min(p,2)} c^{-\min(p,2)} \mu$.

A typical example for the functional setting above is given by the initial-boundary value problem for the nonlinear differential equation for $u = u(x,t)$

$$\partial_t u - \nabla \cdot \left(\varphi(x, t, |\nabla u|^{p-1}) |\nabla u|^{p-2} \nabla u\right) + a_0(x, t, u, \nabla u) + b(x, t, u, \nabla u) = f$$

with $p \geqslant 2$ in a bounded, sufficiently smooth domain $\Omega \subset \mathbb{R}^d$, supplemented by, e.g., homogeneous Dirichlet boundary conditions. The bounded function $\varphi : \overline{\Omega} \times [0,T] \times \mathbb{R}_0^+ \to \mathbb{R}$ with a positive lower bound is supposed to fulfill the Carathéodory condition. Moreover,

$y \mapsto \varphi(\cdot,\cdot,y)y$ is assumed to be monotonically increasing and Lipschitz continuous. Under suitable assumptions on the Carathéodory functions a_0 and b and taking $V = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$, the main part of the spatial differential operator and the semilinearity corresponding to a_0 may be collected within the operator A_0, whereas the semilinearity corresponding to b determines the operator B. Some simple concrete examples for the case $p = 2$ are $a_0(x,t,u,\nabla u) = c(x,t)\cdot\nabla u$ with $\nabla\cdot c(x,t) \leqslant 2\kappa$ a.e. in $\Omega\times(0,T)$ for some $\kappa \geqslant 0$, $a_0(x,t,u,\nabla u) = u|u|^\gamma$ with $\gamma < 4/d$ for $d \geqslant 2$, $b(x,t,u,\nabla u) = \sin u$. See also [16, pp. 68 ff., 215 ff.], [44, pp. 232 ff.], and [55, pp. 567 ff., 590 ff., 779 ff.] for applications.

Some essential properties of the Nemytskii operators A_0, B, A are collected in the following proposition.

Proposition 2.1. *Under assumptions* (H1) *and* (H4)*, the operators A_0, B, and $A = A_0 + B$ map $L^p(0,T;V)\cap L^\infty(0,T;H)$ into $(L^p(0,T;V))^* = L^q(0,T;V^*)$ and are bounded.*

If, in addition, (H5) *holds true, then $A+\kappa I$ fulfills for all $v \in L^p(0,T;V)\cap L^\infty(0,T;H)$*

$$\langle (A+\kappa I)v, v\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)} \geqslant \mu\|v\|^p_{L^p(0,T;V)} - \lambda T.$$

Moreover, (H2) *implies that $A_0+\kappa I : L^p(0,T;V)\cap L^\infty(0,T;H) \subset L^p(0,T;V) \to L^q(0,T;V^*)$ is hemicontinuous and monotone. With* (H2$'$) *instead of* (H2)*, $A_0+\kappa I$ is in addition uniformly monotone such that for all $v, w \in L^p(0,T;V)\cap L^\infty(0,T;H)$*

$$\langle (A_0+\kappa I)v - (A_0+\kappa I)w, v-w\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)} \geqslant \mu_0\|v-w\|^p_{L^p(0,T;V)}.$$

With (H3)*, B is a strongly continuous mapping from $\mathcal{W}$ into $L^r(0,T;H)$ for all $r \in [1, p/(p-\delta))$. With* (H3$''_{p=2}$) *and $\delta \in [1,2]$, B is a continuous mapping from $L^2(0,T;V)\cap L^\infty(0,T;H)$ into $L^r(0,T;H)$ for all $r \in [1, 2/(3-\delta)]$. With (H3$'''_{p=2}$), B is a strongly continuous mapping from $\mathcal{W}(0,T)$ into $L^r(0,T;V^*)$ with $r \in [1, 2/(1-(\delta-1)s))$.*

The proof of the first assertions follows arguments similar to those given in [14, Lemma 8.4.4] and is omitted here. We only note that the Bochner measurability of the images results from Pettis' theorem together with Carathéodory properties that are fulfilled in particular due to the monotonicity and continuity of $A_0(t)+\kappa I$ and $B(t)$ ($t \in [0,T]$), respectively. The proof of the continuity statements for B relies upon the compact embedding of $\mathcal{W}$ into $L^r(0,T;H)$ for arbitrary $r \in [1,\infty)$ and Hölder's inequality.

The following theorem summarises the results on the well-posedness of (1.1) under our structural assumptions.

Theorem 2.1. *If* (H1), (H2), (H3) *or* (H3$'''_{p=2}$), (H4), (H5) *hold true, then, for any $u_0 \in H$ and $f \in \mathcal{X}^*$, there is a unique solution $u \in \mathcal{W}$ to the initial-value problem* (1.1) *such that* (1.1) *holds in $\mathcal{X}^*$.*

Let $f = f_1 + f_2$ with $f_1 \in L^q(0,T;V^*)$, $f_2 \in L^2(0,T;H)$. The solution then satisfies for all $t \in [0,T]$ the a priori estimates

$$|u(t)|^2 + \int_0^t \|u(s)\|^p ds \leqslant c\Big(|u_0|^2 + \int_0^t \big(\|f_1(s)\|_*^q + |f_2(s)|^2\big)\,ds + \lambda t\Big) =: M,$$

$$\int_0^t \|u'(s) - f_2(s)\|_*^q ds + \int_0^t \|A(s)u(s)\|_*^q ds \leqslant M', \tag{2.3}$$

where M' depends on M and is bounded on bounded subsets.

Moreover, the following stability estimates are fulfilled for solutions $u, v \in \mathcal{W}$ *with initial data* $u_0, v_0 \in H$ *and right-hand sides* f, g*: Let* $f, g \in L^2(0,T;H)$*, then*

$$|u(t)-v(t)|^2 \leqslant c\Big(|u_0-v_0|^2 + \int\limits_0^t |f(s)-g(s)|^2 ds\Big),$$

where $c > 0$ *depends on the a priori bounds for* u *and* v*. Let* $f, g \in L^q(0,T;V^*)$ *and assume, in addition,* (H2′) *and* (H3′)*. Then*

$$|u(t)-v(t)|^2 + \int\limits_0^t \|u(s)-v(s)\|^p ds \leqslant c\Big(|u_0-v_0|^2 + \int\limits_0^t \|f(s)-g(s)\|_*^q ds\Big),$$

where $c > 0$ *again depends on the a priori bounds for* u *and* v*.*

Proof. The existence proof can be carried out by means of a Galerkin approximation or the Rothe method but the existence of a solution also follows from Theorem 4.1 below (with somewhat stronger assumptions). Uniqueness immediately follows from (2.4) below. The proof of the *a priori* estimates follows standard arguments.

For the first stability estimate, we subtract the equations for u and v and test with $u-v$. With the monotonicity assumption (H2) and the continuity assumption (H3), we find

$$\frac{1}{2}\frac{d}{dt}|u(t)-v(t)|^2 - \kappa|u(t)-v(t)|^2 - \beta(M)\left(1+\|u(t)\|^{p-\delta}+\|v(t)\|^{p-\delta}\right)|u(t)-v(t)|^2 =$$

$$\langle f(t)-g(t), u(t)-v(t)\rangle \leqslant |f(t)-g(t)||u(t)-v(t)| \leqslant \frac{1}{2}\,|f(t)-g(t)|^2 + \frac{1}{2}|u(t)-v(t)|^2, \quad (2.4)$$

where M denotes the maximum of the two *a priori* bounds from (2.3) for u_0, f and v_0, g. With

$$\frac{d}{dt}\left(e^{-\int_0^t \Lambda(s)ds}|u(t)-v(t)|^2\right) = e^{-\int_0^t \Lambda(s)ds}\left(\frac{d}{dt}|u(t)-v(t)|^2 - \Lambda(t)|u(t)-v(t)|^2\right),$$

$$\Lambda(t) := \left(1+2\kappa+2\beta(M)\left(1+\|u(t)\|^{p-\delta}+\|v(t)\|^{p-\delta}\right)\right),$$

we obtain

$$|u(t)-v(t)|^2 \leqslant e^{\int_0^t \Lambda(s)ds}\Big(|u_0-v_0|^2 + \int\limits_0^t |f(s)-g(s)|^2 ds\Big).$$

This proves, together with (2.3), the stability estimate since

$$\int\limits_0^T \Lambda(s)ds \leqslant (1+2\kappa+2\beta(M))T + 2\beta(M)\int\limits_0^T \left(\|u(s)\|^{p-\delta}+\|v(s)\|^{p-\delta}\right) ds.$$

Assuming ($\mathrm{H3}'''_{p=2}$) instead of (H3), the estimate can be derived in a similar way by employing Young's inequality. Assuming (H2′) and (H3′), the second stability estimate is proven analogously. □

Analogous *a priori* and stability estimates will later be derived for the time-discrete problem.

For $B \equiv 0$ and with a growth condition more restrictive than (H4) (independent on R), the existence and uniqueness are also shown in, e.g.,[5, Thm. 4.2 on p. 167] and [55, Thm. 30.A]. Roubíček [44, Thm. 8.28] provides existence results for a larger class of evolution problems, including the case $B \not\equiv 0$. The case $B \not\equiv 0$ is also considered in [14, Satz 8.4.2] but under more restrictive assumptions. Regularity results for evolution equations governed by a monotone operator are provided, e.g., in [16, pp. 217 ff.] and [44, Thm. 8.16, 8.18].

