

Convergence of the backward Euler scheme for the operator-valued Riccati differential equation with semi-definite data*

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October 8, 2018

For initial value problems associated with operator-valued Riccati differential equations posed in the space of Hilbert–Schmidt operators existence of solutions is studied. An existence result known for algebraic Riccati equations is generalized and used to obtain the existence of a solution to the approximation of the problem via a backward Euler scheme. Weak and strong convergence of the sequence of approximate solutions is established permitting a large class of right-hand sides and initial data.

1 Motivation

In this paper, we prove the existence of a solution to the initial value problem for the operator valued Riccati differential equation

$$\begin{aligned} \mathcal{P}'(t) + \mathcal{A}^*(t)\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A}(t) + \mathcal{P}^2(t) &= \mathcal{Q}(t), \quad t \in (0, T), \\ \mathcal{P}(0) &= \mathcal{P}_0, \end{aligned} \tag{1.1}$$

by showing the convergence of a suitable approximation scheme. A solution of this problem is of importance, for example, in the optimal control of partial differential equations, see [24]. Even though the existence of a solution can be deduced from the optimality conditions of a suitable control problem, it is helpful for applications to construct a solution to the initial value problem directly. This solution can then be used to calculate an optimal solution of the control problem.

*The authors gratefully acknowledge financial support from the Deutsche Forschungsgemeinschaft through the Collaborative Research Center 901 "Control of self-organizing nonlinear systems: Theoretical methods and concepts of application" (projects A2, A8).

In the literature, different ways of examining the solvability of the initial value problem (1.1) have been studied. In [9], an approach using a two-parameter semigroup is proposed. This ansatz is considered and generalized in many works, see, e.g., [11, 12, 16, 17, 21]. A similar approach was used in [10] based on Green functions. In [29], existence was shown by adding a holomorphic function to the linear part of the differential equation. In [22, 23] many further aspects to the approach via mild solution theory to this problem may be found.

For matrix-valued Riccati equations, the backward Euler scheme and BDF methods have been studied e.g. in [2, 5, 6]. Many further aspects related to matrix-valued Riccati equations can be found in [25]. In [6], a spatial discretization is used to obtain results for the operator case, as well. In [19], the convergence analysis of an operator splitting method is considered.

Similarly to our approach, in [31, 32] a numerical method to construct a solution of (1.1) is studied. Here, the existence of a weak solution is proven via a time discretization using a three-step splitting method. In [3, Chapter III.2.3 Example 3], the same result is proven through the solvability of the algebraic Riccati equation

$$\tilde{A}^*P + P\tilde{A} + P^2 = Q$$

for suitable linear operators \tilde{A} and Q and the existence of a mild solution to the initial value problem for differential equations with accretive operators.

In this paper we achieve both, an existence result and a numerical approximation method, via a discretization in time of the initial value problem (1.1) using the backward Euler scheme with constant step size $\tau = \frac{T}{N}$, $N \in \mathbb{N}$. This leads to the discretized system

$$\frac{P_n - P_{n-1}}{\tau} + A_n^*P_n + P_nA_n + P_n^2 = Q_n, \quad n = 1, 2, \dots, N,$$

with $P_0 = \mathcal{P}_0$ and suitable operators A_n^* , A_n , and Q_n . Here P_n denotes an approximation of $\mathcal{P}(t_n)$ with $t_n = n\tau$ ($n = 1, 2, \dots, N$). Rewriting this algebraic operator equation leads to a system of algebraic Riccati equations of the form

$$\left(A_n + \frac{1}{2\tau}I\right)^* P_n + P_n \left(A_n + \frac{1}{2\tau}I\right) + P_n^2 = Q_n + \frac{1}{\tau}P_{n-1}, \quad n = 1, 2, \dots, N,$$

with $P_0 = \mathcal{P}_0$, where in contrast to the approach in [3] the right-hand side is more complicated. To deal with this difficulty is a major component of our work which is assembled as follows.

In Section 2, we begin with a short introduction to the concept of Hilbert–Schmidt operators and the function spaces required for the weak solution approach. In the next section, a generalization of the existence result for an algebraic Riccati equation from [3, Chapter II.3, Theorem 3.9] is considered. Here, we take the right-hand side from a class of operators that are bounded from below by a constant depending on the operator $\mathcal{A} = \mathcal{A}(t)$. The class of possible right-hand sides includes operators which are not necessarily positive definite. In Section 4, our main result Theorem 4.1 is presented.

We prove the existence of a solution in the weak sense, i.e., the existence of a locally integrable function $\mathcal{P} = \mathcal{P}(t)$ taking values in a suitable space of linear operators and fulfilling the initial condition $\mathcal{P}(0) = P_0$ in an appropriate sense such that

$$\begin{aligned} - \int_0^T \langle \mathcal{P}(t), R \rangle \varphi'(t) dt + \int_0^T \langle \mathcal{A}^*(t)\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A}(t) + \mathcal{P}^2(t), R \rangle \varphi(t) dt \\ = \int_0^T \langle \mathcal{Q}(t), R \rangle \varphi(t) dt \end{aligned}$$

is fulfilled for every R from a suitable space of linear operators and every smooth test function $\varphi : (0, T) \rightarrow \mathbb{R}$ with compact support. Here, $\langle \cdot, \cdot \rangle$ is a suitable duality pairing that we will introduce in Section 2 in more detail. To this extent, we construct a solution of the initial value problem for the Riccati differential equation using the backward Euler scheme. The time-discretized equations that occur can be solved using the new existence result for algebraic Riccati equations from Section 3 which allows us to consider a larger class of functions for the right-hand side and the initial value in comparison with [32]. We can even allow indefiniteness for the data as well as a more general condition on the integrability of the right-hand side which are the same as proposed in [30, Section 19–20].

Even though a variational approach is restricted to data that are Hilbert–Schmidt operators, which are compact operators, we believe that this concept of solution is more suitable for numerical examinations. For example, it offers both a constructive scheme and the possibility to fully discretize the problem.

2 Notation and preliminaries

In order to state the weak formulation of the problem, a Hilbert space setting within the space of linear operators is required. To proceed like this, we briefly introduce the space of Hilbert–Schmidt operators. A complete introduction can be found in [15].

For real, separable Hilbert spaces $(X, (\cdot, \cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$, let $\mathcal{L}(X, Y)$ be the space of linear bounded operators mapping X into Y . We denote the *Hilbert–Schmidt norm* of a compact operator $T \in \mathcal{L}(X, Y)$ by

$$\|T\|_{\mathcal{HS}(X, Y)} = \left(\sum_{n=1}^{\infty} \|Te_n\|_Y^2 \right)^{\frac{1}{2}},$$

where $(e_n)_{n \in \mathbb{N}}$ is an arbitrary orthonormal basis of X . This norm is induced by the inner product

$$(T, S)_{\mathcal{HS}(X, Y)} = \sum_{n=1}^{\infty} (Te_n, Se_n)_Y = \text{tr}(T^*S) = \text{tr}(ST^*), \quad S, T \in \mathcal{L}(X, Y),$$

where the operator T^* denotes the Hilbert space adjoint of T and tr is the trace of an operator. Equipped with this inner product and norm, the linear space

$$\mathcal{HS}(X, Y) = \{T \in \mathcal{L}(X, Y) : \|T\|_{\mathcal{HS}(X, Y)} < \infty\}$$

of Hilbert–Schmidt operators is a separable Hilbert space. Note that both the norm $\|\cdot\|_{\mathcal{HS}(X,Y)}$ and the inner product $(\cdot, \cdot)_{\mathcal{HS}(X,Y)}$ are independent of the choice of basis $(e_n)_{n \in \mathbb{N}}$ as shown in [28, Theorem 3.6.1].

Using this closed subspace of the space of linear operators, we can introduce a suitable *Gelfand triple* for the following theory. This concept to construct suitable spaces can also be found in [27] and [32].

We begin by determining the exact assumptions on the spaces on which the operators are defined.

Assumption 1. Let $(V, (\cdot, \cdot)_V, \|\cdot\|_V)$ and $(H, (\cdot, \cdot)_H, \|\cdot\|_H)$ be real, separable Hilbert spaces such that V is compactly and densely embedded (denoted as $\xrightarrow{c,d}$) into H .

The embedding constant is denoted by $C_{V,H}$, i.e.,

$$\|v\|_H \leq C_{V,H} \|v\|_V \quad (2.1)$$

holds for every $v \in V$.

Identifying $(H, (\cdot, \cdot)_H, \|\cdot\|_H)$ with its dual $(H^*, (\cdot, \cdot)_{H^*}, \|\cdot\|_{H^*})$, we have the Gelfand triple

$$V \xrightarrow{c,d} H \cong H^* \xrightarrow{c,d} V^*, \quad (2.2)$$

where $(V^*, (\cdot, \cdot)_{V^*}, \|\cdot\|_{V^*})$ denotes the dual space of V . The inner product in H can be extended to the duality pairing between V^* and V , which is denoted by $\langle \cdot, \cdot \rangle_{V^* \times V}$. We introduce the Hilbert spaces \mathcal{V} , \mathcal{H} and \mathcal{V}^* as

$$\mathcal{V} = \mathcal{HS}(V^*, H) \cap \mathcal{HS}(H, V), \quad \mathcal{H} = \mathcal{HS}(H, H) \quad \text{and} \quad \mathcal{V}^* = \mathcal{HS}(V, H) + \mathcal{HS}(H, V^*) \quad (2.3)$$

and identify \mathcal{H} with its dual \mathcal{H}^* . Note that it is possible to define the above intersection of vector spaces, since every appearing space can be interpreted as a subset of $\mathcal{L}(V, V^*)$. For further details, see, e.g. [18, Bemerkung 5.11]. It is important to note that the norm $\|\cdot\|_{\mathcal{H}}$ is submultiplicative (see, e.g., [15, Theorem 4 in Ch. XI.6.3]). The inner product in \mathcal{H} can be extended to the duality pairing between \mathcal{V}^* and \mathcal{V} that is denoted by

$$\langle \cdot, \cdot \rangle : \mathcal{V}^* \times \mathcal{V} \rightarrow \mathbb{R}.$$

As we have required the embedding $V \xrightarrow{c} H$ to be compact, this property extends to the operator spaces \mathcal{V} and \mathcal{H} , i.e., the embeddings

$$\mathcal{V} \xrightarrow{c} \mathcal{H} \cong \mathcal{H}^* \xrightarrow{c} \mathcal{V}^*$$

are compact as well (see [32, Proposition 2.1.]). Also the norm estimate (2.1) transfers to \mathcal{V} and \mathcal{H} : $\|S\|_{\mathcal{H}} \leq \frac{C_{V,H}}{\sqrt{2}} \|S\|_{\mathcal{V}}$ for all $S \in \mathcal{V}$.

For $p \in [1, \infty]$ and a Banach space $(X, \|\cdot\|_X)$, we introduce the Bochner-Lebesgue space

$$L^p(0, T; X) = \{u : [0, T] \rightarrow X : u \text{ is strongly measurable and } \|u\|_{L^p(0, T; X)} < \infty\}$$

with the usual norm given by

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}} & \text{for } p \in [1, \infty), \\ \operatorname{ess\,sup}_{t \in [0,T]} \|u(t)\|_X & \text{for } p = \infty. \end{cases}$$

Further, we introduce the space $L^1(0, T; \mathcal{H}) + L^2(0, T; \mathcal{V}^*)$ which consists of the functions $u : [0, T] \rightarrow \mathcal{V}^*$ such that there exist $u_1 \in L^1(0, T; \mathcal{H})$ and $u_2 \in L^2(0, T; \mathcal{V}^*)$ that fulfill $u = u_1 + u_2$. A norm for this space is given by

$$\|u\|_{L^1(0,T;\mathcal{H})+L^2(0,T;\mathcal{V}^*)} = \inf_{\substack{u_1 \in L^1(0,T;\mathcal{H}) \\ u_2 \in L^2(0,T;\mathcal{V}^*) \\ u = u_1 + u_2}} (\|u_1\|_{L^1(0,T;\mathcal{H})} + \|u_2\|_{L^2(0,T;\mathcal{V}^*)}),$$

see [30, Section 20] for more details. This space in mind, we define

$$\mathcal{W}_1(0, T) = \{u \in L^2(0, T; \mathcal{V}) : u' \in L^1(0, T; \mathcal{H}) + L^2(0, T; \mathcal{V}^*)\} \quad (2.4)$$

equipped with the norm

$$\|u\|_{\mathcal{W}_1(0,T)} = \|u\|_{L^2(0,T;\mathcal{V})} + \|u'\|_{L^1(0,T;\mathcal{H})+L^2(0,T;\mathcal{V}^*)}.$$

Note that $C^\infty([0, T]; \mathcal{V})$ is dense in $\mathcal{W}_1(0, T)$. Here, $C^\infty([0, T]; \mathcal{V})$ is the linear space of all infinitely many times differentiable functions mapping $[0, T]$ into \mathcal{V} . Furthermore, $\mathcal{W}_1(0, T)$ is dense and continuously embedded into the space $C([0, T]; \mathcal{H})$ of continuous functions mapping $[0, T]$ into \mathcal{H} . See [30, Section 20] for more details. By $C_c^\infty(0, T)$, we denote the linear space of infinitely many times differentiable functions mapping $(0, T)$ into \mathbb{R} having compact support.

In the following, it will be of importance to distinguish between different types of convergence of a sequence $(x_n)_{n \in \mathbb{N}}$ to a limit x in a Banach space $(X, \|\cdot\|_X)$. To this end, we will briefly explain the used notation. We denote the (strong) convergence by

$$x_n \rightarrow x \text{ in } X, \quad \text{i.e.,} \quad \|x - x_n\|_X \rightarrow 0$$

as $n \rightarrow \infty$, the weak convergence by

$$x_n \rightharpoonup x \text{ in } X, \quad \text{i.e.,} \quad \langle f, x - x_n \rangle_{X^* \times X} \rightarrow 0 \text{ for every } f \in X^*$$

as $n \rightarrow \infty$ and the weak* convergence of a sequence $(f_n)_{n \in \mathbb{N}}$ to a limit f in X^* by

$$f_n \xrightarrow{*} f \text{ in } X^*, \quad \text{i.e.,} \quad \langle f - f_n, x \rangle_{X^* \times X} \rightarrow 0 \text{ for every } x \in X.$$

Here X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle_{X^* \times X}$ the duality pairing. For more details, see, e.g., [8, Chapter 3]. In the following, $c > 0$ always denotes a generic constant.