3. Time discrete problem

For $N \in \mathbb{N}$, let $\Delta t := T/N$, $t_n := n\Delta t$ $(n = 0, 1, \dots, N)$. For a grid function $\{u^n\}_{n=n_0}^N$ $(n_0 \in \{0, 1, \dots, N\})$ with values in a Banach space $(X, \|\cdot\|_X)$, we write $\{u^n\}_{n=n_0}^N \in l^r(0, T; X)$ $(r \in [1, \infty])$ if

$$\Delta t \sum_{n=n_0}^{N} \|u^n\|_X^r \ (r \in [1,\infty)) \quad \text{and} \quad \max_{n=n_0,\dots,N} \|u^n\|_X \ (r = \infty),$$

respectively, is bounded independently of Δt. Moreover, we define the divided differences

$$\mathrm{D}_1 u^n := \frac{u^n - u^{n-1}}{\Delta t},$$

$$\mathrm{D}_2 u^n := \frac{3}{2}\mathrm{D}_1 u^n - \frac{1}{2}\mathrm{D}_1 u^{n-1} = \frac{1}{\Delta t}\left(\frac{3}{2}u^n - 2u^{n-1} + \frac{1}{2}u^{n-2}\right),$$

$$\mathrm{D}^2 u^n := \frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2},$$

and the extrapolation

$$\mathrm{E}u^n := 2u^{n-1} - u^{n-2}.$$

For a Bochner integrable function f, we define the natural restrictions

$$\mathrm{R}_1^n f := \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} f(t)\,dt, \quad \mathrm{R}_2^n f := \frac{3}{2}\mathrm{R}_1^n f - \frac{1}{2}\mathrm{R}_1^{n-1} f.$$

For a smooth function u, we have $\mathrm{R}_k^n u' = \mathrm{D}_k u(t_n) = u'(t_n) + \mathcal{O}\left((\Delta t)^k\right)$ $(k \in \{1, 2\})$, $\mathrm{D}^2 u(t_n) = u''(t_n) + \mathcal{O}\left((\Delta t)^2\right)$ as well as $\mathrm{E}u(t_n) = u(t_n) + \mathcal{O}\left((\Delta t)^2\right)$. Let $f = f_1 + f_2$ with $f_1 \in L^q(0, T; V^*)$, $f_2 \in L^2(0, T; H)$. By standard arguments, it can be shown that for $k \in \{1, 2\}$, $n = k, k+1, \dots, N$,

$$\Delta t \sum_{j=k}^{n} \|\mathrm{R}_k^j f_1\|_*^q \leqslant c \int_0^{t_n} \|f_1(t)\|_*^q dt, \quad \Delta t \sum_{j=k}^{n} |\mathrm{R}_k^j f_2|^2 \leqslant c \int_0^{t_n} |f_2(t)|^2 dt,$$

and so $\{\mathrm{R}_k^n f\}_{n=k}^N \in l^q(0, T; V^*) + l^2(0, T; H)$. Here and in the following, we denote by c a generic positive constant that is independent of Δt.

The G-stability of the two-step BDF on an equidistant time grid follows from the algebraic identity

$$4\left(\frac{3}{2}a - 2b + \frac{1}{2}c\right)a = a^2 + (2a-b)^2 - b^2 - (2b-c)^2 + (a - 2b + c)^2, \quad a, b, c \in \mathbb{R},$$

which immediately proves for $\{u^n\}_{n=2}^N \subset H$ the important relation

$$4(\mathrm{D}_2 u^n, u^n) = \mathrm{D}_1 \left(|u^n|^2 + |\mathrm{E}u^{n+1}|^2\right) + (\Delta t)^3 |\mathrm{D}^2 u^{n-1}|^2, \quad n = 2, 3, \ldots, N, \tag{3.1}$$

which will be frequently used in what follows.

The temporal approximation (1.2) of (1.1) we wish to study now reads as

$$\mathrm{D}_2 u^n + A(t_n) u^n = f^n, \quad n = 2, 3, \ldots, N, \tag{3.2}$$

with given approximations f^n $(n = 2, 3, \ldots, N)$ for the right-hand side f and initial values $u^0 \approx u_0$, $u^1 \approx u(t_1)$. The value u^1 can be computed from u^0 by means of the implicit Euler step

$$\mathrm{D}_1 u^1 + A(t_1) u^1 = f^1. \tag{3.3}$$

Theorem 3.1. *Assume* (H2), (H5) *and let $B(t) : V \to V^*$ $(t \in [0, T])$ be strongly continuous. For any $u^0, u^1 \in H$ and $\{f^n\}_{n=2}^N \subset V^*$ there is at least one solution $\{u^n\}_{n=2}^N \subset V$ to* (3.2) *if $\Delta t \leqslant 3/(2\kappa)$. The solution is unique if $\Delta t < 3/(2\kappa)$ and $B \equiv 0$.*

Proof. In each time step, (3.2) is equivalent to the operator equation

$$\frac{3}{2\Delta t}\, u^n + A_0(t_n) u^n + B(t_n) u^n = f^n + \frac{2}{\Delta t}\, u^{n-1} - \frac{1}{2\Delta t}\, u^{n-2}$$

with the right-hand side in V^* since $V \subseteq H \subseteq V^*$. Its solvability follows from Brézis' theorem on pseudomonotone operators (see e.g. Zeidler [55, Thm. 27.A]) since for each time level t_n the operator $\frac{3}{2\Delta t} I + A_0(t_n) : V \to V^*$ is hemicontinuous and monotone if $\Delta t \leqslant 3/(2\kappa)$, the operator $B(t_n) : V \to V^*$ is strongly continuous, and the sum of these operators is coercive if $\Delta t \leqslant 3/(2\kappa)$. The uniqueness in the case $B \equiv 0$ follows from the strict monotonicity of $\frac{3}{2\Delta t} I + A_0(t_n) : V \to V^*$ if $\Delta t < 3/(2\kappa)$. □

Uniqueness for $B \neq 0$ will follow from the stability results in Section 5 (see Remark 5.1).

4. A priori estimates and convergence

Proposition 4.1. *Assume* (H5) *and let $\Delta t \leqslant \tau < 1/(4\kappa)$ for some $\tau < T$. If $u^0, u^1 \in H$ and $f^n = f_1^n + f_2^n$ $(n = 2, 3, \ldots, N)$ with $\{f_1^n\}_{n=2}^N \in l^q(0, T; V^*)$, $\{f_2^n\}_{n=2}^N \in l^2(0, T; H)$ then any solution $\{u^n\}_{n=2}^N$ to* (3.2) *is in $l^\infty(0, T; H) \cap l^p(0, T; V)$ and satisfies the estimate*

$$\max_{j=2,\ldots,N} |u^j|^2 + (\Delta t)^4 \sum_{j=2}^N |\mathrm{D}^2 u^{j-1}|^2 + \Delta t \sum_{j=2}^N \|u^j\|^p \leqslant$$

$$c\left(|u^0|^2 + |u^1|^2 + \Delta t \sum_{j=2}^N (\|f_1^j\|_*^q + |f_2^j|^2) + \lambda T \right) =: M.$$

Assume, in addition, (H4). *Then $\{A(t_n)u^n\}_{n=2}^N \in l^q(0, T; V^*)$, $\{\mathrm{D}_2 u^n\}_{n=2}^N \in l^q(0, T; V^*) + l^2(0, T; H)$, and the estimate*

$$\Delta t \sum_{j=2}^N \|\mathrm{D}_2 u^j - f_2^j\|_*^q + \Delta t \sum_{j=2}^N \|A(t_j) u^j\|_*^q \leqslant M'$$

holds true, where M' depends on M and is bounded on bounded subsets.

Proof. Testing (3.2) by u^n, employing (3.1) and the coercivity condition (H5) as well as Young's inequality leads to

$$\frac{1}{4}\,\mathrm{D}_1\left(|u^n|^2+|\mathrm{E}u^{n+1}|^2\right)+\frac{(\Delta t)^3}{4}\,|\mathrm{D}^2u^{n-1}|^2+\mu\|u^n\|^p-\lambda-\kappa|u^n|^2\leqslant$$

$$c\|f_1^n\|_*^q+\frac{\mu}{2}\,\|u^n\|^p+c|f_2^n|^2+\varepsilon|u^n|^2,$$

where $\varepsilon>0$ is supposed to be sufficiently small such that $4(\kappa+\varepsilon)\tau<1$. The first assertion follows from summing up upon noting that for any grid function $\{a^n\}$ and $\nu\geqslant 0$ with $\nu\Delta t<1$

$$\mathrm{D}_1\left((1-\nu\Delta t)^n a^n\right)=(1-\nu\Delta t)^{n-1}\left(\mathrm{D}_1a^n-\nu a^n\right). \tag{4.1}$$

We take here $a^n:=|u^n|^2+|\mathrm{E}u^{n+1}|^2$ and $\nu=4(\kappa+\varepsilon)$. Note also that $(1-\nu\Delta t)^{-n}\leqslant(1-\nu\Delta t)^{-N}\leqslant(1-\nu\tau)^{-T/\tau}\leqslant\exp\left(\nu T/(1-\nu\tau)\right)$ $(n=1,2,\ldots,N)$.

For $n=2,3,\ldots,N$, the growth condition (H4) yields with $(p-1)q=p$

$$\Delta t\sum_{j=2}^{n}\|A(t_j)u^j\|_*^q\leqslant c\alpha(\sqrt{M})^q\Bigg(T+\Delta t\sum_{j=2}^{n}\|u^j\|^p\Bigg).$$

Moreover, we have

$$\mathrm{D}_2u^n=f_1^n+f_2^n-A(t_n)u^n$$

and

$$\Delta t\sum_{j=2}^{n}\|\mathrm{D}_2u^j-f_2^j\|_*^q=\Delta t\sum_{j=2}^{n}\|f_1^j-A(t_j)u^j\|_*^q\leqslant c\Delta t\sum_{j=2}^{n}\|f_1^j\|_*^q+c\Delta t\sum_{j=2}^{n}\|A(t_j)u^j\|_*^q.$$

This, together with the first estimate, proves the second assertion. □

From the discrete solution $\{u^n\}_{n=0}^N$ of (3.2) corresponding to the partition of $[0,T]$ with the step size Δt, we now construct functions $U_{\Delta t}$ and $V_{\Delta t}$ defined on $[0,T]$. We then study the convergence of $U_{\Delta t}$ and $V_{\Delta t}$ towards the exact solution as $\Delta t\to 0$.