3 An algebraic operator Riccati equation

In this section we prove a generalization of the existence result for algebraic Riccati equations given in [3, Chapter II.3, Lemma 3.2 and Theorem 3.9].

Assumption 2. Let $A \in \mathcal{L}(V, V^*)$ be a *strongly positive operator*, i.e., there exists a constant $\mu_V > 0$ such that

$$\langle Au, u \rangle_{V^* \times V} \geq \mu_V \|u\|_V^2 \quad (3.1)$$

holds for all $u \in V$.

Remark 3.1. Note that Assumptions 1. and 2. immediately imply that

$$\langle Au, u \rangle_{V^* \times V} \geq \mu_H \|u\|_H^2 \quad (3.2)$$

for all $u \in V$ with $\mu_H = \frac{\mu_V}{C_{V,H}^2}$. Nevertheless (3.2) may also be fulfilled with a constant μ_H much larger than $\frac{\mu_V}{C_{V,H}^2}$. This, indeed, will be the case in our application in Lemma 4.2 below.

Remark 3.2.

Lemma 3.3. *Let Assumptions 1. and 2. be satisfied. Then the estimate*

$$\langle PA, P \rangle \geq \mu_V \|P\|_{\mathcal{HS}(V^*, H)}^2$$

holds for all . Moreover,

$$\langle A^*P, P \rangle \geq \mu_V \|P\|_{\mathcal{HS}(H, V)}^2$$

holds for all , where A^ is the dual operator of A . In particular,*

$$\langle A^*P_1 + P_1A, P_1 \rangle \geq \mu_V \|P_1\|_{\mathcal{V}}^2 \quad \text{and} \quad \langle A^*P_2 + P_2A, P_2 \rangle \geq 2\mu_H \|P_2\|_{\mathcal{H}}^2$$

*hold for all $P_1 \in \mathcal{V}$ and $P_2 \in \{R \in \mathcal{V} : A^*R + RA \in \mathcal{H}\}$.*

Proof. A proof can be found in [3, Chapter II.3, Lemma 3.4]. □

The following results will be stated for (indefinite) self-adjoint operators in \mathcal{H} that have a lower bound for the possibly negative eigenvalues. Let $P \in \mathcal{H}$ be self-adjoint. Then it is also compact, since P is a Hilbert–Schmidt operator. Applying the Hilbert–Schmidt theorem (see [14, Theorem 5, Chapter VII.4.5]) assures the existence of an orthonormal system $(e_n)_{n \in \mathbb{N}}$ of eigenvectors of P in H . Using the eigenvectors $(e_n)_{n \in \mathbb{N}}$, P can be represented through

$$P = \sum_{n=1}^{\infty} \alpha_n (e_n, \cdot)_H e_n, \quad (3.3)$$

where each $\alpha_n \in \mathbb{R}$, $n \in \mathbb{N}$, is an eigenvalue of the operator P . As every self-adjoint operator in \mathcal{H} has a representation like this, we can introduce a lower bound for self-adjoint operators. For $\gamma \in \mathbb{R}$, we write $P \geq \gamma$ if every eigenvalue of P is greater or equal to γ . This is equivalent to the condition $P - \gamma I \geq 0$, i.e.,

$$((P - \gamma I)u, u)_H \geq 0 \quad (3.4)$$

for all $u \in H$. This in mind, we introduce the set

$$C_{-\gamma} = \{P \in \mathcal{H} : P = P^*, P \geq -\gamma\}$$

of operators whose eigenvalues are bounded from below by $-\gamma$.

For the linear part of the Riccati equation we introduce the linear *Lyapunov* operator

$$\mathfrak{A}_0 : \mathcal{V} \rightarrow \mathcal{V}^*, \quad P \mapsto A^*P + PA$$

but use the same notation also for its restriction

$$\mathfrak{A}_0 : \text{dom}(\mathfrak{A}_0) \subseteq \mathcal{H} \rightarrow \mathcal{H}, \quad P \mapsto A^*P + PA$$

with $\text{dom}(\mathfrak{A}_0) = \{P \in \mathcal{V} : \mathfrak{A}_0 P \in \mathcal{H}\}$. If Assumption 2. holds, then due to the positivity of A and A^* and Lemma 3.3, the operator \mathfrak{A}_0 is *m-accretive* and $(\mathfrak{A}_0 P, P) \geq 0$ is fulfilled for every $P \in \text{dom}(\mathfrak{A}_0)$. Therefore, $\lambda \mathfrak{A}_0 + I$ is surjective for every $\lambda > 0$, see, e.g., [4, Theorem 2.2]. The following lemma provides some important properties for the resolvent of $\lambda \mathfrak{A}_0$, $\lambda > 0$.

Lemma 3.4. *Let Assumptions 1. and 2. be satisfied and $\lambda > 0$. Then $(\lambda \mathfrak{A}_0 + I)^{-1} : \mathcal{H} \rightarrow \text{dom}(\mathfrak{A}_0)$ is well-defined, linear, bounded and fulfills the estimate*

$$\|(\lambda \mathfrak{A}_0 + I)^{-1} Q\|_{\mathcal{H}} \leq \frac{1}{1 + 2\lambda\mu_H} \|Q\|_{\mathcal{H}}$$

for all $Q \in \mathcal{H}$. Furthermore, for $Q \in C_{-\gamma}$

$$(\lambda \mathfrak{A}_0 + I)^{-1} Q \in C_{-\frac{\gamma}{1+2\lambda\mu_H}}$$

is fulfilled.

Proof. The operator

$$\lambda \mathfrak{A}_0 + I : \mathcal{V} \rightarrow \mathcal{V}^*, \quad R \mapsto \lambda(A^*R + RA) + R$$

is linear, bounded, and strongly positive,

$$\langle (\lambda \mathfrak{A}_0 + I)R, R \rangle \geq \lambda\mu_V \|R\|_{\mathcal{V}}^2 + \|R\|_{\mathcal{H}}^2 \geq \lambda\mu_V \|R\|_{\mathcal{V}}^2$$

Therefore, $\lambda \mathfrak{A}_0 + I : \mathcal{V} \rightarrow \mathcal{V}^*$ is bijective due to the Lax-Milgram lemma, and for every $Q \in \mathcal{H} \subseteq \mathcal{V}^*$ there exists a unique $P \in \text{dom}(\mathfrak{A}_0)$ such that

$$(\lambda \mathfrak{A}_0 + I)P = Q.$$

This proves the existence of the linear operator $(\lambda\mathfrak{A}_0 + I)^{-1}$ in \mathcal{H} . For a self-adjoint operator Q , it follows that

$$(\lambda\mathfrak{A}_0 + I)P = Q = Q^* = ((\lambda\mathfrak{A}_0 + I)P)^* = \lambda(A^*P^* + P^*A) + P^* = (\lambda\mathfrak{A}_0 + I)P^*.$$

Since the mapping $\lambda\mathfrak{A}_0 + I : \mathcal{V} \rightarrow \mathcal{V}^*$ is injective, the operators P and P^* coincide, i.e., P is self-adjoint as it is a bounded, symmetric operator in H . In the following, we represent P as in (3.3). Exploiting the fact that the elements e_n , $n \in \mathbb{N}$, from (3.3) are eigenvectors of P , we obtain that

$$\begin{aligned} (Qe_n, e_n)_H &= \lambda\langle A^*Pe_n, e_n \rangle_{V^* \times V} + \lambda\langle PAe_n, e_n \rangle_{V^* \times V} + (Pe_n, e_n)_H \\ &= 2\lambda\alpha_n\langle Ae_n, e_n \rangle_{V^* \times V} + \alpha_n. \end{aligned}$$

This implies that

$$\alpha_n = (1 + 2\lambda\langle Ae_n, e_n \rangle_{V^* \times V})^{-1} (Qe_n, e_n)_H,$$

which leads to the estimate

$$\begin{aligned} \|(\lambda\mathfrak{A}_0 + I)^{-1}Q\|_{\mathcal{H}}^2 &= \|P\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \alpha_n^2 = \sum_{n=1}^{\infty} \left(\frac{(Qe_n, e_n)_H}{1 + 2\lambda\langle Ae_n, e_n \rangle_{V^* \times V}} \right)^2 \\ &\leq \sum_{n=1}^{\infty} \left(\frac{(Qe_n, e_n)_H}{1 + 2\lambda\mu_H\|e_n\|_H^2} \right)^2 \leq \left(\frac{1}{1 + 2\lambda\mu_H} \right)^2 \|Q\|_{\mathcal{H}}^2. \end{aligned}$$

Since we have the lower bound $(Qe_n, e_n)_H \geq -\gamma$ for $Q \in C_{-\gamma}$, we obtain that

$$\alpha_n = (1 + 2\lambda\langle Ae_n, e_n \rangle_{V^* \times V})^{-1} (Qe_n, e_n)_H \geq -\frac{\gamma}{1 + 2\lambda\mu_H}$$

for every $n \in \mathbb{N}$. As $P = (\lambda\mathfrak{A}_0 + I)^{-1}Q$ is self-adjoint by definition and has a suitable lower bound for every eigenvalue, this proves the second assertion of the lemma. \square

In the following, we derive some corresponding results for the full Riccati equation. In order to do this, we define the nonlinear operator

$$\mathcal{B} : \mathcal{H} \rightarrow \mathcal{H}, \quad P \mapsto P^2. \tag{3.5}$$

Lemma 3.5. *Let Assumption 1. be satisfied and let $\gamma \geq 0$ be given. If $P \in C_{-\gamma}$ then*

$$(PR, R) \geq -\gamma\|R\|_{\mathcal{H}}^2 \quad \text{and} \quad (RP, R) \geq -\gamma\|R\|_{\mathcal{H}}^2$$

is fulfilled for every $R \in \mathcal{H}$ and

$$\langle PR + RP, R \rangle \geq -\gamma C_{V,H}^2 \|R\|_{\mathcal{V}}^2$$

is fulfilled for every $R \in \mathcal{V}$.

Proof. Since we assume that $P \in C_{-\gamma}$, the operator $P + \gamma I$ is positive such that

$$((P + \gamma I)R, R) \geq 0$$

for all $R \in \mathcal{H}$. This implies, in particular, the first assertion. The other assertions follow in an analogous manner. \square

For $\lambda > 0$, the operator $\lambda \mathcal{B} + I$ is neither linear nor Lipschitz continuous on \mathcal{H} . Still the following lemmas provide a suitable boundedness and Lipschitz continuity of the inverse of the restriction of the operator onto a suitable domain $C_{-\gamma}$, $\gamma > 0$.

Remark 3.6. In order to get a better intuition for such an inverse of a restriction of the operator $(\lambda \mathcal{B} + I)$ for given $\lambda > 0$, it is helpful to consider the problem in \mathbb{R} at first. We want to find $P \in \mathbb{R}$ such that for given $Q \in \mathbb{R}$

$$(\lambda \mathcal{B} + I)P = \lambda P^2 + P = Q.$$

This equation can easily be solved in \mathbb{R} if $\lambda > 0$ is small enough and has the solutions

$$P = \frac{1}{2\lambda} \left(-1 \pm \sqrt{1 + 4\lambda Q} \right).$$

For $Q > 0$, this admits one positive and one negative solution. For $\lambda Q \in (-\frac{1}{4}, 0)$, the problem has two different negative solutions.

Analogously to Remark 3.6, looking at the operator setting again, the equation

$$(\lambda \mathcal{B} + I)P = \lambda P^2 + P = Q. \tag{3.6}$$

possesses only one positive solution P if $Q \geq 0$ but more than one negative solution if $Q < 0$. In the following, we will always work with the solution.

Lemma 3.7. *Let Assumption 1. be satisfied and let $\gamma \geq 0$ be given. Then for $\lambda \in (0, \frac{1}{4\gamma})$ if $\gamma > 0$, and $\lambda \in (0, \infty)$ if $\gamma = 0$, and every $Q \in C_{-\gamma}$ there exists a unique $P \in C_{\frac{-2\gamma}{1 + \sqrt{1 - 4\lambda\gamma}}}$ such that (3.6) is fulfilled. In the following, P is denoted by $(\lambda \mathcal{B} + I)^{-1}Q$. Moreover, for $Q, Q_1, Q_2 \in C_{-\gamma}$ the estimates*

$$\|(\lambda \mathcal{B} + I)^{-1}Q\|_{\mathcal{H}} \leq \frac{2}{1 + \sqrt{1 - 4\lambda\gamma}} \|Q\|_{\mathcal{H}} \tag{3.7}$$

and

$$\|(\lambda \mathcal{B} + I)^{-1}Q_1 - (\lambda \mathcal{B} + I)^{-1}Q_2\|_{\mathcal{H}} \leq \frac{1 + \sqrt{1 - 4\lambda\gamma}}{1 + \sqrt{1 - 4\lambda\gamma} - 4\lambda\gamma} \|Q_1 - Q_2\|_{\mathcal{H}} \tag{3.8}$$

hold true.