Let

$$U_{\Delta t}(t):=\begin{cases}u^1, & \text{if }\ t\in[0,t_1],\\ u^n, & \text{if }\ t\in(t_{n-1},t_n]\ (n=2,3,\ldots,N);\end{cases}$$

$$V_{\Delta t}(t):=\begin{cases}\frac{1}{2}\left(u^1+\mathrm{E}u^2\right)\quad+\mathrm{D}_1u^1\,(t-t_1), & \text{if }\ t\in[0,t_1],\\ \frac{1}{2}\left(u^n+\mathrm{E}u^{n+1}\right)+\mathrm{D}_2u^n\,(t-t_n), & \text{if }\ t\in(t_{n-1},t_n]\ \ (n=2,3,\ldots,N).\end{cases}$$

The construction of $V_{\Delta t}$ reflects the choice of the method: The value u^1 is thought to be computed by the implicit Euler method (3.3); if another method is used, then the definition above has to be modified appropriately. The slope of $V_{\Delta t}$ in $(t_{n-1},t_n]$ is D_2u^n for $n=2,3,\ldots,N$, and the function is continuous. However, $V_{\Delta t}$ does not interpolate as $V_{\Delta t}(t_n)=\frac{3}{2}\,u^n-\frac{1}{2}\,u^{n-1}$ for $n=2,3,\ldots,N$.

There are other possible prolongations that we will not consider here, e.g., the interpolating linear spline, the discontinuous interpolating piecewise linear function with slope D_2u^n in $(t_{n-1},t_n]$ $(n=2,3,\ldots,N)$, or a continuous piecewise quadratic interpolation. Results for such prolongations, however, rely upon the functions $U_{\Delta t}$ and $V_{\Delta t}$ above.

For a null sequence $\{(\Delta t)_k\}$ of time steps $(\Delta t)_k:=T/N_k$ ($\{N_k\}\subset\mathbb{N}$ with $N_k\to\infty$ as $k\to\infty$) and corresponding problems (3.2) with initial values $u^0_{(\Delta t)_k}$, $u^1_{(\Delta t)_k}$, we always assume $u^0_{(\Delta t)_k}, u^1_{(\Delta t)_k} \in V$. Moreover, we suppose that the right-hand side of (3.2), (3.3) is given such that the representation

$$f^n_{(\Delta t)_k} = f^n_{1,(\Delta t)_k} + f^n_{2,(\Delta t)_k} \quad (n = 1, 2, \ldots, N_k) \tag{4.2}$$

with $\{f^n_{1,(\Delta t)_k}\}_{n=1}^{N_k} \in l^q(0,T;V^*)$, $\{f^n_{2,(\Delta t)_k}\}_{n=1}^{N_k} \in l^2(0,T;H)$ holds true. Finally, we assume that

$$\begin{gathered} |u^0_{(\Delta t)_k}|^2 + |u^1_{(\Delta t)_k}|^2 + (\Delta t)_k\|u^0_{(\Delta t)_k}\|^p + (\Delta t)_k\|u^1_{(\Delta t)_k}\|^p + (\Delta t)_k\|\mathrm{D}_1 u^1_{(\Delta t)_k} - f^1_{2,(\Delta t)_k}\|^q_* \leqslant c, \\ (\Delta t)_k|\mathrm{D}_1 u^1_{(\Delta t)_k}| \to 0. \end{gathered} \tag{4.3}$$

The first part of assumption (4.3) is fulfilled if $u^0_{(\Delta t)_k} := u^0 \in V$ and $u^1_{(\Delta t)_k} \in V$ is computed from (3.3). The proof is analogous to the one of Proposition 4.1 and relies upon the algebraic identity

$$2(a-b)a = a^2 - b^2 + (a-b)^2, \quad a, b, c \in \mathbb{R},$$

which gives

$$2\Delta t(\mathrm{D}_1 u^1, u^1) = |u^1|^2 - |u^0|^2 + |u^1 - u^0|^2.$$

In view of the stability estimates for the continuous problem (see Theorem 2.1), the assumption $u^0 \in V$ is not a restriction as we can always approximate $u_0 \in H$ by an element of V. The second part of assumption (4.3) then follows from the first part if $(\Delta t)_k|f^1_{(\Delta t)_k}|^2 \to 0$. This can be seen by testing (3.3) with $u^1 - u^0$ and employing the coercivity of $A(t_1) + \kappa I$ and the growth condition for $A(t_1)$. The condition on $f^1_{(\Delta t)_k}$ is fulfilled if $f^1_{(\Delta t)_k}$ is, e.g., the natural restriction $\mathrm{R}^1_1 f$ of $f \in L^2(0,T;H)$ on $(0,(\Delta t)_k)$. If (H2′) and (H3′) are fulfilled, then only $u^0 \in V$ and $(\Delta t)_k\|f^1_{(\Delta t)_k}\|^q_* \to 0$ is required in order to ensure the second part of assumption (4.3). The condition on $f^1_{(\Delta t)_k}$ is fulfilled if $f^1_{(\Delta t)_k}$ is, e.g., the natural restriction of $f \in L^q(0,T;V^*)$ on $(0,(\Delta t)_k)$.

Proposition 4.2. *Let $\{(\Delta t)_k\}$ be a null sequence of time steps. Under the assumptions of Proposition 4.1 and if (4.2), (4.3) are fulfilled, there is a subsequence $\{(\Delta t)_{k'}\}$ and an element $U \in \mathcal{W}$ such that*

$$U_{(\Delta t)_{k'}} \rightharpoonup U \;\text{ in } L^p(0,T;V), \quad U_{(\Delta t)_{k'}} \overset{*}{\rightharpoonup} U \;\text{ in } L^\infty(0,T;H),$$

$$V_{(\Delta t)_{k'}} \rightharpoonup U \;\text{ in } L^p(0,T;V), \quad V_{(\Delta t)_{k'}} \overset{*}{\rightharpoonup} U \;\text{ in } L^\infty(0,T;H),$$
$$V'_{(\Delta t)_{k'}} \rightharpoonup U' \;\text{ in } \mathcal{X}^* = L^q(0,T;V^*) + L^2(0,T;H).$$

If, in addition, $V \overset{c}{\hookrightarrow} H$, then $\{V_{(\Delta t)_{k'}}\}$ also converges strongly in $L^r(0,T;H)$ towards U for all $r \in [1,\infty)$.

Proof. From the definition of $\{U_{(\Delta t)_k}\}$ and $\{V_{(\Delta t)_k}\}$ and with Proposition 4.1 together with (4.3), it follows that $\{U_{(\Delta t)_k}\}$ and $\{V_{(\Delta t)_k}\}$ are bounded in $L^\infty(0,T;H)$ and $L^p(0,T;V)$. Moreover, the sequence of derivatives $\{V'_{(\Delta t)_k}\}$ is bounded in $\mathcal{X}^*$. The existence of a weakly* in $L^\infty(0,T;H)$ and weakly in $L^p(0,T;V)$ convergent subsequence of $\{U_{(\Delta t)_k}\}$ and $\{V_{(\Delta t)_k}\}$, respectively, follows now from standard compactness arguments (see [6, Thm. III.26 f.]). Moreover, there exists a subsequence of $\{V'_{(\Delta t)_k}\}$ that converges weakly in $\mathcal{X}^*$.

Let $\{(\Delta t)_{k'}\}$ be a suitable subsequence of $\{(\Delta t)_k\}$ and let $U \in L^p(0,T;V) \cap L^\infty(0,T;H)$ be the (weak and weak*) limit of $\{U_{(\Delta t)_{k'}}\}$, whereas the limit of $\{V_{(\Delta t)_{k'}}\}$ is denoted by

$\widetilde{U} \in L^p(0,T;V) \cap L^\infty(0,T;H)$. It can easily be shown that $\widetilde{U}' \in \mathcal{X}^*$ is the weak limit of $\{V'_{(\Delta t)_{k'}}\}$ and thus $\widetilde{U} \in \mathcal{W}$.

We show that $U = \widetilde{U}$ in $L^p(0,T;V) \cap L^\infty(0,T;H)$. In what follows, we omit the subscripts k and k'. Since

$$V_{\Delta t}(t) - U_{\Delta t}(t) = \begin{cases} \mathrm{D}_1 u^1 \left(t - t_1 + \frac{\Delta t}{2}\right), & \text{if } \ t \in [0, t_1], \\ \frac{\Delta t}{2} \mathrm{D}_1 u^n + \mathrm{D}_2 u^n \, (t - t_n), & \text{if } \ t \in (t_{n-1}, t_n] \ \ (n = 2, 3, \ldots, N), \end{cases}$$

and since

$$\mathrm{D}_1 u^n = \mathrm{D}_2 u^n - \frac{\Delta t}{2} \mathrm{D}^2 u^{n-1},$$

we find (remember representation (4.2))

$$V_{\Delta t}(t) - U_{\Delta t}(t) =$$

$$\begin{cases} (\mathrm{D}_1 u^1 - f_2^1) \left(t - t_1 + \frac{\Delta t}{2}\right) + f_2^1 \left(t - t_1 + \frac{\Delta t}{2}\right), & \text{if } \ t \in [0, t_1], \\ (\mathrm{D}_2 u^n - f_2^n) \left(t - t_n + \frac{\Delta t}{2}\right) - \frac{(\Delta t)^2}{4} \mathrm{D}^2 u^{n-1} + & \\ f_2^n \left(t - t_n + \frac{\Delta t}{2}\right), & \text{if } \ t \in (t_{n-1}, t_n], \quad n = 2, 3, \ldots, N. \end{cases} \tag{4.4}$$

It follows that

$$\|V_{\Delta t} - U_{\Delta t}\|_{\mathcal{X}^*} \leqslant c \max \Bigg(\Bigg((\Delta t)^{q+1} \|\mathrm{D}_1 u^1 - f_2^1\|_*^q + (\Delta t)^{q+1} \sum_{j=2}^{N} \|\mathrm{D}_2 u^j - f_2^j\|_*^q + $$
$$(\Delta t)^{2q+1} \sum_{j=2}^{N} \|\mathrm{D}^2 u^{j-1}\|_*^q \Bigg)^{1/q}, \left((\Delta t)^3 \sum_{j=1}^{N} |f_2^j|^2 \right)^{1/2} \Bigg). \tag{4.5}$$

By assumption, $(\Delta t)^3 \sum_{j=1}^{N} |f_2^j|^2$ tends to zero as $\Delta t \to 0$. In view of (4.3) and Proposition 4.1, also $(\Delta t)^{q+1} \|\mathrm{D}_1 u^1 - f_2^1\|_*^q$ and $(\Delta t)^{q+1} \sum_{j=2}^{N} \|\mathrm{D}_2 u^j - f_2^j\|_*^q$ tend to zero as $\Delta t \to 0$. For the remaining term, we observe with $H \hookrightarrow V^*$ and Hölder's inequality that

$$(\Delta t)^{2q+1} \sum_{j=2}^{N} \|\mathrm{D}^2 u^{j-1}\|_*^q \leqslant c (\Delta t)^{2q+1} \sum_{j=2}^{N} |\mathrm{D}^2 u^{j-1}|^q \leqslant$$

$$c \begin{cases} (\Delta t)^{2q+1} \max\limits_{j=0,\ldots,N} |(\Delta t)^{-2} u^j|^{q-2} \sum\limits_{j=2}^{N} |\mathrm{D}^2 u^{j-1}|^2, & \text{if } \ p < 2, \\ (\Delta t)^5 \sum\limits_{j=2}^{N} |\mathrm{D}^2 u^{j-1}|^2, & \text{if } \ p = 2, \\ (\Delta t)^{2q+1} (\Delta t)^{-1+q/2} \left(\sum\limits_{j=2}^{N} |\mathrm{D}^2 u^{j-1}|^2 \right)^{q/2}, & \text{if } \ p > 2, \end{cases} =$$

$$c \begin{cases} (\Delta t)^5 \sum\limits_{j=2}^{N} |\mathrm{D}^2 u^{j-1}|^2 \max\limits_{j=0,\ldots,N} |u^j|^{q-2}, & \text{if } \ p < 2, \\ (\Delta t)^5 \sum\limits_{j=2}^{N} |\mathrm{D}^2 u^{j-1}|^2, & \text{if } \ p = 2, \\ \left((\Delta t)^5 \sum\limits_{j=2}^{N} |\mathrm{D}^2 u^{j-1}|^2 \right)^{q/2}, & \text{if } \ p > 2. \end{cases}$$

So, because of the *a priori* estimate in Proposition 4.1, also this term converges towards zero as $\Delta t \to 0$. This shows that the difference $V_{\Delta t} - U_{\Delta t}$ converges strongly in $\mathcal{X}^*$ towards zero as $\Delta t \to 0$. Because of $U_{\Delta t} \rightharpoonup U$, $V_{\Delta t} \rightharpoonup \widetilde{U}$ in $L^p(0,T;V)$ and $U_{\Delta t} \overset{*}{\rightharpoonup} U$, $V_{\Delta t} \overset{*}{\rightharpoonup} \widetilde{U}$ in $L^\infty(0,T;H)$, we also have $U_{\Delta t} \rightharpoonup U$, $V_{\Delta t} \rightharpoonup \widetilde{U}$ in $L^{\min(p,q)}(0,T;V^*) = \big(L^{\max(p,q)}(0,T;V)\big)^*$. Since $\mathcal{X}^* \hookrightarrow L^{\min(p,q)}(0,T;V^*)$, the limits U and $\widetilde{U}$ are equal at least as elements of $L^{\min(p,q)}(0,T;V^*)$. However, $L^{\max(p,q)}(0,T;V)$ is dense in $L^q(0,T;V^*)$ as well as in $L^1(0,T;H)$ and so we find $U = \widetilde{U}$ in $L^p(0,T;V) \cap L^\infty(0,T;H) = (L^q(0,T;V^*) + L^1(0,T;H))^*$.

If $V \overset{c}{\hookrightarrow} H$, the strong convergence of a subsequence of $\{V_{(\Delta t)_k}\}$ in $L^p(0,T;H)$ follows since $\{V_{(\Delta t)_k}\}$ is bounded in $\mathcal{W} \overset{c}{\hookrightarrow} L^p(0,T;H)$. The boundedness in $L^\infty(0,T;H)$ then implies convergence in $L^r(0,T;H)$ for all $r < \infty$. □

Theorem 4.1. *Assume* (H1′), (H2), (H3) *with* $\delta > \min(1, p/2)$ *or* (H3$'''_{p=2}$) *with* $\delta > 1$, (H4), (H5). *If* $B \neq 0$ *or* $\kappa \neq 0$, *then assume that* V *is compactly embedded in* H. *Let* $u_0 \in H$ *and* $f \in \mathcal{X}^*$ *be given. For a null sequence* $\{(\Delta t)_k\}$ *of time steps, consider the corresponding sequence of problems* (3.2), (3.3) *with* $u^0_{(\Delta t)_k}, u^1_{(\Delta t)_k} \in V$ *fulfilling* (4.3) *as well as*

$$\frac{1}{2}(u^1_{(\Delta t)_k} + u^0_{(\Delta t)_k}) \to u_0 \quad \text{in} \quad H \tag{4.6}$$

and the right-hand side $\{f^n_{(\Delta t)_k}\}_{n=1}^{N_k}$ *being given by the natural restrictions of* f. *Then the limit* $U \in \mathcal{W}$ *from Proposition* 4.2 *is a solution to the initial-value problem* (1.1) *such that* (1.1) *holds in* $\mathcal{X}^*$ *and both* $\{U_{(\Delta t)_k}\}$ *and* $\{V_{(\Delta t)_k}\}$ *converge weakly in* $L^p(0,T;V)$ *and weakly* in* $L^\infty(0,T;H)$ *towards* U.

Proof. In what follows, we omit the subscripts k and k' for a suitable subsequence. With the aid of the functions $U_{\Delta t}$ and $V_{\Delta t}$, the numerical scheme (3.2), (3.3) can be rewritten as the differential equation

$$V'_{\Delta t} + A_{\Delta t} U_{\Delta t} = f_{\Delta t}\,, \quad 0 < t \leqslant T, \tag{4.7}$$

where $f_{\Delta t} : [0,T] \to V^*$ is defined via $f_{\Delta t}(t) := \mathrm{R}^n_2 f$ for $t \in (t_{n-1}, t_n]$ ($n = 2, 3, \ldots, N$) and $f_{\Delta t}(t) := \mathrm{R}^n_1 f$ for $t \in [0, t_1]$, $A_{\Delta t} := A_{0,\Delta t} + B_{\Delta t}$ with $A_{0,\Delta t}$ being piecewise constant such that $A_{0,\Delta t}(t) = A_0(t_n)$ for $t \in (t_{n-1}, t_n]$ ($n = 1, 3, \ldots, N$) and $B_{\Delta t}$ being defined analogously. The properties of the Nemytskii operator $A_{0,\Delta t}$ and $B_{\Delta t}$ are analogous to those of A_0 and B, respectively, as stated in Proposition 2.1. In particular, $A_{0,\Delta t}$ and $B_{\Delta t}$ are well-defined on $L^p(0,T;V) \cap L^\infty(0,T;H)$ with range in $\mathcal{X}^*$.

By standard arguments, we find the strong convergence $f_{\Delta t} \to f$ in $\mathcal{X}^*$.

Because of assumption (H4), (4.3), and Proposition 4.1, we know that $\{A_{\Delta t} U_{\Delta t}\}$ is bounded in $L^q(0,T;V^*)$. So, we can extract a subsequence of time steps such that we have the convergence results from Proposition 4.2 as well as

$$A_{\Delta t} U_{\Delta t} \rightharpoonup b \quad \text{in} \quad L^q(0,T;V^*) \tag{4.8}$$

for some $b \in L^q(0,T;V^*)$. With Proposition 4.2, we then obtain

$$0 = V'_{\Delta t} + A_{\Delta t} U_{\Delta t} - f_{\Delta t} \rightharpoonup U' + b - f \quad \text{in} \quad \mathcal{X}^*$$

and thus

$$U' + b = f \quad \text{in} \quad \mathcal{X}^*. \tag{4.9}$$

In what follows, we show that $U \in \mathcal{W}$ fulfills the initial condition and that $b = AU$. This, finally, proves that U is a weak solution to the initial-value problem (1.1).

Since $\{V_{\Delta t}\}$ is bounded in $\mathcal{W} \hookrightarrow \mathcal{C}([0,T];H)$, for any $t \in [0,T]$, the sequence $\{V_{\Delta t}(t)\}$ is bounded in H and (together with (4.6)) we have for a suitable subsequence

$$V_{\Delta t}(0) = \frac{1}{2}(u^1_{\Delta t} + u^0_{\Delta t}) \to u_0 \quad \text{in} \quad H$$

as well as

$$V_{\Delta t}(T) = \frac{3}{2}u^N_{\Delta t} - \frac{1}{2}u^{N-1}_{\Delta t} \rightharpoonup \theta \quad \text{in} \quad H$$

for some $\theta \in H$. Since $U, V_{\Delta t} \in \mathcal{W}$, we can employ integration by parts and obtain for all $v \in V$ and $\phi \in \mathcal{C}^1([0,T])$

$$(U(T),v)\phi(T) - (U(0),v)\phi(0) = \int_0^T \Big(\langle U'(t),v\rangle\phi(t) + \langle U(t),v\rangle\phi'(t)\Big)dt =$$

$$\int_0^T \Big(\langle f(t) - b(t),v\rangle\phi(t) + \langle U(t),v\rangle\phi'(t)\Big)dt =$$

$$\int_0^T \Big(\langle f(t) - f_{\Delta t}(t) + V'_{\Delta t}(t) + A_{\Delta t}(t)U_{\Delta t}(t) - b(t),v\rangle\phi(t) + \langle U(t),v\rangle\phi'(t)\Big)dt =$$

$$\int_0^T \Big(\langle f(t) - f_{\Delta t}(t) + A_{\Delta t}(t)U_{\Delta t}(t) - b(t),v\rangle\phi(t) + \langle U(t) - V_{\Delta t}(t),v\rangle\phi'(t)\Big)dt +$$

$$(V_{\Delta t}(T),v)\phi(T) - (V_{\Delta t}(0),v)\phi(0).$$

Taking the limit on the right-hand side, we come up with

$$(U(T),v)\phi(T) - (U(0),v)\phi(0) = (\theta,v)\phi(T) - (u_0,v)\phi(0).$$

Choosing $\phi(T) = 0$ and $\phi(0) = 0$, respectively, we find that $U(0) = u_0$ and $U(T) = \theta$ in H since $V \ni v$ is dense in H.

The method for proving $b = AU$ is similar to Minty's well-known monotonicity trick.