Proof. For $Q \in C_{-\gamma}$, we begin by constructing a solution P for (3.6). We demonstrate the uniqueness of such an element P at the end of the proof. Since $Q \in \mathcal{H}$ is self-adjoint and there exists an orthonormal system $(e_n)_{n \in \mathbb{N}}$ of eigenvectors and a sequence $(\nu_n)_{n \in \mathbb{N}}$ of real eigenvalues such that

$$Q = \sum_{n=1}^{\infty} \nu_n (e_n, \cdot)_H e_n,$$

where $\nu_n \geq -\gamma$. Then the self-adjoint operator P given by

$$P = \sum_{n=1}^{\infty} \alpha_n (e_n, \cdot)_H e_n \quad \text{with} \quad \alpha_n = \frac{1}{2\lambda} \left(-1 + \sqrt{1 + 4\lambda\nu_n} \right) \quad (3.9)$$

fulfills (3.6), as can be seen by simply inserting P into (3.6). Since $\inf_{n \in \mathbb{N}} \nu_n \geq -\gamma$ and $\lambda < \frac{1}{4\gamma}$ imply that $1 + 4\lambda\nu_n > 0$ for every $n \in \mathbb{N}$, the numbers α_n , $n \in \mathbb{N}$, are indeed real. We obtain that the eigenvalues of P satisfy

$$\begin{aligned} \alpha_n &= \frac{1}{2\lambda} \left(-1 + \sqrt{1 + 4\lambda\nu_n} \right) \\ &= \frac{(-1 + \sqrt{1 + 4\lambda\nu_n})(1 + \sqrt{1 + 4\lambda\nu_n})}{2\lambda(1 + \sqrt{1 + 4\lambda\nu_n})} \\ &= \frac{2\nu_n}{1 + \sqrt{1 + 4\lambda\nu_n}} \\ &\geq \frac{-2\gamma}{1 + \sqrt{1 + 4\lambda\nu_n}} \geq \frac{-2\gamma}{1 + \sqrt{1 - 4\lambda\gamma}}. \end{aligned}$$

Thus, $P \in C_{\frac{-2\gamma}{1 + \sqrt{1 - 4\lambda\gamma}}}$ is fulfilled if

$$\|P\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \alpha_n^2 = \sum_{n=1}^{\infty} \left(\frac{2\nu_n}{1 + \sqrt{1 + 4\lambda\nu_n}} \right)^2 \leq \left(\frac{2}{1 + \sqrt{1 - 4\lambda\gamma}} \right)^2 \|Q\|_{\mathcal{H}}^2 < \infty, \quad (3.10)$$

In the following, we consider $Q_1, Q_2 \in C_{-\gamma}$ and choose $P_1, P_2 \in C_{\frac{-2\gamma}{1 + \sqrt{1 - 4\lambda\gamma}}}$ such that

$$(\lambda\mathcal{B} + I)P_1 = Q_1 \quad \text{and} \quad (\lambda\mathcal{B} + I)P_2 = Q_2$$

and obtain that

$$\begin{aligned} \|Q_1 - Q_2\|_{\mathcal{H}} \|P_1 - P_2\|_{\mathcal{H}} &= \|(\lambda\mathcal{B} + I)P_1 - (\lambda\mathcal{B} + I)P_2\|_{\mathcal{H}} \|P_1 - P_2\|_{\mathcal{H}} \\ &\geq \|(\lambda\mathcal{B} + I)P_1 - (\lambda\mathcal{B} + I)P_2, P_1 - P_2\| \\ &= \lambda (P_1^2 - P_2^2, P_1 - P_2) + (P_1 - P_2, P_1 - P_2). \end{aligned} \quad (3.11)$$

Since $P_1, P_2 \geq \frac{-2\gamma}{1 + \sqrt{1 - 4\lambda\gamma}}$, the first summand on the right-hand side of (3.11) can be estimated by

$$(P_1^2 - P_2^2, P_1 - P_2) = (P_1(P_1 - P_2), P_1 - P_2) + ((P_1 - P_2)P_2, P_1 - P_2)$$

$$\geq \frac{-4\gamma}{1 + \sqrt{1 - 4\lambda\gamma}} \|P_1 - P_2\|_{\mathcal{H}}^2.$$

Altogether, this implies

$$\|Q_1 - Q_2\|_{\mathcal{H}} \|P_1 - P_2\|_{\mathcal{H}} \geq \frac{-4\lambda\gamma}{1 + \sqrt{1 - 4\lambda\gamma}} \|P_1 - P_2\|_{\mathcal{H}}^2 + \|P_1 - P_2\|_{\mathcal{H}}^2,$$

as well as

$$\|Q_1 - Q_2\|_{\mathcal{H}} \geq \frac{1 + \sqrt{1 - 4\lambda\gamma} - 4\lambda\gamma}{1 + \sqrt{1 - 4\lambda\gamma}} \|P_1 - P_2\|_{\mathcal{H}}. \quad (3.12)$$

This inequality in mind, it now follows directly that the operator P defined in (3.9) is the unique solution to $(\lambda\mathcal{B} + I)P = Q$ within the set $C_{\frac{-2\gamma}{1 + \sqrt{1 - 4\lambda\gamma}}}$, since

$$\frac{1 + \sqrt{1 - 4\lambda\gamma} - 4\lambda\gamma}{1 + \sqrt{1 - 4\lambda\gamma}} > 0.$$

The estimates (3.7) and (3.8) are given by (3.10) and (3.12), respectively. \square

The previous lemmas in mind, we can now prove the main result of this section.

Theorem 3.8. *Let Assumptions 1. and 2. be satisfied and let γ with $0 \leq \gamma < \mu_H$ be given. Then for $Q \in C_{-\gamma^2}$ there exists $P \in \mathcal{V} \cap C_{-\gamma}$ such that*

$$A^*P + PA + P^2 = Q.$$

Proof. The proof is a generalization of the proof in [3, Chapter II.3, Lemma 3.2 and Theorem 3.9].

define the operator $\mathcal{G} : C_{-\gamma} \rightarrow \mathcal{H}$ for $\lambda \in (0, \frac{1}{4\gamma})$ if $\gamma > 0$, and $\lambda \in (0, \infty)$ if $\gamma = 0$, through

$$\mathcal{G}(P) = \lambda(\lambda\mathfrak{A}_0 + I)^{-1}Q + (\lambda\mathfrak{A}_0 + I)^{-1}(\lambda\mathcal{B} + I)^{-1}P.$$

In the following, we use the Banach fixed-point theorem to prove the existence of a fixed-point. We begin by proving that \mathcal{G} maps the closed subset $C_{-\gamma}$ of \mathcal{H} into itself. For $Q \in C_{-\gamma^2}$ and $P \in C_{-\gamma}$, from Lemma 3.4 and Lemma 3.7 we obtain that

$$\begin{aligned} \lambda(\lambda\mathfrak{A}_0 + I)^{-1}Q &\geq \frac{-\lambda\gamma^2}{1 + 2\lambda\mu_H}, \\ (\lambda\mathcal{B} + I)^{-1}P &\geq \frac{-2\gamma}{1 + \sqrt{1 - 4\lambda\gamma}}, \end{aligned}$$

and

$$(\lambda\mathfrak{A}_0 + I)^{-1}(\lambda\mathcal{B} + I)^{-1}P \geq \frac{1}{1 + 2\lambda\mu_H} \cdot \frac{-2\gamma}{1 + \sqrt{1 - 4\lambda\gamma}}.$$

If $\lambda > 0$ is sufficiently small, then

$$\gamma \geq \frac{\lambda\gamma^2}{1 + 2\lambda\mu_H} + \frac{2\gamma}{(1 + 2\lambda\mu_H)(1 + \sqrt{1 - 4\lambda\gamma})}.$$

This yields $\mathcal{G}(P) \geq -\gamma$ and thus shows that \mathcal{G} maps $C_{-\gamma}$ into itself.

The next step is to prove that \mathcal{G} is a contraction on $C_{-\gamma}$. Using Lemma 3.4 and Lemma 3.7, for $P_1, P_2 \in C_{-\gamma}$ it follows that

$$\begin{aligned} & \|\mathcal{G}(P_1) - \mathcal{G}(P_2)\|_{\mathcal{H}} \\ &= \|(\lambda\mathfrak{A}_0 + I)^{-1}(\lambda\mathcal{B} + I)^{-1}P_1 - (\lambda\mathfrak{A}_0 + I)^{-1}(\lambda\mathcal{B} + I)^{-1}P_2\|_{\mathcal{H}} \\ &\leq \frac{1}{1 + 2\lambda\mu_H} \cdot \frac{1 + \sqrt{1 - 4\lambda\gamma}}{1 + \sqrt{1 - 4\lambda\gamma} - 4\lambda\gamma} \|P_1 - P_2\|_{\mathcal{H}}, \end{aligned}$$

where

$$\begin{aligned} & \frac{1}{1 + 2\lambda\mu_H} \cdot \frac{1 + \sqrt{1 - 4\lambda\gamma}}{1 + \sqrt{1 - 4\lambda\gamma} - 4\lambda\gamma} \\ &= \frac{1 + \sqrt{1 - 4\lambda\gamma}}{1 + \sqrt{1 - 4\lambda\gamma} + 2\lambda(\mu_H + \mu_H\sqrt{1 - 4\lambda\gamma} - 2\gamma - 4\lambda\mu_H\gamma)} < 1, \end{aligned}$$

since

$$\mu_H + \mu_H\sqrt{1 - 4\lambda\gamma} - 2\gamma - 4\lambda\mu_H\gamma > 0$$

for $\lambda > 0$ small enough. So if $\lambda > 0$ is sufficiently small then Banach's fixed-point theorem yields the existence of a unique $P_\lambda \in C_{-\gamma} \cap \text{dom}(\mathfrak{A}_0)$ such that $\mathcal{G}(P_\lambda) = P_\lambda$. It remains to prove that $\{P_\lambda\}_{\lambda>0}$ converges strongly in \mathcal{H} as $\lambda \rightarrow 0$ to $P \in C_{-\gamma} \cap \mathcal{V}$ and that the limit P fulfills

$$A^*P + PA + P^2 = Q.$$

Applying $\frac{1}{\lambda}(\lambda\mathfrak{A}_0 + I)$ to both sides of the equation $P_\lambda = \mathcal{G}(P_\lambda)$ shows that

$$\frac{1}{\lambda}(\lambda\mathfrak{A}_0 + I)P_\lambda = Q + \frac{1}{\lambda}(\lambda\mathcal{B} + I)^{-1}P_\lambda,$$

which can be rearranged as

$$\mathfrak{A}_0P_\lambda + \frac{1}{\lambda}\left(I - (\lambda\mathcal{B} + I)^{-1}\right)P_\lambda = Q. \quad (3.13)$$

The nonlinearity in this equation is the Yosida approximation for the quadratic term of the Riccati equation. To abbreviate this term, we write in the following

$$J_\lambda = (\lambda\mathcal{B} + I)^{-1} \quad \text{and} \quad \mathcal{B}_\lambda = \frac{1}{\lambda}(I - J_\lambda) = \mathcal{B}J_\lambda.$$

Testing (3.13) with P_λ , we obtain

$$(\mathfrak{A}_0 P_\lambda, P_\lambda) + (\mathcal{B}_\lambda P_\lambda, P_\lambda) = (Q, P_\lambda). \quad (3.14)$$

For the first summand on the left-hand side, the estimate

$$(\mathfrak{A}_0 P_\lambda, P_\lambda) \geq 2\mu_H \|P_\lambda\|_{\mathcal{H}}^2$$

is fulfilled. Using Lemma 3.7 for the second summand, it follows that

$$\begin{aligned} (\mathcal{B}_\lambda P_\lambda, P_\lambda) &= \frac{1}{\lambda} \left((I - (\lambda\mathcal{B} + I)^{-1}) P_\lambda, P_\lambda \right) \\ &\geq \frac{1}{\lambda} \|P_\lambda\|_{\mathcal{H}}^2 - \frac{1}{\lambda} \|(\lambda\mathcal{B} + I)^{-1} P_\lambda\|_{\mathcal{H}} \|P_\lambda\|_{\mathcal{H}} \\ &\geq \frac{1}{\lambda} \left(1 - \frac{2}{1 + \sqrt{1 - 4\lambda\gamma}} \right) \|P_\lambda\|_{\mathcal{H}}^2. \end{aligned}$$

Inserting these estimates into (3.14) yields

$$\left(2\mu_H + \frac{1}{\lambda} \left(1 - \frac{2}{1 + \sqrt{1 - 4\lambda\gamma}} \right) \right) \|P_\lambda\|_{\mathcal{H}} \leq \|Q\|_{\mathcal{H}}.$$

Applying L'Hôpital's rule shows that

$$\frac{1}{\lambda} \left(1 - \frac{2}{1 + \sqrt{1 - 4\lambda\gamma}} \right) \rightarrow -\gamma$$

as $\lambda \rightarrow 0$. Hence, for $c_1 \in (0, 2\mu_H - \gamma)$ there exists $\varepsilon > 0$ such that

$$2\mu_H + \frac{1}{\lambda} \left(1 - \frac{2}{1 + \sqrt{1 - 4\lambda\gamma}} \right) > c_1$$

is fulfilled if $\lambda \in (0, \varepsilon)$. Thus,

$$c_1 \|P_\lambda\|_{\mathcal{H}} \leq \|Q\|_{\mathcal{H}} \quad (3.15)$$

holds for $\lambda > 0$ small enough. This proves that $\{P_\lambda\}_{\lambda>0}$ is bounded in \mathcal{H} for sufficiently small $\lambda > 0$. It remains to show that $\{P_\lambda\}_{\lambda>0}$ converges in \mathcal{H} as $\lambda \rightarrow 0$. Using (3.15) and Lemma 3.7, we obtain

$$\begin{aligned} \|J_\lambda P_\lambda - P_\lambda\|_{\mathcal{H}} &= \lambda \left\| \frac{1}{\lambda} (I - J_\lambda) P_\lambda \right\|_{\mathcal{H}} = \lambda \| \mathcal{B} (\lambda\mathcal{B} + I)^{-1} P_\lambda \|_{\mathcal{H}} \leq \lambda \| (\lambda\mathcal{B} + I)^{-1} P_\lambda \|_{\mathcal{H}}^2 \\ &\leq \frac{4\lambda}{(1 + \sqrt{1 - 4\lambda\gamma})^2} \|P_\lambda\|_{\mathcal{H}}^2 \leq \frac{4\lambda}{(1 + \sqrt{1 - 4\lambda\gamma})^2} \frac{\|Q\|_{\mathcal{H}}^2}{c_1^2} \leq c\lambda \|Q\|_{\mathcal{H}}^2 \end{aligned} \quad (3.16)$$

for a certain constant $c > 0$ if $\lambda > 0$ is sufficiently small. Now we show the Cauchy property of $\{P_\lambda\}_{\lambda>0}$. As $P_\lambda, P_\nu \geq -\gamma$ for sufficiently small $\lambda, \nu > 0$, using Lemma 3.7, it follows that

$$J_\lambda P_\lambda = (\lambda\mathcal{B} + I)^{-1} P_\lambda \geq \frac{-2\gamma}{1 + \sqrt{1 - 4\lambda\gamma}} \quad \text{and} \quad J_\nu P_\nu = (\nu\mathcal{B} + I)^{-1} P_\nu \geq \frac{-2\gamma}{1 + \sqrt{1 - 4\nu\gamma}}.$$

Exploiting these lower bounds and assuming without loss of generality that $\nu < \lambda$, we obtain that

$$\begin{aligned} & (\mathcal{B}_\lambda P_\lambda - \mathcal{B}_\nu P_\nu, J_\lambda P_\lambda - J_\nu P_\nu) \\ &= (\mathcal{B}J_\lambda P_\lambda - \mathcal{B}J_\nu P_\nu, J_\lambda P_\lambda - J_\nu P_\nu) \\ &\geq \frac{-4\gamma}{1 + \sqrt{1 - 4\lambda\gamma}} \|J_\lambda P_\lambda - J_\nu P_\nu\|_{\mathcal{H}}^2. \end{aligned}$$

Subtracting the equations (3.13) for P_λ and P_ν and testing with $P_\lambda - P_\nu$, it follows that

$$2\mu_H \|P_\lambda - P_\nu\|_{\mathcal{H}}^2 \leq -(\mathcal{B}_\lambda P_\lambda - \mathcal{B}_\nu P_\nu, P_\lambda - P_\nu).$$

Altogether this yields

$$\begin{aligned} & (\mathcal{B}_\lambda P_\lambda - \mathcal{B}_\nu P_\nu, (J_\lambda P_\lambda - P_\lambda) - (J_\nu P_\nu - P_\nu)) \\ &= (\mathcal{B}_\lambda P_\lambda - \mathcal{B}_\nu P_\nu, J_\lambda P_\lambda - J_\nu P_\nu) - (\mathcal{B}_\lambda P_\lambda - \mathcal{B}_\nu P_\nu, P_\lambda - P_\nu) \\ &\geq 2\mu_H \|P_\lambda - P_\nu\|_{\mathcal{H}}^2 - \frac{4\gamma}{1 + \sqrt{1 - 4\lambda\gamma}} \|J_\lambda P_\lambda - J_\nu P_\nu\|_{\mathcal{H}}^2 \\ &\geq 2\mu_H \|P_\lambda - P_\nu\|_{\mathcal{H}}^2 - \frac{4\gamma}{1 + \sqrt{1 - 4\lambda\gamma}} (\|J_\lambda P_\lambda - P_\lambda\|_{\mathcal{H}} + \|P_\lambda - P_\nu\|_{\mathcal{H}} + \|P_\nu - J_\nu P_\nu\|_{\mathcal{H}})^2 \\ &= \left(2\mu_H - \frac{4\gamma}{1 + \sqrt{1 - 4\lambda\gamma}}\right) \|P_\lambda - P_\nu\|_{\mathcal{H}}^2 \\ &\quad - \frac{4\gamma}{1 + \sqrt{1 - 4\lambda\gamma}} (\|J_\lambda P_\lambda - P_\lambda\|_{\mathcal{H}}^2 + \|P_\nu - J_\nu P_\nu\|_{\mathcal{H}}^2 + \|J_\lambda P_\lambda - P_\lambda\|_{\mathcal{H}} \|P_\lambda - P_\nu\|_{\mathcal{H}} \\ &\quad + \|P_\lambda - P_\nu\|_{\mathcal{H}} \|P_\nu - J_\nu P_\nu\|_{\mathcal{H}} + \|J_\lambda P_\lambda - P_\lambda\|_{\mathcal{H}} \|P_\nu - J_\nu P_\nu\|_{\mathcal{H}}). \end{aligned}$$

On the other hand, we have the upper bound

$$\begin{aligned} & (\mathcal{B}_\lambda P_\lambda - \mathcal{B}_\nu P_\nu, (J_\lambda P_\lambda - P_\lambda) - (J_\nu P_\nu - P_\nu)) \\ &\leq \|\mathcal{B}_\lambda P_\lambda - \mathcal{B}_\nu P_\nu\|_{\mathcal{H}} (\|J_\lambda P_\lambda - P_\lambda\|_{\mathcal{H}} + \|J_\nu P_\nu - P_\nu\|_{\mathcal{H}}). \end{aligned}$$

Both together imply that

$$\begin{aligned} & \left(2\mu_H - \frac{4\gamma}{1 + \sqrt{1 - 4\lambda\gamma}}\right) \|P_\lambda - P_\nu\|_{\mathcal{H}}^2 \\ &\leq \|\mathcal{B}_\lambda P_\lambda - \mathcal{B}_\nu P_\nu\|_{\mathcal{H}} (\|J_\lambda P_\lambda - P_\lambda\|_{\mathcal{H}} + \|J_\nu P_\nu - P_\nu\|_{\mathcal{H}}) \\ &\quad + \frac{4\gamma}{1 + \sqrt{1 - 4\lambda\gamma}} (\|J_\lambda P_\lambda - P_\lambda\|_{\mathcal{H}}^2 + \|P_\nu - J_\nu P_\nu\|_{\mathcal{H}}^2 + \|J_\lambda P_\lambda - P_\lambda\|_{\mathcal{H}} \|P_\lambda - P_\nu\|_{\mathcal{H}} \\ &\quad + \|P_\lambda - P_\nu\|_{\mathcal{H}} \|P_\nu - J_\nu P_\nu\|_{\mathcal{H}} + \|J_\lambda P_\lambda - P_\lambda\|_{\mathcal{H}} \|P_\nu - J_\nu P_\nu\|_{\mathcal{H}}). \end{aligned}$$

Since $\gamma < \mu_H$, for $c_2 \in (2\gamma, 2\mu_H)$, we have

$$2\mu_H - \frac{4\gamma}{1 + \sqrt{1 - 4\lambda\gamma}} > c_2$$

if $\lambda > 0$ sufficiently small. Employing (3.15) and (3.16), this proves

$$c_2 \|P_\lambda - P_\nu\|_{\mathcal{H}}^2 \rightarrow 0 \quad \text{as } \lambda, \nu \rightarrow 0.$$

Thus, there exists $P \in \mathcal{H}$ such that $P_\lambda \rightarrow P$ in \mathcal{H} as $\lambda \rightarrow 0$. Since $C_{-\gamma}$ is closed, we find that $P \in C_{-\gamma}$.

It remains to prove that $A^*P + PA + P^2 = Q$ is satisfied and $P \in \mathcal{V}$. Because of (3.16), it follows that

$$J_\lambda P_\lambda \rightarrow P \quad \text{in } \mathcal{H} \text{ as } \lambda \rightarrow 0.$$

Since \mathcal{B} is continuous, we obtain that

$$\mathcal{B}_\lambda(P_\lambda) = \mathcal{B}J_\lambda(P_\lambda) \rightarrow \mathcal{B}P = P^2 \quad \text{in } \mathcal{H} \text{ as } \lambda \rightarrow 0.$$

For the linear part, it can be concluded that

$$\mathfrak{A}_0 P_\lambda = Q - \mathcal{B}_\lambda(P_\lambda) \rightarrow Q - P^2 \quad \text{in } \mathcal{H} \text{ as } \lambda \rightarrow 0.$$

Since

$$\mu_V \|P_\lambda\|_{\mathcal{V}}^2 \leq (\mathfrak{A}_0 P_\lambda, P_\lambda) \leq \|\mathfrak{A}_0 P_\lambda\|_{\mathcal{H}} \|P_\lambda\|_{\mathcal{H}},$$

and since $\{\mathfrak{A}_0 P_\lambda\}_{\lambda>0}$ and $\{P_\lambda\}_{\lambda>0}$ are convergent and thus bounded in \mathcal{H} , we see that $\{P_\lambda\}_{\lambda>0}$ is also bounded in \mathcal{V} . Therefore, there exists a weakly convergent subsequence in \mathcal{V} . The uniqueness of the limit implies that this limit has to be P and that P is an element of \mathcal{V} . Since $\mathfrak{A}_0 : \mathcal{V} \rightarrow \mathcal{V}^*$ is a linear and bounded operator, it is weakly-weakly continuous, see [8, Theorem 3.10]. Therefore, $\{\mathfrak{A}_0 P_\lambda\}_{\lambda>0}$ converges weakly to $\mathfrak{A}_0 P$ in \mathcal{V}^* and the convergence of $\{\mathfrak{A}_0 P_\lambda\}_{\lambda>0}$ in \mathcal{H} implies that

$$\mathfrak{A}_0 P_\lambda \rightarrow \mathfrak{A}_0 P \quad \text{in } \mathcal{V}^* \text{ as } \lambda \rightarrow 0$$

which yields that $\mathfrak{A}_0 P + \mathcal{B}P = Q$. □

4 Weak solution of the Riccati equation

In this section, we consider the Hilbert spaces V and H as stated in Assumption 1. Further, we introduce the following mappings.

Assumption 3. Let $a : [0, T] \times V \times V \rightarrow \mathbb{R}$ be given such that for every $t \in [0, T]$ the form $a(t; \cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is bilinear. There exist constants $\mu, \eta > 0$ such that for every $u, v \in V$ and every $t \in [0, T]$

$$\begin{aligned} a(t; u, u) &\geq \mu \|u\|_V^2, & \text{(i.e. } a \text{ is uniformly strongly positive),} \\ |a(t; u, v)| &\leq \eta \|u\|_V \|v\|_V, & \text{(i.e. } a \text{ is uniformly bounded).} \end{aligned}$$

Further, let $a(\cdot; u, v) : [0, T] \rightarrow \mathbb{R}$ be Lebesgue-measurable for fixed $u, v \in V$.

This assumption in mind, for every $t \in [0, T]$ we introduce the operators $\mathcal{A}(t), \mathcal{A}^*(t) : V \rightarrow V^*$ given by

$$\langle \mathcal{A}(t)u, v \rangle_{V^* \times V} = a(t; u, v), \quad \text{and} \quad \langle \mathcal{A}^*(t)u, v \rangle_{V^* \times V} = a(t; v, u),$$

with $u, v \in V$. Note that $\mathcal{A}^*(t)$ is the dual operator of $\mathcal{A}(t)$ for every $t \in [0, T]$.

We consider the initial value problem (1.1) for the Riccati differential equation and we use the backward Euler scheme to obtain a time discretization. The resulting semi-discrete problem can be solved using the existence result for algebraic Riccati equations from Section 3. This approach will lead to the following result, which we then prove in detail.

Theorem 4.1. *Let Assumptions 1. and 3. be satisfied. For γ with , $\mathcal{Q} \in L^1(0, T; \mathcal{H}) + L^2(0, T; \mathcal{V}^*)$ with $\mathcal{Q}(t) = \mathcal{Q}^*(t) \in \mathcal{H}$ and $\mathcal{Q}(t) \geq -\gamma^2$ for almost every $t \in [0, T]$ as well as $\mathcal{P}_0 \in \mathcal{H}$ with $\mathcal{P}_0 = \mathcal{P}_0^*$ and $\mathcal{P}_0 \geq -\gamma$, there exists a weak solution $\mathcal{P} \in \mathcal{W}_1(0, T)$ to the initial value problem (1.1)*

Again, let us remark that we construct the maximal solution.

4.1 Time discrete problem

For the time discretization, we consider the equidistant partition $0 = t_0 < \dots < t_N = T$ with $\tau = \frac{T}{N}$ and $t_n = n\tau$ ($n = 0, 1, \dots, N$). We always assume that τ is sufficiently small such that $\tau < \frac{\mu}{2C_{V,H}^2}$. We consider the semi-discrete problem

$$\frac{P_n - P_{n-1}}{\tau} + \mathfrak{A}_n P_n = Q_n, \quad n = 1, 2, \dots, N, \quad (4.1)$$

with $P_0 = \mathcal{P}_0$, where $P_n \in \mathcal{V}$ ($n = 1, 2, \dots, N$) denotes an approximation of $\mathcal{P}(t_n)$. Here the right-hand side is given by $Q_n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \mathcal{Q}(t) dt \in \mathcal{H}$. Moreover, we define $\mathfrak{A}_n P_n = A_n^* P_n + P_n A_n + P_n^2 \in \mathcal{V}^*$ with

$$A_n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \mathcal{A}(t) dt$$

for $n = 1, 2, \dots, N$. In order to show the existence of a solution $(P_n)_{n=1}^N$ to the semi-discrete problem (4.1), we make use of Theorem 3.8. We first consider a somewhat more regular right-hand side $\mathcal{Q} \in L^2(0, T; \mathcal{H})$.

Lemma 4.2. *Let Assumptions 1. and 3. be satisfied and let $\tau < \frac{\mu}{2C_{V,H}^2}$. For γ with $0 \leq \gamma < \frac{\mu}{C_{V,H}^2}$, $\mathcal{Q} \in L^2(0, T; \mathcal{H})$ with $\mathcal{Q}(t) = \mathcal{Q}^*(t)$, $\mathcal{Q}(t) \geq -\gamma^2$ for almost every $t \in [0, T]$, $\mathcal{P}_0 \in \mathcal{H}$ with $\mathcal{P}_0 = \mathcal{P}_0^*$, and $\mathcal{P}_0 \geq -\gamma$, the semi-discrete problem (4.1) admits a solution $(P_n)_{n=1}^N$ such that $P_n \in \mathcal{V}$ with $P_n \geq -\gamma$, $n = 1, 2, \dots, N$, that fulfills the a priori estimate*

$$\begin{aligned}
& \|P_n\|_{\mathcal{H}}^2 + \sum_{k=1}^n \|P_k - P_{k-1}\|_{\mathcal{H}}^2 + \left(\mu - \frac{\gamma C_{V,H}^2}{2} \right) \tau \sum_{k=1}^n \|P_k\|_{\mathcal{V}}^2 \\
& \leq \|P_0\|_{\mathcal{H}}^2 + \frac{1}{\mu - \frac{\gamma C_{V,H}^2}{2}} \|\mathcal{Q}\|_{L^2(0,T;\mathcal{V}^*)}^2, \quad n = 1, 2, \dots, N.
\end{aligned} \tag{4.2}$$

Proof. Considering the semi-discrete problem (4.1) for each step $n = 1, 2, \dots, N$, we obtain an algebraic Riccati equation of the form

$$\left(A_n + \frac{1}{2\tau} I \right)^* P_n + P_n \left(A_n + \frac{1}{2\tau} I \right) + P_n^2 = Q_n + \frac{1}{\tau} P_{n-1} \tag{4.3}$$

with

$$\left\langle \left(A_n + \frac{1}{2\tau} I \right) P, P \right\rangle \geq \mu \|P\|_{\mathcal{V}}^2 + \frac{1}{2\tau} \|P\|_{\mathcal{H}}^2 \geq \left(\frac{2\mu}{C_{V,H}^2} + \frac{1}{2\tau} \right) \|P\|_{\mathcal{H}}^2$$

for $P \in \mathcal{V}$. We can then apply Theorem 3.8 (choosing γ in Theorem 3.8 appropriately with $\mu_H := \frac{2\mu}{C_{V,H}^2} + \frac{1}{2\tau}$) if

$$Q_n + \frac{1}{\tau} P_{n-1} \geq - \left(\gamma + \frac{1}{2\tau} \right)^2 > - \left(\frac{2\mu}{C_{V,H}^2} + \frac{1}{2\tau} \right)^2 = -\mu_H^2$$

is fulfilled for every $n = 1, 2, \dots, N$. To prove this condition, we argue inductively. Since $P_0 \geq -\gamma$ and $Q_1 \geq -\gamma^2$, we obtain that

$$Q_1 + \frac{1}{\tau} P_0 \geq -\gamma^2 - \frac{\gamma}{\tau} > - \left(\left(\frac{2\mu}{C_{V,H}^2} \right)^2 + \frac{1}{\tau} \frac{2\mu}{C_{V,H}^2} + \frac{1}{4\tau^2} \right) = - \left(\frac{2\mu}{C_{V,H}^2} + \frac{1}{2\tau} \right)^2.$$