Let us show that $B_{\Delta t}U_{\Delta t} \to BU$ in $\mathcal{X}^*$. So, let $B \neq 0$. We then have additionally the compact embedding $V \overset{c}{\hookrightarrow} H$ at hand. We firstly show that $U_{\Delta t} \to U$ in $L^r(0,T;H)$ for all $r \in [1,\infty)$ (remember that already $V_{\Delta t} \to U$ in $L^r(0,T;H)$ for all $r \in [1,\infty)$). Since $\{U_{\Delta t}\}$ is bounded in $L^\infty(0,T;H)$ and $U \in L^\infty(0,T;H)$, it suffices to show $U_{\Delta t} \to U$ in $L^2(0,T;H)$. Because of $U_{\Delta t} \overset{*}{\rightharpoonup} U$ in $L^\infty(0,T;H)$, we already know that $U_{\Delta t} \rightharpoonup U$ in $L^2(0,T;H)$ and it remains to show $\|U_{\Delta t}\|_{L^2(0,T;H)} \to \|U\|_{L^2(0,T;H)}$ as $L^2(0,T;H)$ is a Hilbert space. We observe

$$\Big|\|U_{\Delta t}\|^2_{L^2(0,T;H)} - \|U\|^2_{L^2(0,T;H)}\Big| = \Big|(U_{\Delta t},U_{\Delta t})_{L^2(0,T;H)} - (U,U)_{L^2(0,T;H)}\Big| =$$

$$\Big|(U_{\Delta t} - V_{\Delta t},U_{\Delta t})_{L^2(0,T;H)} + (V_{\Delta t} - U,U_{\Delta t})_{L^2(0,T;H)} + (U_{\Delta t} - U,U)_{L^2(0,T;H)}\Big| \leqslant$$

$$\|U_{\Delta t} - V_{\Delta t}\|_{\mathcal{X}^*}\|U_{\Delta t}\|_{\mathcal{X}} + \|V_{\Delta t} - U\|_{L^2(0,T;H)}\|U_{\Delta t}\|_{L^2(0,T;H)} + \Big|(U_{\Delta t} - U,U)_{L^2(0,T;H)}\Big|.$$

Each of the terms on the right-hand side converges towards zero as $\Delta t \to 0$ since $U_{\Delta t} - V_{\Delta t} \to 0$ in $\mathcal{X}^*$ (see estimate (4.5) and the according arguments), $V_{\Delta t} \to U$ in $L^2(0,T;H)$ if $V \overset{c}{\hookrightarrow} H$, $U_{\Delta t} \rightharpoonup U$ in $L^2(0,T;H)$, and $\{U_{\Delta t}\}$ is bounded in $L^p(0,T;V)$ and $L^2(0,T;H)$.

With (H3) and the boundedness of $\{U_{\Delta t}\}$ in $L^p(0,T;V)$, we obtain $B_{\Delta t}U_{\Delta t} - B_{\Delta t}U \to 0$ in $L^r(0,T;H)$ for all $r \in [1, p/(p-\delta))$ and thus in $L^q(0,T;V^*)$ if $\delta > 1$ as well as in $L^2(0,T;H)$ if $\delta > p/2$. Hence, $B_{\Delta t}U_{\Delta t} - B_{\Delta t}U \to 0$ in $\mathcal{X}^*$ if $\delta > \min(1, p/2)$. With (H3$'''_{p=2}$) instead of (H3) and $\delta > 1$, it immediately follows $B_{\Delta t}U_{\Delta t} - B_{\Delta t}U \to 0$ in $L^2(0,T;V^*)$. In view of (H1$'$), we have for almost all $t \in (0,T)$

$$\|B_{\Delta t}(t)U(t) - B(t)U(t)\|_* \to 0.$$

Since also (see the growth condition (H4)) where the right-hand side is integrable, we find by Lebesgue's theorem that $B_{\Delta t}U \to BU$ in $L^q(0,T;V^*)$.

It follows $B_{\Delta t}U_{\Delta t} \to BU$ in $\mathcal{X}^*$ and thus (because of $U_{\Delta t} \rightharpoonup U$ in $\mathcal{X}$)

$$\langle B_{\Delta t}U_{\Delta t}, U_{\Delta t}\rangle \to \langle BU, U\rangle, \tag{4.10}$$

where here and in the following $\langle\cdot,\cdot\rangle$ is the dual pairing between $\mathcal{X}^*$ and $\mathcal{X}$.

Since $U, V_{\Delta t} \in \mathcal{W}$, we find with integration by parts

$$\langle U', U\rangle = \frac{1}{2}\left(|U(T)|^2 - |U(0)|^2\right) = \frac{1}{2}\left(|U(T)|^2 - |u_0|^2\right)$$

as well as

$$\langle V'_{\Delta t}, V_{\Delta t}\rangle = \frac{1}{2}\left(|V_{\Delta t}(T)|^2 - |V_{\Delta t}(0)|^2\right)$$

and hence (remember $V_{\Delta t}(T) \rightharpoonup U(T)$, $V_{\Delta t}(0) \to u_0$ in H)

$$\langle U', U\rangle \leqslant \liminf \langle V'_{\Delta t}, V_{\Delta t}\rangle.$$

Together with $f_{\Delta t} \to f$ in $\mathcal{X}^*$ and $U_{\Delta t} \rightharpoonup U$ in $\mathcal{X}$, we obtain from (4.9), (4.10), and (4.7)

$$\begin{aligned}\langle b - BU, U\rangle = \langle f - U' - BU, U\rangle \geqslant \\ \limsup\Big(\langle f_{\Delta t} - V'_{\Delta t} - B_{\Delta t}U_{\Delta t}, U_{\Delta t}\rangle + \langle V'_{\Delta t}, U_{\Delta t} - V_{\Delta t}\rangle\Big) = \\ \limsup\Big(\langle A_{0,\Delta t}U_{\Delta t}, U_{\Delta t}\rangle + \langle V'_{\Delta t}, U_{\Delta t} - V_{\Delta t}\rangle\Big).\end{aligned} \tag{4.11}$$

The monotonicity of $A_0(t) + \kappa I : V \to V^*$ ($t \in [0,T]$) implies the monotonicity of $A_{0,\Delta t} + \kappa I : L^p(0,T;V) \cap L^\infty(0,T;H) \subset \mathcal{X} \to \mathcal{X}^*$ and shows for all $w \in L^p(0,T;V) \cap L^\infty(0,T;H)$

$$\begin{aligned}\langle A_{0,\Delta t}U_{\Delta t}, U_{\Delta t}\rangle \geqslant \langle A_{0,\Delta t}U_{\Delta t}, U_{\Delta t}\rangle - \langle A_{0,\Delta t}U_{\Delta t} - A_{0,\Delta t}w, U_{\Delta t} - w\rangle - \kappa\|U_{\Delta t} - w\|^2_{L^2(0,T;H)} = \\ \langle A_{0,\Delta t}U_{\Delta t}, w\rangle + \langle A_{0,\Delta t}w, U_{\Delta t} - w\rangle - \kappa\|U_{\Delta t} - w\|^2_{L^2(0,T;H)}.\end{aligned} \tag{4.12}$$

We then observe that

$$\langle A_{0,\Delta t}U_{\Delta t}, w\rangle = \langle A_{\Delta t}U_{\Delta t}, w\rangle - \langle B_{\Delta t}U_{\Delta t}, w\rangle \to \langle b, w\rangle - \langle BU, w\rangle = \langle b - BU, w\rangle.$$

With (H1$'$), we also have $A_{0,\Delta t}w \to A_0 w$ in $L^q(0,T;V^*)$ (as was shown for $B_{\Delta t}$ above) and thus $U_{\Delta t} \rightharpoonup U$ in $\mathcal{X}$ implies

$$\langle A_{0,\Delta t}w, U_{\Delta t} - w\rangle \to \langle A_0 w, U - w\rangle.$$

Since $U_{\Delta t} \to U$ in $L^2(0,T;H)$ if $V \overset{c}{\hookrightarrow} H$, we also have

$$\|U_{\Delta t} - w\|_{L^2(0,T;H)} \to \|U - w\|_{L^2(0,T;H)}. \tag{4.13}$$

Altogether, we find from (4.12)

$$\limsup \langle A_{0,\Delta t} U_{\Delta t}, U_{\Delta t}\rangle \geqslant \langle b - BU, w\rangle + \langle A_0 w, U - w\rangle - \kappa \|U - w\|^2_{L^2(0,T;H)}. \tag{4.14}$$

It remains to analyse $\langle V'_{\Delta t}, U_{\Delta t} - V_{\Delta t}\rangle$. A straightforward calculation shows that

$$\left(\frac{3}{2}\, a - 2b + \frac{1}{2}\, c\right) (a - 2b + c) = \frac{3}{2}\, (a-b)^2 - 2(a-b)(b-c) + \frac{1}{2}(b-c)^2 \geqslant$$

$$\frac{1}{2}\, (a-b)^2 - \frac{1}{2}\, (b-c)^2, \quad a, b, c \in \mathbb{R}.$$

With the definition of $V_{\Delta t}$ and (4.4), we thus obtain

$$\langle V'_{\Delta t}, U_{\Delta t} - V_{\Delta t}\rangle = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \langle V'_{\Delta t}(t), U_{\Delta t}(t) - V_{\Delta t}(t)\rangle dt =$$

$$-\int_0^{t_1} \left(t - t_1 + \frac{\Delta t}{2}\right) dt |\mathrm{D}_1 u^1_{\Delta t}|^2 - \sum_{n=2}^{N} \int_{t_{n-1}}^{t_n} \left(t - t_n + \frac{\Delta t}{2}\right) dt |\mathrm{D}_2 u^n_{\Delta t}|^2 + \frac{(\Delta t)^3}{4} \sum_{n=2}^{N} \left(\mathrm{D}_2 u^n_{\Delta t}, \mathrm{D}^2 u^n_{\Delta t}\right) =$$

$$\frac{1}{4} \sum_{n=2}^{N} \left(\frac{3}{2}\, u^n_{\Delta t} - 2u^{n-1}_{\Delta t} + \frac{1}{2}\, u^{n-2}_{\Delta t}, u^n_{\Delta t} - 2u^{n-1}_{\Delta t} + u^{n-2}_{\Delta t}\right) =$$

$$\frac{1}{4} \sum_{n=2}^{N} \left(\frac{3}{2}\, |u^n_{\Delta t} - u^{n-1}_{\Delta t}|^2 - 2(u^n_{\Delta t} - u^{n-1}_{\Delta t}, u^{n-1}_{\Delta t} - u^{n-2}_{\Delta t}) + \frac{1}{2}\, |u^{n-1}_{\Delta t} - u^{n-2}_{\Delta t}|^2\right) \geqslant$$

$$\frac{1}{8} \sum_{n=2}^{N} \left(|u^n_{\Delta t} - u^{n-1}_{\Delta t}|^2 - |u^{n-1}_{\Delta t} - u^{n-2}_{\Delta t}|^2\right) \geqslant -\frac{1}{8}\, |u^1_{\Delta t} - u^0_{\Delta t}|^2,$$

which converges by assumption (4.3) towards zero.