The existence of $P_1 \in \mathcal{V}$ then follows using Theorem 3.8. It remains to prove that $P_{n-1} \geq -\gamma$ implies $P_n \geq -\gamma$ for arbitrary $n = 1, 2, \dots, N$. Since the existence of a compact and self-adjoint operator P_n already follows from using $P_{n-1} \geq -\gamma$, there exists an orthonormal system $(e_i)_{i \in \mathbb{N}}$ of eigenvectors in H and real eigenvalues $(\alpha_i)_{i \in \mathbb{N}}$ in \mathbb{R} such that

$$P_n = \sum_{i=1}^{\infty} \alpha_i (e_i, \cdot)_H e_i.$$

Note that the eigenvectors and eigenvalues depend on n but to keep the notation simple, we will not state this dependence. Testing (4.3) with e_i , $i \in \mathbb{N}$, it follows that

$$(P_n^2 e_i, e_i)_H + 2 \left(\left(A_n + \frac{1}{2\tau} I \right)^* P_n e_i, e_i \right)_H - (Q_n e_i, e_i)_H - \frac{1}{\tau} (P_{n-1} e_i, e_i)_H = 0.$$

Abbreviating the terms $(Q_n e_i, e_i)_H = q_i$, $(P_{n-1} e_i, e_i)_H = p_i$, and $\langle A_n e_i, e_i \rangle_{V^* \times V} = \mathbf{a}_i$, the equation can be simplified to

$$\alpha_i^2 + 2\alpha_i \left(\mathbf{a}_i + \frac{1}{2\tau} \right) - q_i - \frac{p_i}{\tau} = 0.$$

Note that the discriminant is larger than $\frac{1}{4\tau^2}$ so that the roots are real. Since we know from Theorem 3.8 (choosing γ in Theorem 3.8 appropriately) that $\alpha_i > -\left(\frac{2\mu}{C_{V,H}^2} + \frac{1}{2\tau}\right)$, we only have to consider the larger of the two solutions of this quadratic equation in \mathbb{R} given by

$$-\left(\mathbf{a}_i + \frac{1}{2\tau}\right) + \sqrt{\left(\mathbf{a}_i + \frac{1}{2\tau}\right)^2 + q_i + \frac{p_i}{\tau}} = \frac{q_i + \frac{1}{\tau}p_i}{\mathbf{a}_i + \frac{1}{2\tau} + \sqrt{\left(\mathbf{a}_i + \frac{1}{2\tau}\right)^2 + q_i + \frac{p_i}{\tau}}}$$

if τ is sufficiently small such that $\tau < \frac{C_{V,H}^2}{2\mu}$.

Due to Assumption 3, the values \mathbf{a}_i fulfill the estimate $\mathbf{a}_i \geq \mu \|e_i\|_V^2 \geq \frac{\mu}{C_{V,H}^2}$ for every $i \in \mathbb{N}$. Since we have assumed that $P_{n-1} \geq -\gamma$ and $Q_{n-1} \geq -\gamma^2$, it follows that $p_i \geq -\gamma$ and $q_i \geq -\gamma^2$. Therefore, we find

$$\begin{aligned} \alpha_i &= \frac{q_i + \frac{1}{\tau}p_i}{\mathbf{a}_i + \frac{1}{2\tau} + \sqrt{\left(\mathbf{a}_i + \frac{1}{2\tau}\right)^2 + q_i + \frac{1}{\tau}p_i}} \\ &\geq \frac{-\gamma^2 - \frac{1}{\tau}\gamma}{\mathbf{a}_i + \frac{1}{2\tau} + \sqrt{\mathbf{a}_i^2 + \frac{1}{\tau}\mathbf{a}_i + \frac{1}{4\tau^2} + q_i + \frac{1}{\tau}p_i}} \\ &\geq \frac{-\gamma\left(\gamma + \frac{1}{\tau}\right)}{\mathbf{a}_i + \frac{1}{2\tau} + \sqrt{\frac{1}{4\tau^2}}} \geq \frac{-\gamma\left(\frac{\mu}{C_{V,H}^2} + \frac{1}{\tau}\right)}{\frac{\mu}{C_{V,H}^2} + \frac{1}{\tau}} = -\gamma, \end{aligned}$$

where we have employed that

$$\mathbf{a}_i + p_i > \gamma - \gamma \geq 0 \quad \text{and} \quad \mathbf{a}_i^2 + q_i > \gamma^2 - \gamma^2 \geq 0.$$

This proves that $P_n \geq -\gamma$ and thus the existence of all P_n for $n = 1, 2, \dots, N$.

Let us now derive an a priori bound. Testing (4.1) with P_n , and using both

$$(P_n - P_{n-1}, P_n) = \frac{1}{2}\|P_n\|_{\mathcal{H}}^2 - \frac{1}{2}\|P_{n-1}\|_{\mathcal{H}}^2 + \frac{1}{2}\|P_n - P_{n-1}\|_{\mathcal{H}}^2$$

and Young's inequality, we obtain that

$$\begin{aligned} &\frac{1}{2\tau} \left(\|P_n\|_{\mathcal{H}}^2 - \|P_{n-1}\|_{\mathcal{H}}^2 + \|P_n - P_{n-1}\|_{\mathcal{H}}^2 \right) + \mu \|P_n\|_V^2 + (P_n^2, P_n) \\ &\leq \frac{1}{2\left(\mu - \frac{\gamma C_{V,H}^2}{2}\right)} \|Q_n\|_{V^*}^2 + \frac{\mu - \frac{\gamma C_{V,H}^2}{2}}{2} \|P_n\|_V^2. \end{aligned} \tag{4.4}$$

Using Lemma 3.5, we can estimate the nonlinearity by

$$(P_n^2, P_n) \geq -\gamma \|P_n\|_{\mathcal{H}}^2 \geq -\frac{\gamma C_{V,H}^2}{2} \|P_n\|_{\mathcal{V}}^2.$$

Inserting this estimate into (4.4), it follows that

$$\begin{aligned} & \frac{1}{2\tau} (\|P_n\|_{\mathcal{H}}^2 - \|P_{n-1}\|_{\mathcal{H}}^2 + \|P_n - P_{n-1}\|_{\mathcal{H}}^2) + \left(\mu - \frac{\gamma C_{V,H}^2}{2} \right) \|P_n\|_{\mathcal{V}}^2 \\ & \leq \frac{1}{2 \left(\mu - \frac{\gamma C_{V,H}^2}{2} \right)} \|Q_n\|_{\mathcal{V}^*}^2 + \frac{\mu - \frac{\gamma C_{V,H}^2}{2}}{2} \|P_n\|_{\mathcal{V}}^2 \end{aligned}$$

and, therefore,

$$\|P_n\|_{\mathcal{H}}^2 - \|P_{n-1}\|_{\mathcal{H}}^2 + \|P_n - P_{n-1}\|_{\mathcal{H}}^2 + \left(\mu - \frac{\gamma C_{V,H}^2}{2} \right) \tau \|P_n\|_{\mathcal{V}}^2 \leq \frac{\tau}{\mu - \frac{\gamma C_{V,H}^2}{2}} \|Q_n\|_{\mathcal{V}^*}^2.$$

Summing up leads to the estimate

$$\begin{aligned} & \|P_n\|_{\mathcal{H}}^2 + \sum_{k=1}^n \|P_k - P_{k-1}\|_{\mathcal{H}}^2 + \left(\mu - \frac{\gamma C_{V,H}^2}{2} \right) \tau \sum_{k=1}^n \|P_k\|_{\mathcal{V}}^2 \\ & \leq \|P_0\|_{\mathcal{H}}^2 + \frac{\tau}{\mu - \frac{\gamma C_{V,H}^2}{2}} \sum_{k=1}^N \|Q_k\|_{\mathcal{V}^*}^2, \quad n = 1, 2, \dots, N. \end{aligned} \tag{4.5}$$

Since

$$\tau \sum_{i=1}^N \|Q_n\|_{\mathcal{V}^*}^2 \leq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|\mathcal{Q}(t)\|_{\mathcal{V}^*}^2 dt = \|\mathcal{Q}\|_{L^2(0,T;\mathcal{V}^*)}^2, \tag{4.6}$$

the right-hand side of (4.5) can be simplified and we obtain the desired a priori estimate (4.2), recalling that $\mathcal{P}_0 = P_0$. \square

Using the solution $(P_n)_{n=1}^N$ of the semi-discrete problem (4.1), we define both a piecewise constant and a piecewise linear interpolation. For $t \in (t_{n-1}, t_n]$, $n = 1, 2, \dots, N$, define

$$\mathcal{P}_\tau(t) = P_n \quad \text{and} \quad \widehat{\mathcal{P}}_\tau(t) = \frac{P_n - P_{n-1}}{\tau} (t - t_{n-1}) + P_{n-1}, \tag{4.7}$$

with $\widehat{\mathcal{P}}_\tau(0) = \mathcal{P}_\tau(0) = P_0$. Further, we define the piecewise constant interpolations for the discrete right-hand side $(Q_n)_{n=1}^N$ and for $(\mathfrak{A}_n)_{n=1}^N$: For $t \in (t_{n-1}, t_n]$, $n = 1, 2, \dots, N$, we define

$$\mathcal{Q}_\tau(t) = Q_n, \quad \mathcal{A}_\tau(t) = A_n, \quad \mathfrak{A}_\tau(t)P = \mathcal{A}_\tau^*(t)P + P\mathcal{A}_\tau(t) + P^2$$

for $P \in \mathcal{V}$. As the function $\widehat{\mathcal{P}}_\tau : [0, T] \rightarrow \mathcal{H}$ is piecewise linear and continuous, it is weakly differentiable with the derivative $\widehat{\mathcal{P}}'_\tau(t) = \frac{P_n - P_{n-1}}{\tau}$ for $t \in (t_{n-1}, t_n)$, $n = 1, 2, \dots, N$. Using the functions \mathcal{P}_τ and $\widehat{\mathcal{P}}_\tau$, the semi-discrete problem (4.1) reads

$$\begin{aligned} \widehat{\mathcal{P}}'_\tau(t) + \mathfrak{A}_\tau(t)\mathcal{P}_\tau(t) &= \mathcal{Q}_\tau(t) \quad \text{for almost every } t \in (0, T), \\ \mathcal{P}_\tau(0) &= P_0. \end{aligned} \quad (4.8)$$

4.2 Limiting process

The approximate solutions $\widehat{\mathcal{P}}_\tau$ and \mathcal{P}_τ from the last subsection provide a sequence of functions that will be shown to converge to a weak solution of (1.1) as $\tau \rightarrow 0$. In this subsection, we examine each term of (4.8) for the limiting process $\tau \rightarrow 0$.

Lemma 4.3. *Let Assumptions 1. and 3. be satisfied. Further let $0 \leq \gamma < \frac{\mu}{C_{\mathcal{V},H}^2}$, $\mathcal{Q} \in L^2(0, T; \mathcal{H})$ with $\mathcal{Q}(t) = \mathcal{Q}^*(t)$ and $\mathcal{Q}(t) \geq -\gamma^2$ for almost every $t \in [0, T]$ and $\mathcal{P}_0 \in \mathcal{H}$ with $\mathcal{P}_0 = \mathcal{P}_0^*$, and $\mathcal{P}_0 \geq -\gamma$. Let $(N_k)_{k \in \mathbb{N}}$ be a sequence of positive integers such that $N_k \rightarrow \infty$ as $k \rightarrow \infty$ and let $(\tau_k)_{k \in \mathbb{N}}$ be the sequence of step sizes $\tau_k = \frac{T}{N_k}$ such that $\sup_{k \in \mathbb{N}} \tau_k < \frac{\mu}{2C_{\mathcal{V},H}^2}$. Then there exists a subsequence $(N_{k'})_{k' \in \mathbb{N}}$ and $\mathcal{P} \in L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V})$ with $\mathcal{P}' \in L^2(0, T; \mathcal{V}^*)$ (so that $\mathcal{P} \in \mathcal{W}_1(0, T)$) such that the sequences $(\mathcal{P}_{\tau_{k'}})_{k' \in \mathbb{N}}$ and $(\widehat{\mathcal{P}}_{\tau_{k'}})_{k' \in \mathbb{N}}$ of approximate solutions to (1.1) satisfy*

$$\begin{aligned} \mathcal{P}_{\tau_{k'}} &\rightharpoonup \mathcal{P} \text{ in } L^2(0, T; \mathcal{V}), \\ \widehat{\mathcal{P}}_{\tau_{k'}}, \mathcal{P}_{\tau_{k'}} &\overset{*}{\rightharpoonup} \mathcal{P} \text{ in } L^\infty(0, T; \mathcal{H}), \quad \text{and} \\ \widehat{\mathcal{P}}'_{\tau_{k'}} &\rightharpoonup \mathcal{P}' \text{ in } L^2(0, T; \mathcal{V}^*) \end{aligned}$$

as $k' \rightarrow \infty$.