From (4.11) and (4.14), we now obtain for all $w \in L^p(0,T;V) \cap L^\infty(0,T;H)$

$$\langle b - BU, U\rangle \geqslant \langle b - BU, w\rangle + \langle A_0 w, U - w\rangle - \kappa \|U - w\|^2_{L^2(0,T;H)}$$

and thus

$$\langle b - BU, U - w\rangle \geqslant \langle A_0 w, U - w\rangle - \kappa \|U - w\|^2_{L^2(0,T;H)}.$$

With $w = U \pm sv$ ($v \in L^p(0,T;V) \cap L^\infty(0,T;H)$) and $s \to 0+$, the hemicontinuity of A_0 proves

$$\langle b - BU, v\rangle = \langle A_0 U, v\rangle$$

and thus, by density, $b = AU$ in $L^q(0,T;V^*)$. So, $U \in \mathcal{W}$ is a weak solution to (1.1).

By contradiction, we can show that the whole sequences $\{V_{\Delta t}\}$ and $\{U_{\Delta t}\}$ converge towards U since a solution to (1.1) is unique in $\mathcal{W}$. □

Note that assumption (4.6) follows from (4.3) if $u^0_{(\Delta t)_k} \to u_0$ in H. The compact embedding $V \overset{c}{\hookrightarrow} H$ is also employed if $B \equiv 0$ but $\kappa \neq 0$ in order to have (4.13).

Assumption (H3) with $\delta > \min(1, p/2)$ or ($\mathrm{H3}'''_{p=2}$) with $\delta > 1$ on $B(t)$ ($t \in [0,T]$) can often be relaxed: As one can infer from the proof above, we only need that $U_{\Delta t} \rightharpoonup U$ in $L^p(0,T;V)$, $U_{\Delta t} \overset{*}{\rightharpoonup} U$ in $L^\infty(0,T;H)$, and $U_{\Delta t} \to U$ in $L^r(0,T;H)$ ($r \in [1,\infty)$) imply $\langle BU, U\rangle \leqslant \liminf \langle B_{\Delta t} U_{\Delta t}, U_{\Delta t}\rangle$.

Remark 4.1. The statement of Theorem 4.1 remains true under assumption (H3) with $\delta > 0$ instead of $\delta > \min(1, p/2)$ if $t \mapsto B(t)v$ is, as a mapping with values in H, demicontinuous a.e. in $(0, T)$ for all $v \in V$, i.e., if $t \mapsto (B(t)v, w)$ is continuous a.e. in $(0, T)$ for all $v \in V, w \in H$.

The remark above can be seen as follows: Since $B(t)$ $(t \in [0, T])$ has range in H, we find with the demicontinuity, Lebesgue's theorem, and the growth condition (H4) that $B_{\Delta t}U \rightharpoonup BU$ in $L^{r'}(0, T; H)$ for all $r' \in [1, q]$. We already know from the original proof that $B_{\Delta t}U_{\Delta t} - B_{\Delta t}U \to 0$ in $L^{r'}(0, T; H)$ for all $r' \in [1, p/(p-\delta))$. Taking a suitable exponent $r' > 1$ with the conjugated exponent $r > 1$ and remembering that $U_{\Delta t} \to U$ in $L^r(0, T; H)$, it follows

$$\langle B_{\Delta t}U_{\Delta t}, U_{\Delta t}\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)} = \langle B_{\Delta t}U_{\Delta t}, U_{\Delta t}\rangle_{L^{r'}(0,T;H)\times L^r(0,T;H)} \to$$
$$\langle BU, U\rangle_{L^{r'}(0,T;H)\times L^r(0,T;H)} = \langle BU, U\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)},$$

which gives (4.10).

Remark 4.2. Let the operators $A_0(t) : V \to V^*$ $(t \in [0, T])$ be linear and assume (H2$'$) (i.e., the operators fulfill, uniformly in t, a Gårding inequality). Instead of (H1$'$), let $t \mapsto A_0(t)^*v$ (with $A_0(t)^* : V \to V^*$ denoting the dual operator of $A(t)$) be continuous and let $t \mapsto A_0(t)v$, $t \mapsto B(t)v$ be demicontinuous, as mappings with values in V^*, a.e. on $(0, T)$ for all $v \in V$. Then the statement of Theorem 4.1 remains true if (H3$'''_{p=2}$) holds for arbitrary $\delta \in (0, 2]$.

The remark above is based upon the following observations: Because of the linearity, we can consider the dual operators $A^*_{0,\Delta t} : L^2(0, T; V) \to L^2(0, T; V^*)$ that are bounded uniformly with respect to Δt. It can be shown that $A^*_{0,\Delta t}v - A^*_0 v \to 0$ in $L^2(0, T; V^*)$ for all $v \in L^2(0, T; V)$. Since $U_{\Delta t} \rightharpoonup U$ in $L^2(0, T; V)$, we find for all $v \in L^2(0, T; V)$

$$\langle A_{0,\Delta t}U_{\Delta t} - A_{0,\Delta t}U, v\rangle = \langle A^*_{0,\Delta t}v, U_{\Delta t} - U\rangle \to 0$$

such that $A_{0,\Delta t}U_{\Delta t} - A_{0,\Delta t}U \rightharpoonup 0$ in $L^2(0, T; V^*)$. Because of the demicontinuity property, we have $A_{0,\Delta t}U \rightharpoonup A_0U$ in $L^2(0, T; V^*)$. So, we come up with $A_{0,\Delta t}U_{\Delta t} \rightharpoonup A_0U$ in $L^2(0, T; V^*)$. But then, in view of (4.8), it remains to show $b - A_0U = BU$, i.e. $B_{\Delta t}U_{\Delta t} \rightharpoonup BU$ in $L^2(0, T; V^*)$. The demicontinuity property for B shows that $B_{\Delta t}U \rightharpoonup BU$ in $L^2(0, T; V^*)$. Moreover, (a subsequence of) $\{B_{\Delta t}U_{\Delta t} - B_{\Delta t}U\}$ converges weakly in $L^2(0, T; V^*)$ (the sequence is bounded because of the growth condition (H4)). The limit can only be 0 since from (H3$'''_{p=2}$) (with arbitrary $\delta \in (0, 2]$), we also know that $B_{\Delta t}U_{\Delta t} - B_{\Delta t}U \to 0$ in $L^{r'}(0, T; V^*)$ for some $1 < r' < 2$ (depending on δ and s) but the dual space $L^r(0, T; V)$ with the conjugated exponent r is dense in $L^2(0, T; V)$.

As an example, let us consider the semilinear incompressible Navier-Stokes problem in a bounded, sufficiently smooth, two-dimensional domain Ω described by the differential equation

$$\partial_t u - \mathrm{Re}^{-1}\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0,$$

subject to homogeneous Dirichlet boundary conditions, where $\mathrm{Re} > 0$ denotes the Reynolds number. With $V = \{v \in H^1_0(\Omega)^2 : \nabla \cdot v = 0\}$ and $H = \{v \in L^2(\Omega)^2 : \nabla \cdot v = 0,\ \gamma_n v = 0\}$ (γ_n denotes the trace in the normal direction), the existence and uniqueness of the velocity $u \in \mathcal{W}$ with $p = 2$ that fulfills the corresponding evolution equation (1.1) in $L^2(0, T; V^*)$ is wellknown (see [51, Thm. 3.1 on p. 282, Thm. 3.2 on p. 294]). For the time-independent semilinearity B with

$$\langle Bu, v\rangle := \int_\Omega (u \cdot \nabla)u \cdot v dx, \tag{4.15}$$

it can be shown that

$$\|Bv - Bw\|_* \leqslant c\left(|v|^{1/2}\|v\|^{1/2} + |w|^{1/2}\|w\|^{1/2}\right)|v-w|^{1/2}\|v-w\|^{1/2}, \quad v,w \in V,$$

which implies ($\mathrm{H3}'''_{p=2}$) with $\delta = 1$ and $s = 1/2$.

The three-dimensional Navier — Stokes problem requires, however, a more refined technique since the existence of a velocity $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ with $u' \in L^{4/3}(0,T;V^*)$ can be shown but not the existence in a space $\mathcal{W}$. Nevertheless, a convergence result analogous to Theorem 4.1 has been shown by the author in [13].

Let us finally remark that the convergence of the two-step BDF was already postulated in a remark in [44, Rem. 8.20] for an evolution problem with A_0, B being independent of time and A_0 being a monotone potential operator such that $A_0 v = \Phi'(v)$. Here, $\Phi : V \to \mathbb{R}$ is a convex functional such that for all $v \in V$

$$\Phi(v) \geqslant \mu\|v\|_V^p - \kappa|v|^2.$$

The proof of convergence shall then rely upon the testing of the discrete equation (3.2) by $u^n - u^{n-1}$, the algebraic relation

$$\left(\frac{3}{2}a - 2b + \frac{1}{2}c\right)(a-b) = \frac{3}{2}(a-b)^2 - \frac{1}{2}(a-b)(b-c) \geqslant$$

$$(a-b)^2 - \frac{1}{8}(b-c)^2, \quad a,b,c \in \mathbb{R},$$

which gives for $n = 2,3,\ldots,N$

$$\Delta t(\mathrm{D}_2 u^n, u^n - u^{n-1}) \geqslant |u^n - u^{n-1}|^2 - \frac{1}{8}|u^{n-1} - u^{n-2}|^2,$$

and the convexity of Φ, which gives for all $u, v \in V$

$$\langle \Phi'(u), u - v\rangle \geqslant \Phi(u) - \Phi(v)\,.$$

This technique, however, does not apply to the more general case considered here.