Proof. In the following, we will omit the indices k to keep the notation simple. Using the a priori estimate (4.2), we obtain the uniform boundedness of the piecewise constant and the piecewise linear interpolation with respect to the norm of $L^\infty(0, T; \mathcal{H})$:

$$\|\mathcal{P}_\tau\|_{L^\infty(0, T; \mathcal{H})} \leq c, \quad \|\widehat{\mathcal{P}}_\tau\|_{L^\infty(0, T; \mathcal{H})} \leq c,$$

where $c > 0$ only depends on the data of the problem but not on τ . In an analogous manner, (4.2) yields the uniform boundedness of the piecewise constant interpolation in $L^2(0, T; \mathcal{V})$,

$$\|\mathcal{P}_\tau\|_{L^2(0, T; \mathcal{V})}^2 = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|\mathcal{P}_\tau(t)\|_{\mathcal{V}}^2 dt = \tau \sum_{n=1}^N \|P_n\|_{\mathcal{V}}^2 \leq c.$$

For the $L^2(0, T; \mathcal{V}^*)$ -norm of the derivative of $\widehat{\mathcal{P}}_\tau$, it follows that

$$\|\widehat{\mathcal{P}}'_\tau\|_{L^2(0, T; \mathcal{V}^*)}^2 = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left\| \frac{P_n - P_{n-1}}{\tau} \right\|_{\mathcal{V}^*}^2 dt = \tau \sum_{n=1}^N \|Q_n - \mathfrak{A}_n P_n\|_{\mathcal{V}^*}^2$$

$$\leq 2\tau \sum_{n=1}^N (\|Q_n\|_{\mathcal{V}^*}^2 + \|\mathfrak{A}_n P_n\|_{\mathcal{V}^*}^2).$$

Moreover, we recall (4.6) and, using Assumption 3, we see that

$$\begin{aligned} \|\mathfrak{A}_n P_n\|_{\mathcal{V}^*} &\leq \|A_n^* P_n\|_{\mathcal{V}^*} + \|P_n A_n\|_{\mathcal{V}^*} + \|P_n^2\|_{\mathcal{V}^*} \\ &\leq 2\eta \|P_n\|_{\mathcal{V}} + \frac{C_{V,H}}{\sqrt{2}} \|P_n^2\|_{\mathcal{H}} \leq 2\eta \|P_n\|_{\mathcal{V}} + \frac{C_{V,H}}{\sqrt{2}} \|P_n\|_{\mathcal{H}}^2, \end{aligned}$$

and thus

$$\tau \sum_{n=1}^N \|\mathfrak{A}_n P_n\|_{\mathcal{V}^*}^2 \leq c \left(\tau \sum_{n=1}^N \|P_n\|_{\mathcal{V}}^2 + \max_{n=1,2,\dots,N} \|P_n\|_{\mathcal{H}}^4 \right) \leq c.$$

Because of the reflexivity of $L^2(0, T; \mathcal{V})$ and its dual and the fact that $L^\infty(0, T; \mathcal{H})$ is the dual of the separable normed space $L^1(0, T; \mathcal{H})$, there exist $\mathcal{P} \in L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V})$, $\widehat{\mathcal{P}} \in L^\infty(0, T; \mathcal{H})$ and $\mathcal{R} \in L^2(0, T; \mathcal{V}^*)$ such that

$$\begin{aligned} \mathcal{P}_\tau &\xrightarrow{*} \mathcal{P} \text{ in } L^\infty(0, T; \mathcal{H}), \\ \widehat{\mathcal{P}}_\tau &\xrightarrow{*} \widehat{\mathcal{P}} \text{ in } L^\infty(0, T; \mathcal{H}), \\ \mathcal{P}_\tau &\rightharpoonup \mathcal{P} \text{ in } L^2(0, T; \mathcal{V}), \quad \text{and} \\ \widehat{\mathcal{P}}'_\tau &\rightharpoonup \mathcal{R} \text{ in } L^2(0, T; \mathcal{V}^*) \end{aligned}$$

as $\tau \rightarrow 0$ (passing to a subsequence if necessary), compare [8, Chapter 3] for further details. The next step is to prove that the limit \mathcal{P} of the constant interpolation coincides with the limit $\widehat{\mathcal{P}}$ of the linear interpolation. The a priori estimate (4.2) implies that

$$\begin{aligned} \int_0^T \|\mathcal{P}_\tau(t) - \widehat{\mathcal{P}}_\tau(t)\|_{\mathcal{H}}^2 dt &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left\| \frac{P_n - P_{n-1}}{\tau} (t_n - t) \right\|_{\mathcal{H}}^2 dt \\ &= \frac{\tau}{3} \sum_{n=1}^N \|P_n - P_{n-1}\|_{\mathcal{H}}^2 \leq \tau c \rightarrow 0 \end{aligned}$$

as $\tau \rightarrow 0$. Therefore, the limits of (\mathcal{P}_τ) and $(\widehat{\mathcal{P}}_\tau)$ coincide. The last step to prove the assertion is to show that $\mathcal{P}' = \mathcal{R}$. For arbitrary $S \in \mathcal{V}$ and $\varphi \in C_c^\infty(0, T)$, it follows that

$$\begin{aligned} \int_0^T \langle \mathcal{R}(t), S \rangle \varphi(t) dt &= \int_0^T \langle \mathcal{R}(t) - \widehat{\mathcal{P}}'_\tau(t), S \rangle \varphi(t) dt - \int_0^T \langle \widehat{\mathcal{P}}_\tau(t), S \rangle \varphi'(t) dt \\ &\rightarrow - \int_0^T \langle \mathcal{P}(t), S \rangle \varphi'(t) dt \end{aligned}$$

as $\tau \rightarrow 0$. Thus, $\mathcal{R} \in L^2(0, T; \mathcal{V}^*)$ is the weak derivative \mathcal{P}' of $\mathcal{P} \in L^2(0, T; \mathcal{V}) \subseteq L^2(0, T; \mathcal{V}^*)$. This finally shows that $\mathcal{P} \in \mathcal{W}_1(0, T)$. \square

The next step is to study the behaviour of the operators $\mathcal{A}_\tau^*(t)$ and $\mathcal{A}_\tau(t)$ as $\tau \rightarrow 0$.

Lemma 4.4. *Let Assumptions 1. and 3. be satisfied. Further, let $(N_k)_{k \in \mathbb{N}}$ be a sequence of positive integers such that $N_k \rightarrow \infty$ as $k \rightarrow \infty$ and let $(\tau_k)_{k \in \mathbb{N}}$ be the sequence of step sizes $\tau_k = \frac{T}{N_k}$ such that $\sup_{k \in \mathbb{N}} \tau_k < \frac{\mu}{2C_{V,H}^2}$. Then for all $P \in \mathcal{V}$ both*

$$\mathcal{A}_{\tau_k}^*(t)P \rightarrow \mathcal{A}^*(t)P \text{ in } \mathcal{V}^* \quad \text{and} \quad P\mathcal{A}_{\tau_k}(t) \rightarrow P\mathcal{A}(t) \text{ in } \mathcal{V}^*$$

hold for almost every $t \in [0, T]$ as $k \rightarrow \infty$. Furthermore, also

$$\mathcal{A}_{\tau_k}^* P \rightarrow \mathcal{A}^* P \text{ in } L^r(0, T; \mathcal{V}^*) \quad \text{and} \quad P\mathcal{A}_{\tau_k} \rightarrow P\mathcal{A} \text{ in } L^r(0, T; \mathcal{V}^*)$$

for every $r \in [1, \infty)$ as $k \rightarrow \infty$.

We omit the proof of Lemma 4.4. The assertion can be verified employing that both $t \mapsto \mathcal{A}(t)P$ and $t \mapsto P\mathcal{A}(t)$ are elements of $L^r(0, T; \mathcal{V}^*)$ for every $r \in [1, \infty)$ and the fact that $C^\infty([0, T]; \mathcal{V}^*)$ is a dense subspace of $L^r(0, T; \mathcal{V}^*)$. See, e.g., [26, Remark 8.15 and Remark 8.21] for more details. Using that the space $C^\infty([0, T]; \mathcal{H})$ is dense in $L^1(0, T; \mathcal{H}) + L^2(0, T; \mathcal{V}^*)$, the same argumentation yields the strong convergence of \mathcal{Q}_τ to \mathcal{Q} in $L^1(0, T; \mathcal{H}) + L^2(0, T; \mathcal{V}^*)$ as $\tau \rightarrow 0$.

4.3 Weak solution of the Riccati equation

Using the results of the previous subsections, the assertion of Theorem 4.1 can be proven. First, we show the result for a smaller class of right-hand sides \mathcal{Q} .

Lemma 4.5. *Let Assumptions 1. and 3. be satisfied and let γ , with $0 \leq \gamma < \frac{\mu}{C_{V,H}^2}$, be given. Then for $\mathcal{Q} \in L^2(0, T; \mathcal{H})$ with $\mathcal{Q}(t) = \mathcal{Q}^*(t)$ and $\mathcal{Q}(t) \geq -\gamma^2$ for almost every $t \in [0, T]$ and $\mathcal{P}_0 \in \mathcal{H}$ with $\mathcal{P}_0 = \mathcal{P}_0^*$ and $\mathcal{P}_0 \geq -\gamma$, the initial value problem (1.1) possesses a weak solution $\mathcal{P} \in \mathcal{W}_1(0, T)$.*

Note that under the assumptions of Lemma 4.5 one even obtains that \mathcal{P}' is an element of $L^2(0, T; \mathcal{V}^*)$.

Proof of Lemma 4.5. Again, we use the notation from Lemma 4.3 and its proof. In Lemma 4.3, we have shown that there exists $\mathcal{P} \in \mathcal{W}_1(0, T)$ such that

$$\begin{aligned} \mathcal{P}_\tau &\rightharpoonup \mathcal{P} \text{ in } L^2(0, T; \mathcal{V}), \\ \widehat{\mathcal{P}}_\tau, \mathcal{P}_\tau &\overset{*}{\rightharpoonup} \mathcal{P} \text{ in } L^\infty(0, T; \mathcal{H}), \quad \text{and} \\ \widehat{\mathcal{P}}'_\tau &\rightharpoonup \mathcal{P}' \text{ in } L^2(0, T; \mathcal{V}^*) \end{aligned}$$

as $\tau \rightarrow 0$. This in mind, we start proving that \mathcal{P} fulfills the initial condition. In order to do so, let $\mathcal{R}(t) = \frac{T-t}{T} R$ for arbitrary $R \in \mathcal{V}$, $t \in [0, T]$. Then we obtain that

$$-\int_0^T \langle \widehat{\mathcal{P}}'_\tau(t), \mathcal{R}(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt - \int_0^T \langle \widehat{\mathcal{P}}_\tau(t), \mathcal{R}'(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt$$

$$= - \left(\widehat{\mathcal{P}}_\tau(T), \mathcal{R}(T) \right) - \left(\widehat{\mathcal{P}}_\tau(0), \mathcal{R}(0) \right) = (P_0, R)$$

and also

$$- \int_0^T \langle \mathcal{P}'(t), \mathcal{R}(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt - \int_0^T \langle \mathcal{P}(t), \mathcal{R}'(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt = (\mathcal{P}(0), R).$$

Since

$$\begin{aligned} & - \int_0^T \langle \widehat{\mathcal{P}}'_\tau(t), \mathcal{R}(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt - \int_0^T \langle \widehat{\mathcal{P}}_\tau(t), \mathcal{R}'(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt \\ & \rightarrow - \int_0^T \langle \mathcal{P}'(t), \mathcal{R}(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt - \int_0^T \langle \mathcal{P}(t), \mathcal{R}'(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt \end{aligned}$$

as $\tau \rightarrow 0$, it follows that $\mathcal{P}(0) = P_0$.

Using $\mathcal{R}(t) = \frac{t}{T} R$ for arbitrary $R \in \mathcal{V}$, $t \in [0, T]$, we obtain similarly that

$$\begin{aligned} (P_N, R) &= \int_0^T \langle \widehat{\mathcal{P}}'_\tau(t), \mathcal{R}(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt + \int_0^T \langle \widehat{\mathcal{P}}_\tau(t), \mathcal{R}'(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt \\ &\rightarrow \int_0^T \langle \mathcal{P}'(t), \mathcal{R}(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt + \int_0^T \langle \mathcal{P}(t), \mathcal{R}'(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt = (\mathcal{P}(T), R), \end{aligned}$$

which means that $P_N \rightarrow \mathcal{P}(T)$ in \mathcal{H} as $\tau \rightarrow 0$. This will be employed later in the proof.

Using Mazur's lemma, see for example [8, Corollary 3.8], there exists a sequence (\mathcal{K}_τ) of convex combinations of the elements of the sequence (\mathcal{P}_τ) such that

$$\mathcal{K}_\tau \rightarrow \mathcal{P} \quad \text{in } L^2(0, T; \mathcal{V}) \text{ as } \tau \rightarrow 0.$$

Since every function \mathcal{P}_τ is piecewise constant with values $P_n \geq -\gamma$ ($n = 0, 1, \dots, N$), it is clear that $\mathcal{P}_\tau(t) \geq -\gamma$ for every $t \in [0, T]$. Thus, every convex combination of functions \mathcal{P}_τ is pointwise greater or equal than $-\gamma$. Due to the strong convergence of the sequence (\mathcal{K}_τ) in $L^2(0, T; \mathcal{V})$, there exists a subsequence which converges pointwise in \mathcal{V} to the limit \mathcal{P} almost everywhere in $[0, T]$. Since $\mathcal{K}_\tau(t) \geq -\gamma$ for every $t \in [0, T]$, the limit \mathcal{P} possesses the same lower bound almost everywhere.

The sequence $(\mathfrak{A}_\tau \mathcal{P}_\tau)$ is bounded in $L^2(0, T; \mathcal{V}^*)$, since

$$\begin{aligned} \|\mathfrak{A}_\tau \mathcal{P}_\tau\|_{L^2(0, T; \mathcal{V}^*)} &= \|\mathcal{A}_\tau^* \mathcal{P}_\tau + \mathcal{P}_\tau \mathcal{A}_\tau + \mathcal{P}_\tau^2\|_{L^2(0, T; \mathcal{V}^*)} \\ &\leq \|\mathcal{A}_\tau^* \mathcal{P}_\tau\|_{L^2(0, T; \mathcal{V}^*)} + \|\mathcal{P}_\tau \mathcal{A}_\tau\|_{L^2(0, T; \mathcal{V}^*)} + \|\mathcal{P}_\tau^2\|_{L^2(0, T; \mathcal{V}^*)} \\ &\leq 2\eta \|\mathcal{P}_\tau\|_{L^2(0, T; \mathcal{V})} + \sqrt{T} \frac{C_{V, H}}{\sqrt{2}} \|\mathcal{P}_\tau\|_{L^\infty(0, T; \mathcal{H})}^2, \end{aligned}$$

where η is introduced in Assumption 3. Thus, there exists a subsequence that converges weakly to an element $\mathcal{S} \in L^2(0, T; \mathcal{V}^*)$. Since $\mathfrak{A}_\tau \mathcal{P}_\tau = \mathcal{Q}_\tau - \widehat{\mathcal{P}}'_\tau$, it is clear that \mathcal{S} coincides with $\mathcal{Q} - \mathcal{P}'$.