5. Stability and error estimates

For brevity, we write in the following only $\{v^n\}$ instead of $\{v^n\}_{n=n_0}^N$ for a grid function.

Proposition 5.1. *Assume* (H2), (H3), *and* (H5). *Let $\Delta t \leqslant \tau$ with $\tau < T$ being sufficiently small. Let $\{u^n\}$ and $\{v^n\}$ be a solution to* (3.2) *with right-hand side $\{f^n\} \in l^2(0,T;H)$ and $\{g^n\} \in l^2(0,T;H)$ as well as initial values $u^0, u^1 \in H$ and $v^0, v^1 \in H$, respectively. Then*

$$\max_{n=2,\ldots,N} |u^n - v^n|^2 + (\Delta t)^4 \sum_{j=2}^N |\mathrm{D}^2(u^{j-1} - v^{j-1})|^2 \leqslant$$

$$c\left(1 - c_1\Delta t - c_2(\Delta t)^{\delta/p}\right)^{-T/\Delta t}\left(|u^0 - v^0|^2 + |u^1 - v^1|^2 + \Delta t\sum_{j=2}^N |f^j - g^j|^2\right),$$

where c, c_1, c_2 are positive constants not depending on Δt.

Proof. We subtract the equations for $\{u^n\}$ and $\{v^n\}$ and test by $u^n - v^n$. In view of the monotonicity assumption (H2), we find

$$(\mathrm{D}_2(u^n - v^n), u^n - v^n) - \kappa|u^n - v^n|^2 + (B(t_n)u^n - B(t_n)v^n, u^n - v^n) \leqslant (f^n - g^n, u^n - v^n).$$

Identity (3.1) together with (H3), the Cauchy — Schwarz and the Young inequality gives

$$\frac{1}{4}\,\mathrm{D}_1\left(|u^n - v^n|^2 + |\mathrm{E}(u^{n+1} - v^{n+1})|^2\right) + \frac{(\Delta t)^3}{4}\,|\mathrm{D}^2(u^{n-1} - v^{n-1})|^2 - \kappa|u^n - v^n|^2 -$$
$$\beta(M)\left(1 + \|u^n\|^{p-\delta} + \|v^n\|^{p-\delta}\right)|u^n - v^n|^2 \leqslant |f^n - g^n|^2 + \frac{1}{4}\,|u^n - v^n|^2.$$

Here, M is the maximum of the two *a priori* bounds from Proposition 4.1 corresponding to either of the two sets of initial values and right-hand sides.

From Proposition 4.1, we also infer with inequality (2.2) that for $n = 2, 3, \ldots, N$

$$\|u^n\|^{p-\delta} + \|v^n\|^{p-\delta} \leqslant 2^{\delta/p}\left(\sum_{j=2}^{N}\left(\|u^j\|^p + \|u^j\|^p\right)\right)^{(p-\delta)/p} \leqslant 2^{\delta/p}\left(\frac{2M}{\Delta t}\right)^{(p-\delta)/p} = 2\left(\frac{M}{\Delta t}\right)^{(p-\delta)/p}.$$

So, we come up with

$$\mathrm{D}_1\left(|u^n - v^n|^2 + |\mathrm{E}(u^{n+1} - v^{n+1})|^2\right) + (\Delta t)^3|\mathrm{D}^2(u^{n-1} - v^{n-1})|^2 -$$
$$\Delta t\left(1 + 4\kappa + 4\beta(M)\left(1 + 2\left(\frac{M}{\Delta t}\right)^{(p-\delta)/p}\right)\right)|u^n - v^n|^2 \leqslant 4|f^n - g^n|^2.$$

Let

$$\Delta t \leqslant \tau < \min\left((2(1 + 4\kappa + 4\beta(M)))^{-1}, \left(16\beta(M)M^{(p-\delta)/p}\right)^{-\delta/p}\right).$$

Then

$$\Delta t\,\nu(\Delta t) := \Delta t\left(1 + 4\kappa + 4\beta(M)\left(1 + 2\left(\frac{M}{\Delta t}\right)^{(p-\delta)/p}\right)\right) \leqslant \tau\,\nu(\tau) < \frac{1}{2} + \frac{1}{2} = 1,$$

and we can apply (4.1). The assertion now follows from (4.1) and summing up, which leads on the right-hand side to the factor

$$(1 - \Delta t\,\nu(\Delta t))^{-T/\Delta t} = \left(1 - c_1\Delta t - c_2(\Delta t)^{\delta/p}\right)^{-T/\Delta t},$$
$$c_1 = 1 + 4\kappa + 4\beta(M), \quad c_2 = 8\beta(M)M^{(p-\delta)/p}.$$

□

For the following proposition, note again that assumption (H2′) implies $p \geqslant 2$ and so $l^2(0,T;H) \subseteq l^q(0,T;V^*)$.

Proposition 5.2. *Assume* (H2′), (H3′), *and* (H5). *Let* $\Delta t \leqslant \tau$ *with* $\tau < T$ *being sufficiently small. Let* $\{u^n\}$ *and* $\{v^n\}$ *be a solution to* (3.2) *with the right-hand side* $\{f^n\} \in l^q(0,T;V^*)$ *and* $\{g^n\} \in l^q(0,T;V^*)$, *as well as the initial values* $u^0, u^1 \in H$ *and* $v^0, v^1 \in H$, *respectively. Then*

$$\max_{n=2,\ldots,N}|u^n - v^n|^2 + (\Delta t)^4\sum_{j=2}^{N}|\mathrm{D}^2(u^{j-1} - v^{j-1})|^2 + \Delta t\sum_{j=2}^{N}\|u^j - v^j\|^p \leqslant$$
$$c\left(1 - c_1\Delta t - c_2(\Delta t)^{\delta/p}\right)^{-T/\Delta t}\left(|u^0 - v^0|^2 + |u^1 - v^1|^2 + \Delta t\sum_{j=2}^{N}\|f^j - g^j\|_*^q\right),$$

where c, c_1, c_2 *are positive constants not depending on* Δt.

Proof. We subtract the equations for $\{u^n\}$ and $\{v^n\}$ and test by $u^n - v^n$. In view of the monotonicity assumption (H2′), we find

$$(\mathrm{D}_2(u^n - v^n), u^n - v^n) + \mu_0 \|u^n - v^n\|^p - \kappa |u^n - v^n|^2 + \\ \langle B(t_n)u^n - B(t_n)v^n, u^n - v^n\rangle \leqslant \langle f^n - g^n, u^n - v^n\rangle.$$

For $p > 2$, assumption (H3′) together with (3.1) and Young inequality gives

$$\frac{1}{4}\,\mathrm{D}_1\left(|u^n - v^n|^2 + |\mathrm{E}(u^{n+1} - v^{n+1})|^2\right) + \frac{(\Delta t)^3}{4}\,|\mathrm{D}^2(u^{n-1} - v^{n-1})|^2 + \mu_0\|u^n - v^n\|^p - \\ \kappa|u^n - v^n|^2 - \beta(M)\left(1 + \|u^n\|^{p-\delta} + \|v^n\|^{p-\delta}\right)|u^n - v^n|^2 \leqslant c\|f^n - g^n\|_*^q + \frac{\mu_0}{2}\,\|u^n - v^n\|^p,$$

where M is the maximum of the two *a priori* bounds from Proposition 4.1 corresponding to either of the two sets of initial values and right-hand sides. Let

$$\Delta t \leqslant \tau < \min\left\{(8(\kappa + \beta(M)))^{-1}, \left(16\beta(M)M^{(p-\delta)/p}\right)^{-\delta/p}\right\}.$$

The rest of the proof then is as in the proof of Proposition 5.1 with $c_1 = 4(\kappa + \beta(M))$ and the same c_2.

For $p = 2$, we observe that

$$\frac{1}{4}\,\mathrm{D}_1\left(|u^n - v^n|^2 + |\mathrm{E}(u^{n+1} - v^{n+1})|^2\right) + \frac{(\Delta t)^3}{4}\,|\mathrm{D}^2(u^{n-1} - v^{n-1})|^2 + \mu_0\|u^n - v^n\|^2 - \\ \kappa|u^n - v^n|^2 - \beta(M)\left(1 + \|u^n\|^{2-\delta} + \|v^n\|^{2-\delta}\right)^s |u^n - v^n|^s\|u^n - v^n\|^{2-s} \leqslant \\ c\|f^n - g^n\|_*^2 + \frac{\mu_0}{4}\,\|u^n - v^n\|^2.$$

With Young's inequality, we find

$$\beta(M)\left(1 + \|u^n\|^{2-\delta} + \|v^n\|^{2-\delta}\right)^s |u^n - v^n|^s\|u^n - v^n\|^{2-s} \leqslant \\ C(\mu_0, s)\beta(M)^{2/s}\left(1 + \|u^n\|^{2-\delta} + \|v^n\|^{2-\delta}\right)|u^n - v^n|^2 + \frac{\mu_0}{2}\|u^n - v^n\|^2,$$

where

$$C(\mu_0, s) = \frac{s}{2}\left(\frac{2-s}{\mu_0}\right)^{(2-s)/s},$$

and the proof can be finished as before with $\beta(M)$ being replaced by $C(\mu_0, s)\beta(M)^{2/s}$. □

Remark 5.1. The propositions above show stability with respect to the right-hand side and the initial values for sufficiently small but fixed Δt; the stability estimates are in general not uniform in Δt. However, if $\delta = p$ in (H3) and (H3′), respectively, then the stability constant is bounded for $\Delta t \to 0$ since then

$$\left(1 - c_1\Delta t - c_2(\Delta t)^{\delta/p}\right)^{-T/\Delta t} = (1 - (c_1 + c_2)\Delta t)^{-T/\Delta t} \leqslant \exp\left(\frac{(c_1 + c_2)T}{1 - (c_1 + c_2)\tau}\right).$$

Moreover, uniqueness of a solution to (3.2) can immediately be inferred from the proofs of Propositions 5.1 and 5.2.

Stability estimates for the discrete derivative $\mathrm{D}_2(u^n - v^n)$ can be obtained if the operators $A(t)$ ($t \in [0, T]$) fulfill some Hölder-type condition.