Using this subsequence from now on, it remains to prove that $\mathfrak{A}\mathcal{P} = \mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} + \mathcal{P}^2$ coincides with the limit \mathcal{S} of $(\mathfrak{A}_\tau \mathcal{P}_\tau)$. We show this using the Minty trick (see for example

). For a function $\mathcal{R}_\theta = \mathcal{P} - \theta\mathcal{U}$ with $\mathcal{U} \in L^\infty(0, T; \mathcal{V})$ and $\theta > 0$ small enough, one obtains that

$$\begin{aligned}
& \int_0^T \langle \mathfrak{A}_\tau(t) \mathcal{P}_\tau(t) - \mathfrak{A}_\tau(t) \mathcal{R}_\theta(t), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle dt \\
&= \int_0^T \langle \mathcal{A}_\tau^*(t) (\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)) + (\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)) \mathcal{A}_\tau(t), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle dt \\
&\quad + \int_0^T \langle \mathcal{P}_\tau^2(t) - \mathcal{R}_\theta^2(t), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle dt \\
&\geq \int_0^T \mu \|\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)\|_{\mathcal{V}}^2 dt + \int_0^T \langle \mathcal{P}_\tau(t) (\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle dt \\
&\quad + \int_0^T \langle (\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)) \mathcal{P}(t), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle dt \\
&\quad - \int_0^T \theta \langle (\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)) \mathcal{U}(t), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle dt.
\end{aligned}$$

An application of Lemma 3.5 then yields

$$\begin{aligned}
& \langle \mathcal{P}_\tau(t) (\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle \\
&\geq -\gamma \|\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)\|_{\mathcal{H}}^2 \geq -\frac{\gamma C_{V,H}^2}{2} \|\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)\|_{\mathcal{V}}^2.
\end{aligned}$$

An analogous argument yields that

$$\langle (\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)) \mathcal{P}(t), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle \geq -\frac{\gamma C_{V,H}^2}{2} \|\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)\|_{\mathcal{V}}^2.$$

Altogether, we then get

$$\begin{aligned}
& \int_0^T \langle \mathfrak{A}_\tau(t) \mathcal{P}_\tau(t) - \mathfrak{A}_\tau(t) \mathcal{R}_\theta(t), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle dt \\
&\geq (\mu - \gamma C_{V,H}^2) \int_0^T \|\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)\|_{\mathcal{V}}^2 dt - \int_0^T \theta \|\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)\|_{\mathcal{H}}^2 \|\mathcal{U}(t)\|_{\mathcal{H}} dt \\
&\geq \left(\mu - \gamma C_{V,H}^2 - \theta \frac{C_{V,H}^2}{2} \|\mathcal{U}\|_{L^\infty(0,T;\mathcal{H})} \right) \int_0^T \|\mathcal{P}_\tau(t) - \mathcal{R}_\theta(t)\|_{\mathcal{V}}^2 dt \geq 0
\end{aligned}$$

for $\theta \frac{C_{V,H}^2}{2} \|\mathcal{U}\|_{L^\infty(0,T;\mathcal{H})} \leq \mu - \gamma C_{V,H}^2$. Thus,

$$\begin{aligned}
& \int_0^T \langle \mathfrak{A}_\tau(t) \mathcal{P}_\tau(t), \mathcal{P}_\tau(t) \rangle dt \\
&\geq \int_0^T \langle \mathfrak{A}_\tau(t) \mathcal{R}_\theta(t), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle dt + \int_0^T \langle \mathfrak{A}_\tau(t) \mathcal{P}_\tau(t), \mathcal{R}_\theta(t) \rangle dt
\end{aligned}$$

holds for $\theta > 0$ sufficiently small. Let us look at the summands of the right-hand side separately. For the second summand, the weak convergence of $(\mathfrak{A}_\tau \mathcal{P}_\tau)$ implies that

$$\int_0^T \langle \mathfrak{A}_\tau(t) \mathcal{P}_\tau(t), \mathcal{R}_\theta(t) \rangle dt \rightarrow \int_0^T \langle \mathcal{S}(t), \mathcal{R}_\theta(t) \rangle dt$$

as $\tau \rightarrow 0$. To study the convergence of the first summand, we split it up as

$$\begin{aligned} & \int_0^T \langle \mathfrak{A}_\tau(t) \mathcal{R}_\theta(t), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle dt \\ &= \int_0^T \langle \mathcal{A}_\tau^*(t) \mathcal{R}_\theta(t), \mathcal{P}_\tau(t) \rangle dt + \int_0^T \langle \mathcal{R}_\theta(t) \mathcal{A}_\tau(t), \mathcal{P}_\tau(t) \rangle dt + \int_0^T \langle \mathcal{R}_\theta^2(t), \mathcal{P}_\tau(t) \rangle dt \\ & \quad - \int_0^T \langle \mathcal{A}_\tau^*(t) \mathcal{R}_\theta(t) + \mathcal{R}_\theta(t) \mathcal{A}_\tau(t) + \mathcal{R}_\theta^2(t), \mathcal{R}_\theta(t) \rangle dt \end{aligned} \quad (4.9)$$

Then for the first summand on the right-hand side of (4.9), we use

$$\begin{aligned} & \left| \int_0^T \langle \mathcal{A}_\tau^*(t) \mathcal{R}_\theta(t), \mathcal{P}_\tau(t) \rangle dt - \int_0^T \langle \mathcal{A}^*(t) \mathcal{R}_\theta(t), \mathcal{P}(t) \rangle dt \right| \\ & \leq \left| \int_0^T \langle (\mathcal{A}_\tau^*(t) - \mathcal{A}^*(t)) \mathcal{R}_\theta(t), \mathcal{P}_\tau(t) \rangle dt \right| + \left| \int_0^T \langle \mathcal{A}^*(t) \mathcal{R}_\theta(t), \mathcal{P}_\tau(t) - \mathcal{P}(t) \rangle dt \right| \\ & \leq \left(\int_0^T \|(\mathcal{A}_\tau^*(t) - \mathcal{A}^*(t)) \mathcal{R}_\theta(t)\|_{\mathcal{V}^*}^2 dt \int_0^T \|\mathcal{P}_\tau(t)\|_{\mathcal{V}}^2 dt \right)^{\frac{1}{2}} \\ & \quad + \left| \int_0^T \langle \mathcal{A}^*(t) \mathcal{R}_\theta(t), \mathcal{P}_\tau(t) - \mathcal{P}(t) \rangle dt \right|. \end{aligned} \quad (4.10)$$

The second summand on the right-hand side of (4.10) converges to zero due to the weak convergence of (\mathcal{P}_τ) towards \mathcal{P} in $L^2(0, T; \mathcal{V})$. The weak convergence of the sequence (\mathcal{P}_τ) in $L^2(0, T; \mathcal{V})$ also implies that it is bounded. As shown in Lemma 4.4, $(\mathcal{A}_\tau^*(t) - \mathcal{A}^*(t)) \mathcal{R}_\theta(t) \rightarrow 0$ in \mathcal{V}^* for almost every $t \in [0, T]$ as $\tau \rightarrow 0$. Due to the uniform boundedness of \mathcal{A}^* , we obtain the following bound

$$\|(\mathcal{A}_\tau^*(t) - \mathcal{A}^*(t)) \mathcal{R}_\theta(t)\|_{\mathcal{V}^*}^2 \leq 4\eta \|\mathcal{R}_\theta(t)\|_{\mathcal{V}}^2 \quad \text{for almost every } t \in [0, T],$$

with $4\eta \|\mathcal{R}_\theta\|_{\mathcal{V}}^2 \in L^1(0, T)$. Therefore, the first summand on the right-hand side of (4.10) converges to zero using Lebesgue's theorem on dominated convergence. This proves

$$\int_0^T \langle \mathcal{A}_\tau^*(t) \mathcal{R}_\theta(t), \mathcal{P}_\tau(t) \rangle dt \rightarrow \int_0^T \langle \mathcal{A}^*(t) \mathcal{R}_\theta(t), \mathcal{P}(t) \rangle dt$$

as $\tau \rightarrow 0$. For the second summand on the right-hand side of (4.9), we can follow an analogous argumentation. In this case, we have

$$\left| \int_0^T \langle \mathcal{R}_\theta(t) \mathcal{A}_\tau(t), \mathcal{P}_\tau(t) \rangle dt - \int_0^T \langle \mathcal{R}_\theta(t) \mathcal{A}(t), \mathcal{P}(t) \rangle dt \right|$$

$$\begin{aligned}
&\leq \left| \int_0^T \langle \mathcal{R}_\theta(t)(\mathcal{A}_\tau(t) - \mathcal{A}(t)), \mathcal{P}_\tau(t) \rangle dt \right| + \left| \int_0^T \langle \mathcal{R}_\theta(t)\mathcal{A}(t), \mathcal{P}_\tau(t) - \mathcal{P}(t) \rangle dt \right| \\
&\leq \left(\int_0^T \|\mathcal{R}_\theta(t)(\mathcal{A}_\tau(t) - \mathcal{A}(t))\|_{\mathcal{V}^*}^2 dt \int_0^T \|\mathcal{P}_\tau(t)\|_{\mathcal{V}}^2 dt \right)^{\frac{1}{2}} \\
&\quad + \left| \int_0^T \langle \mathcal{R}_\theta(t)\mathcal{A}(t), \mathcal{P}_\tau(t) - \mathcal{P}(t) \rangle dt \right|.
\end{aligned}$$

As before, the weak convergence of (\mathcal{P}_τ) towards \mathcal{P} in $L^2(0, T; \mathcal{V})$ and Lemma 4.4 imply that

$$\int_0^T \langle \mathcal{R}_\theta(t)\mathcal{A}_\tau(t), \mathcal{P}_\tau(t) \rangle dt \rightarrow \int_0^T \langle \mathcal{R}_\theta(t)\mathcal{A}(t), \mathcal{P}(t) \rangle dt$$

as $\tau \rightarrow 0$.

The third summand on the right-hand side of (4.9) converges to $\int_0^T \langle \mathcal{R}_\theta^2(t), \mathcal{P}(t) \rangle dt$ as $\tau \rightarrow 0$, since (\mathcal{P}_τ) converges weakly to \mathcal{P} in $L^2(0, T; \mathcal{V})$.

For the fourth term on the right-hand side of (4.9), one similarly shows convergence to $-\int_0^T \langle \mathfrak{A}(t)\mathcal{R}_\theta(t), \mathcal{R}_\theta(t) \rangle dt$ as $\tau \rightarrow 0$.

Therefore, we obtain that

$$\lim_{\tau \rightarrow 0} \int_0^T \langle \mathfrak{A}_\tau(t)\mathcal{R}_\theta(t), \mathcal{P}_\tau(t) - \mathcal{R}_\theta(t) \rangle dt = \int_0^T \langle \mathfrak{A}(t)\mathcal{R}_\theta(t), \mathcal{P}(t) - \mathcal{R}_\theta(t) \rangle dt.$$

Altogether, this yields the estimate

$$\begin{aligned}
&\liminf_{\tau \rightarrow 0} \int_0^T \langle \mathfrak{A}_\tau(t)\mathcal{P}_\tau(t), \mathcal{P}_\tau(t) \rangle dt \\
&\geq \int_0^T \langle \mathfrak{A}(t)\mathcal{R}_\theta(t), \mathcal{P}(t) - \mathcal{R}_\theta(t) \rangle dt + \int_0^T \langle \mathcal{S}(t), \mathcal{R}_\theta(t) \rangle dt.
\end{aligned} \tag{4.11}$$

Using (4.8), we obtain that

$$\int_0^T \langle \mathfrak{A}_\tau(t)\mathcal{P}_\tau(t), \mathcal{P}_\tau(t) \rangle dt = \int_0^T (\langle \mathcal{Q}_\tau(t), \mathcal{P}_\tau(t) \rangle - \langle \widehat{\mathcal{P}}'_\tau(t), \mathcal{P}_\tau(t) \rangle) dt.$$

As \mathcal{Q}_τ converges strongly to \mathcal{Q} in $L^2(0, T; \mathcal{H})$ and thus in $L^2(0, T; \mathcal{V}^*)$, we have that

$$\int_0^T \langle \mathcal{Q}_\tau(t), \mathcal{P}_\tau(t) \rangle dt \rightarrow \int_0^T \langle \mathcal{Q}(t), \mathcal{P}(t) \rangle dt$$

holds as $\tau \rightarrow 0$. Furthermore,

$$\begin{aligned}
\int_0^T \langle \widehat{\mathcal{P}}'_\tau(t), \mathcal{P}_\tau(t) \rangle dt &= \sum_{n=1}^N \frac{1}{\tau} \int_{t_{n-1}}^{t_n} dt \langle P_n - P_{n-1}, P_n \rangle \\
&\geq \frac{1}{2} \sum_{n=1}^N (\|P_n\|_{\mathcal{H}}^2 - \|P_{n-1}\|_{\mathcal{H}}^2) = \frac{1}{2} \|P_N\|_{\mathcal{H}}^2 - \frac{1}{2} \|P_0\|_{\mathcal{H}}^2,
\end{aligned}$$

together with the fact that $P_0 = \mathcal{P}(0)$ as well as $P_N \rightarrow \mathcal{P}(T)$ in \mathcal{H} as $\tau \rightarrow 0$ and the weak lower semi-continuity of the norm, yields that

$$\liminf_{\tau \rightarrow 0} \int_0^T \langle \widehat{\mathcal{P}}'_\tau(t), \mathcal{P}_\tau(t) \rangle dt \geq \frac{1}{2} \|\mathcal{P}(T)\|_{\mathcal{H}}^2 - \frac{1}{2} \|\mathcal{P}(0)\|_{\mathcal{H}}^2 = \int_0^T \langle \mathcal{P}'(t), \mathcal{P}(t) \rangle dt.$$

For the last step, we use that $\mathcal{P} \in \mathcal{W}_1(0, T)$, compare, e.g., [30, Section 20]. Altogether, this yields that

$$\limsup_{\tau \rightarrow 0} \int_0^T \langle \mathfrak{A}_\tau(t) \mathcal{P}_\tau(t), \mathcal{P}_\tau(t) \rangle dt \leq \int_0^T \langle \mathcal{Q}(t), \mathcal{P}(t) \rangle dt - \int_0^T \langle \mathcal{P}'(t), \mathcal{P}(t) \rangle dt.$$

Using this estimate and (4.11), we obtain that

$$\begin{aligned} \int_0^T \langle \mathcal{S}(t), \mathcal{P}(t) \rangle dt &= \int_0^T \langle \mathcal{Q}(t) - \mathcal{P}'(t), \mathcal{P}(t) \rangle dt \\ &\geq \int_0^T (\langle \mathfrak{A}(t) \mathcal{R}_\theta(t), \mathcal{P}(t) - \mathcal{R}_\theta(t) \rangle + \langle \mathcal{S}(t), \mathcal{R}_\theta(t) \rangle) dt \end{aligned}$$

which can be rewritten as

$$\int_0^T \langle \mathcal{S}(t), \mathcal{P}(t) - \mathcal{R}_\theta(t) \rangle dt \geq \int_0^T \langle \mathfrak{A}(t) \mathcal{R}_\theta(t), \mathcal{P}(t) - \mathcal{R}_\theta(t) \rangle dt. \quad (4.12)$$

Reinserting $\mathcal{R}_\theta = \mathcal{P} - \theta \mathcal{U}$ for $\theta > 0$ small enough, the estimate has the form

$$\theta \int_0^T \langle \mathcal{S}(t), \mathcal{U}(t) \rangle dt \geq \theta \int_0^T \langle \mathfrak{A}(t) (\mathcal{P}(t) - \theta \mathcal{U}(t)), \mathcal{U}(t) \rangle dt.$$

Dividing this by θ and followed by the limiting process $\theta \rightarrow 0$, this yields

$$\int_0^T \langle \mathcal{S}(t), \mathcal{U}(t) \rangle dt \geq \int_0^T \langle \mathfrak{A}(t) \mathcal{P}(t), \mathcal{U}(t) \rangle dt$$

for all \mathcal{U} and thus $\mathcal{S} = \mathfrak{A} \mathcal{P}$. □

The last lemma in mind, the convergence of the time discretization for a right-hand side $\mathcal{Q} \in L^1(0, T; \mathcal{H}) + L^2(0, T; \mathcal{V}^*)$ can now be deduced.