Proposition 5.3. *In addition to the assumptions of Proposition* 5.2 *assume that there is some* $r \in (0,1]$ *and for any* $R > 0$ *there exists* $\gamma(R) > 0$ *such that for all* $t \in [0,T]$ *and* $v, w \in V$ *with* $\max(|v|,|w|) \leqslant R$

$$\|A(t)v - A(t)w\|_* \leqslant \gamma(R)(1 + \|v\| + \|w\|)^{p-1-r}\|v - w\|^r.$$

Then

$$\Delta t \sum_{j=2}^{N} \|\mathrm{D}_2(u^j - v^j)\|_*^q \leqslant c\Delta t \sum_{j=2}^{N} \|f^j - g^j\|_*^q + c\left(1 - c_1\Delta t - c_2(\Delta t)^{\delta/p}\right)^{-T/\Delta t \cdot r/(p-1)} \times$$

$$\left(|u^0 - v^0|^2 + |u^1 - v^1|^2 + \Delta t \sum_{j=2}^{N} \|f^j - g^j\|_*^q\right)^{r/(p-1)},$$

where c, c_1, c_2 *are positive constants not depending on* Δt.

Proof. From (3.2), we immediately get

$$\Delta t \sum_{j=2}^{N} \|\mathrm{D}_2(u^j - v^j)\|_*^q \leqslant c\Delta t \sum_{j=2}^{N} \|f^j - g^j\|_*^q + c\Delta t \sum_{j=2}^{N} \|A(t_j)u^j - A(t_j)v^j\|_*^q.$$

Hölder's inequality yields (with $\Delta t = (\Delta t)^{(p-1-r)q/p}(\Delta t)^{rq/p}$)

$$\Delta t \sum_{j=2}^{N} \|A(t_j)u^j - A(t_j)v^j\|_*^q \leqslant \gamma(M)^q \Delta t \sum_{j=2}^{N} (1 + \|u^j\| + \|v^j\|)^{(p-1-r)q}\|u^j - v^j\|^{rq} \leqslant$$

$$c\gamma(M)^q \left(\Delta t \sum_{j=2}^{N} (1 + \|u^j\| + \|v^j\|)^p\right)^{(p-1-r)q/p} \left(\Delta t \sum_{j=2}^{N} \|u^j - v^j\|^p\right)^{rq/p} \leqslant$$

$$c\gamma(M)^q (T + 2M)^{(p-1-r)q/p} \left(\Delta t \sum_{j=2}^{N} \|u^j - v^j\|^p\right)^{rq/p}.$$

Here, M is again the *a priori* bound for solutions to (3.2). The assertion now follows from Proposition 5.2 since $q = p/(p-1)$. □

Estimates for the error $u(t_n) - u^n$ ($n = 2, 3, \ldots, N$) between the exact and the numerical solution easily follow from stability estimates that are uniform in Δt because of the error equation

$$\mathrm{D}_2(u(t_n) - u^n) + A(t_n)u(t_n) - A(t_n)u^n = \rho^n := \mathrm{D}_2 u(t_n) - u'(t_n) + f(t_n) - f^n. \tag{5.1}$$

Proposition 5.4. *Let* $f'' - u''' \in L^2(0,T;H)$. *Then*

$$\Delta t \sum_{j=2}^{N} |\rho^j|^2 \leqslant c(\Delta t)^4 \|f'' - u'''\|_{L^2(0,T;H)}^2 + c\Delta t \sum_{j=2}^{N} |\mathrm{R}_2^j f - f^j|^2.$$

Let $f'' - u''' \in L^q(0,T;V^*)$ $(q \in [1,\infty))$. *Then*

$$\Delta t \sum_{j=2}^{N} \|\rho^j\|_*^q \leqslant c(\Delta t)^{2q} \|f'' - u'''\|_{L^q(0,T;V^*)}^q + c\Delta t \sum_{j=2}^{N} \|\mathrm{R}_2^j f - f^j\|_*^q.$$

Proof. Integration by parts shows that

$$\hat{\rho}^n := \rho^n + f^n - \mathrm{R}_2^n f = \mathrm{D}_2 u(t_n) - u'(t_n) + f(t_n) - \mathrm{R}_2^n f =$$

$$\frac{1}{4\Delta t}\Bigg(\int\limits_{t_{n-1}}^{t_n} (t_n - t)(t_n + 3t - 4t_{n-1})(f''(t) - u'''(t))dt + \int\limits_{t_{n-2}}^{t_{n-1}} (t - t_{n-2})^2(f''(t) - u'''(t))dt\Bigg).$$

By standard arguments, we find

$$\Delta t \sum_{j=2}^{N} |\hat{\rho}^j|^2 \leqslant c(\Delta t)^4 \int\limits_0^T |f''(t) - u'''(t)|^2 dt$$

as well as

$$\Delta t \sum_{j=2}^{N} \|\hat{\rho}^j\|_*^q \leqslant c(\Delta t)^{2p/(p-1)} \int\limits_0^T \|f''(t) - u'''(t)\|_*^q dt.$$

Upon noting that $q = p/(p-1)$, the assertion follows. □

The proposition above shows that the consistency error, measured in the discrete $l^2(0,T;H)$- and $l^q(0,T;V^*)$-norm, respectively, is of second order if $\{f^n\}$ is an appropriate approximation. It might be worth to mention that one obtains first-order consistency (and corresponding error estimates) if only $t(f''-u''') \in L^2(0,T;H)$ and $t(f''-u''') \in L^q(0,T;V^*)$, respectively. Moreover, order two can generically not be exceeded even if $f''-u''' \in L^\infty(0,T;H)$ and $f'' - u''' \in L^\infty(0,T;V^*)$, respectively.

The stability results from Propositions 5.1 and 5.2 together with the error equation (5.1) and Proposition 5.4 now immediately prove the following theorem.

Theorem 5.1. *Let $u \in \mathcal{W}$ be the solution to* (1.1) *with $u_0 \in H$ and $f \in \mathcal{X}^*$, and let $\Delta t \leqslant \tau$ with $\tau < T$ being sufficiently small.*

Assume (H2), (H3) *with $\delta = p$,* (H5)*, and let $f, f'' - u''' \in L^2(0,T;H)$, $u^0, u^1 \in H$, $\{f^n\} \in l^2(0,T;H)$. The discrete solution $\{u^n\}$ to* (3.2) *then satisfies the error estimate*

$$\max_{n=2,\dots,N} |u(t_n) - u^n|^2 \leqslant c\Bigg(|u_0 - u^0|^2 + |u(t_1) - u^1|^2 + (\Delta t)^4 \|f'' - u'''\|_{L^2(0,T;H)}^2 + \Delta t \sum_{j=2}^{N} |\mathrm{R}_2^j f - f^j|^2\Bigg).$$

Assume (H2′), (H3′) *with $\delta = p$,* (H5)*, and let $f, f'' - u''' \in L^q(0,T;V^*)$, $u^0, u^1 \in H$, $\{f^n\} \in l^q(0,T;V^*)$. The discrete solution $\{u^n\}$ to* (3.2) *then satisfies the error estimate*

$$\max_{n=2,\dots,N} |u(t_n) - u^n|^2 + \Delta t \sum_{j=2}^{N} \|u(t_j) - u^j\|^p \leqslant c\Bigg(|u_0 - u^0|^2 + |u(t_1) - u^1|^2 +$$

$$(\Delta t)^{2q} \|f'' - u'''\|_{L^q(0,T;V^*)}^q + \Delta t \sum_{j=2}^{N} \|\mathrm{R}_2^j f - f^j\|_*^q\Bigg).$$

The first part of the theorem shows convergence of optimal order $\mathcal{O}((\Delta t)^2)$ in the discrete $l^2(0,T;H)$-norm if the initial approximations and if $\{\mathrm{R}_2^n f - f^n\}$, measured in the discrete $l^2(0,T;H)$-norm, are of second order. The second part shows — under weaker assumptions on the right-hand side and the regularity of the exact solution — convergence of order

$\mathcal{O}((\Delta t)^{p/(p-1)})$ in the discrete $l^2(0,T;H)$-norm and of order $\mathcal{O}((\Delta t)^{2/(p-1)})$ in the discrete $l^p(0,T;V)$-norm, respectively, if $\{\mathrm{R}_2^n f - f^n\}$ is of corresponding order. If $p = 2$ then the order is always $\mathcal{O}((\Delta t)^2)$. With respect to the initial approximation, we remark that u^1 might be computed by means of an implicit Euler step giving a local error of the corresponding order as one can show similarly to the results above for the two-step BDF: Let $f' - u'' \in L^\infty(0,T;H)$, which is the case if $f' - u'', f'' - u''' \in L^2(0,T;H)$ or if $f' - u'' \in L^p(0,T;V)$, $f'' - u''' \in L^q(0,T;V^*)$, then $|u(t_1) - u^1|$ can be shown to be of order $\mathcal{O}((\Delta t)^2)$. Note here that one only needs one step and no summation is carried out.

In particular situations, the assumptions on B can again be relaxed. The semilinearity (4.15) of the Navier — Stokes problem, for instance, satisfies for solenoidal functions $v, w \in V$ the relation

$$\langle Bv - Bw, v - w\rangle = \int\limits_\Omega \big((v-w)\cdot\nabla\big)v\cdot(v-w)dx.$$

The Hölder's and the Young inequality, embedding arguments, and interpolatory inequalities, then allow (with spatial dimension $d \in \{2,3\}$) to estimate for sufficiently smooth v

$$|\langle Bv - Bw, v - w\rangle| \leqslant c\|v\|_{2,2}|v-w|^{3/2}\|v-w\|^{1/2} \leqslant c\|v\|_{2,2}^{4/3}|v-w|^2 + \frac{\mu_0}{4}\|v-w\|^2,$$

where $\|\cdot\|_{2,2}$ denotes the $H^2(\Omega)^d$-norm. With $v = u(t_n)$ and $w = u^n$, error estimates can be obtained as far as the exact solution possesses more spatial regularity. Note, however, that higher-order error estimates for the Navier — Stokes problem require a more refined analysis since a higher regularity of the exact solution is equivalent to certain compatibility conditions on the initial data (over-determined Neumann problem for the initial pressure) that are hard to fulfill (see, e.g., [52]). The estimates then rely upon the parabolic smoothing property and duality arguments (see, e.g., [12]).

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