Proof of Theorem 4.1. For a right-hand side $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2 \in L^1(0, T; \mathcal{H}) + L^2(0, T; \mathcal{V}^*)$, there exist sequences $(\mathcal{Q}_{1,i})_{i \in \mathbb{N}}$ and $(\mathcal{Q}_{2,i})_{i \in \mathbb{N}}$ in $L^2(0, T; \mathcal{H})$ such that

$$\mathcal{Q}_{1,i} \rightarrow \mathcal{Q}_1 \text{ in } L^1(0, T; \mathcal{H}) \quad \text{and} \quad \mathcal{Q}_{2,i} \rightarrow \mathcal{Q}_2 \text{ in } L^2(0, T; \mathcal{V}^*)$$

as $i \rightarrow \infty$. Furthermore, we set $\mathcal{Q}_i = \mathcal{Q}_{1,i} + \mathcal{Q}_{2,i}$ for $i \in \mathbb{N}$. Every problem

$$\begin{aligned} \mathcal{P}'_i(t) + \mathcal{A}^*(t) \mathcal{P}_i(t) + \mathcal{P}_i(t) \mathcal{A}(t) + \mathcal{P}_i^2(t) &= \mathcal{Q}_i(t), \quad t \in (0, T), \\ \mathcal{P}_i(0) &= P_0 \end{aligned} \quad (4.13)$$

has a solution $\mathcal{P}_i \in \mathcal{W}_1(0, T)$.

For arbitrary $i, j \in \mathbb{N}$, we consider the difference of the solutions \mathcal{P}_i and \mathcal{P}_j of the associated problems (4.13). Using Lemma 3.5, it follows that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\mathcal{P}_i(t) - \mathcal{P}_j(t)\|_{\mathcal{H}}^2 + \mu \|\mathcal{P}_i(t) - \mathcal{P}_j(t)\|_{\mathcal{V}}^2 - \gamma C_{V,H}^2 \|\mathcal{P}_i(t) - \mathcal{P}_j(t)\|_{\mathcal{V}}^2 \\
& \leq (\mathcal{P}'_i(t) - \mathcal{P}'_j(t), \mathcal{P}_i(t) - \mathcal{P}_j(t)) \\
& \quad + \langle \mathcal{A}^*(t)(\mathcal{P}_i(t) - \mathcal{P}_j(t)) + (\mathcal{P}_i(t) - \mathcal{P}_j(t))\mathcal{A}(t), \mathcal{P}_i(t) - \mathcal{P}_j(t) \rangle \\
& \quad + (\mathcal{P}_i^2(t) - \mathcal{P}_j^2(t), \mathcal{P}_i(t) - \mathcal{P}_j(t)) \\
& = (\mathcal{Q}_{1,i}(t) - \mathcal{Q}_{1,j}(t), \mathcal{P}_i(t) - \mathcal{P}_j(t)) + (\mathcal{Q}_{2,i}(t) - \mathcal{Q}_{2,j}(t), \mathcal{P}_i(t) - \mathcal{P}_j(t)) \tag{4.14}
\end{aligned}$$

for almost every $t \in (0, T)$. Integrating this estimate and applying Young's inequality yields

$$\begin{aligned}
& \frac{1}{2} \|\mathcal{P}_i(t) - \mathcal{P}_j(t)\|_{\mathcal{H}}^2 + (\mu - \gamma C_{V,H}^2) \int_0^t \|\mathcal{P}_i(s) - \mathcal{P}_j(s)\|_{\mathcal{V}}^2 ds \\
& \leq \max_{t \in [0, T]} \|\mathcal{P}_i(t) - \mathcal{P}_j(t)\|_{\mathcal{H}} \int_0^t \|\mathcal{Q}_{1,i}(s) - \mathcal{Q}_{1,j}(s)\|_{\mathcal{H}} ds \\
& \quad + \int_0^t \left(\frac{1}{2(\mu - \gamma C_{V,H}^2)} \|\mathcal{Q}_{2,i}(s) - \mathcal{Q}_{2,j}(s)\|_{\mathcal{V}^*}^2 + \frac{\mu - \gamma C_{V,H}^2}{2} \|\mathcal{P}_i(s) - \mathcal{P}_j(s)\|_{\mathcal{V}}^2 \right) ds. \tag{4.15}
\end{aligned}$$

As $\mathcal{P}_i, \mathcal{P}_j \in C([0, T]; \mathcal{H})$, there exists $t_0 \in [0, T]$ such that

$$\|\mathcal{P}_i(t_0) - \mathcal{P}_j(t_0)\|_{\mathcal{H}} = \max_{t \in [0, T]} \|\mathcal{P}_i(t) - \mathcal{P}_j(t)\|_{\mathcal{H}}$$

holds. As $t \in [0, T]$ in estimate (4.15) is arbitrary, we can take $t = t_0$ and obtain that

$$\begin{aligned}
\|\mathcal{P}_i(t_0) - \mathcal{P}_j(t_0)\|_{\mathcal{H}}^2 & \leq 2 \|\mathcal{P}_i(t_0) - \mathcal{P}_j(t_0)\|_{\mathcal{H}} \int_0^{t_0} \|\mathcal{Q}_{1,i}(s) - \mathcal{Q}_{1,j}(s)\|_{\mathcal{H}} ds \\
& \quad + \frac{1}{\mu - \gamma C_{V,H}^2} \int_0^{t_0} \|\mathcal{Q}_{2,i}(s) - \mathcal{Q}_{2,j}(s)\|_{\mathcal{V}^*}^2 ds. \tag{4.16}
\end{aligned}$$

Setting

$$\begin{aligned}
x & = \|\mathcal{P}_i(t_0) - \mathcal{P}_j(t_0)\|_{\mathcal{H}}, \\
a & = \int_0^{t_0} \|\mathcal{Q}_{1,i}(s) - \mathcal{Q}_{1,j}(s)\|_{\mathcal{H}} ds, \quad \text{and} \\
b & = \left(\frac{1}{\mu - \gamma C_{V,H}^2} \int_0^{t_0} \|\mathcal{Q}_{2,i}(s) - \mathcal{Q}_{2,j}(s)\|_{\mathcal{V}^*}^2 ds \right)^{\frac{1}{2}}
\end{aligned}$$

for abbreviation, (4.16) is equivalent to

$$x^2 \leq 2ax + b^2.$$

This estimate implies that

$$(x - a)^2 = x^2 - 2ax + a^2 \leq a^2 + b^2.$$

Taking the square root on both sides, this yields

$$x - a \leq \sqrt{a^2 + b^2} \leq a + b.$$

Altogether this leads to the estimate

$$x \leq 2a + b,$$

which implies that

$$\begin{aligned} & \max_{t \in [0, T]} \|\mathcal{P}_i(t) - \mathcal{P}_j(t)\|_{\mathcal{H}} \\ & \leq 2 \int_0^T \|\mathcal{Q}_{1,i}(s) - \mathcal{Q}_{1,j}(s)\|_{\mathcal{H}} ds + \left(\frac{1}{\mu - \gamma C_{V,H}} \int_0^T \|\mathcal{Q}_{2,i}(s) - \mathcal{Q}_{2,j}(s)\|_{\mathcal{V}^*}^2 ds \right)^{\frac{1}{2}} =: C_{max}. \end{aligned}$$

Using this estimate in (4.15), we further obtain that for all $t \in [0, T]$

$$\begin{aligned} & \|\mathcal{P}_i(t) - \mathcal{P}_j(t)\|_{\mathcal{H}}^2 + (\mu - \gamma C_{V,H}^2) \int_0^t \|\mathcal{P}_i(s) - \mathcal{P}_j(s)\|_{\mathcal{V}}^2 ds \\ & \leq 2C_{max} \int_0^T \|\mathcal{Q}_{1,i}(s) - \mathcal{Q}_{1,j}(s)\|_{\mathcal{H}} ds \\ & \quad + \frac{1}{\mu - \gamma C_{V,H}} \int_0^T \|\mathcal{Q}_{2,i}(s) - \mathcal{Q}_{2,j}(s)\|_{\mathcal{V}^*}^2 ds. \end{aligned}$$

This proves that $(\mathcal{P}_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V})$ and therefore convergent to a certain limit $\mathcal{P} \in C([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V})$. This convergence and the estimate

$$\begin{aligned} & \int_0^T \|\mathfrak{A}(t)\mathcal{P}_i(t) - \mathfrak{A}(t)\mathcal{P}(t)\|_{\mathcal{V}^*}^2 dt \\ & \leq c \int_0^T (\eta \|\mathcal{P}_i(t) - \mathcal{P}(t)\|_{\mathcal{V}}^2 + C_{V,H}^4 \|\mathcal{P}_i(t) - \mathcal{P}(t)\|_{\mathcal{V}}^2 (\|\mathcal{P}_i(t)\|_{\mathcal{H}}^2 + \|\mathcal{P}(t)\|_{\mathcal{H}}^2)) dt, \end{aligned}$$

where η is defined in Assumption 3., imply the convergence of the sequence $(\mathfrak{A}\mathcal{P}_i)_{i \in \mathbb{N}}$ to $\mathfrak{A}\mathcal{P}$ in $L^2(0, T; \mathcal{V}^*)$. As the sequence $(\mathcal{Q}_i)_{i \in \mathbb{N}}$ of right-hand sides converges strongly in $L^1(0, T; \mathcal{H}) + L^2(0, T; \mathcal{V}^*)$ towards \mathcal{Q} , it follows that

$$\mathcal{P}'_i = \mathcal{Q}_i - \mathfrak{A}\mathcal{P}_i \rightarrow \mathcal{Q} - \mathfrak{A}\mathcal{P} \text{ in } L^1(0, T; \mathcal{H}) + L^2(0, T; \mathcal{V}^*).$$

This shows that \mathcal{P} possesses a weak derivative that coincides with $\mathcal{Q} - \mathfrak{A}\mathcal{P}$. Moreover, $\mathcal{P} \in \mathcal{W}_1(0, T) \hookrightarrow C([0, T]; \mathcal{H})$. Considering the continuity of the corresponding trace operator mapping onto the evaluation at $t = 0$ immediately shows that $\mathcal{P}(0)$ is the limit of $\mathcal{P}_i(0) = P_0$ in \mathcal{H} . □

So far we have proven that the sequence of interpolations (4.7) has a weakly convergent subsequence. This convergence result can be strengthened.

Theorem 4.6. *Let the assumptions of Theorem 4.1 be satisfied. Further, let $(N_k)_{k \in \mathbb{N}}$ be a sequence of positive integers such that $N_k \rightarrow \infty$ as $k \rightarrow \infty$ and let $(\tau_k)_{k \in \mathbb{N}}$ be the sequence of step sizes $\tau_k = \frac{T}{N_k}$ such that $\sup_{k \in \mathbb{N}} \tau_k < \frac{\mu}{2C_{V,H}^2}$.*

Proof.

$$\begin{aligned}
& (\mu - \gamma C_{V,H}^2) \int_0^T \|\mathcal{P}_\tau(t) - \mathcal{P}(t)\|_{\mathcal{V}}^2 dt \\
& \leq \int_0^T \langle \mathcal{A}_\tau^*(t)(\mathcal{P}_\tau(t) - \mathcal{P}(t)) + (\mathcal{P}_\tau(t) - \mathcal{P}(t))\mathcal{A}_\tau(t), \mathcal{P}_\tau(t) - \mathcal{P}(t) \rangle dt \\
& \quad + \int_0^T \langle \mathcal{P}_\tau^2(t) - \mathcal{P}^2(t), \mathcal{P}_\tau(t) - \mathcal{P}(t) \rangle dt \\
& = \int_0^T \langle \mathfrak{A}_\tau(t)\mathcal{P}_\tau(t), \mathcal{P}_\tau(t) - \mathcal{P}(t) \rangle - \langle \mathfrak{A}_\tau(t)\mathcal{P}(t), \mathcal{P}_\tau(t) - \mathcal{P}(t) \rangle dt.
\end{aligned}$$

In the proof of Lemma 4.5, it is shown with Minty's trick that

$$\int_0^T \langle \mathfrak{A}_\tau(t)\mathcal{P}_\tau(t), \mathcal{P}_\tau(t) \rangle dt \rightarrow \int_0^T \langle \mathfrak{A}(t)\mathcal{P}(t), \mathcal{P}(t) \rangle dt.$$

This together with the weak convergence of \mathcal{P}_τ towards \mathcal{P} in $L^2(0, T; \mathcal{V})$ and the convergence of $\mathfrak{A}_\tau \mathcal{P}$ towards $\mathfrak{A} \mathcal{P}$ in $L^2(0, T; \mathcal{V}^*)$ implies the assertion. \square

Acknowledgments

The authors would like to thank Christian Kreuzler (Berlin) for helpful discussions and suggestions as well as careful reading of the manuscript.

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