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Operator Differential-Algebraic Equations Arising in Fluid Dynamics

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Dedicated to Rolf Dieter Grigorieff on occasion of his 75th birthday

Abstract — Existence and uniqueness of generalized solutions to initial value problems for a class of abstract differential-algebraic equations (DAEs) is shown. The class of equations covers, in particular, the Stokes and Oseen problem describing the motion of an incompressible or nearly incompressible Newtonian fluid but also their spatial semi-discretization. The equations are governed by a block operator matrix with entries that fulfill suitable inf-sup conditions. The problem data are required to satisfy appropriate consistency conditions. The results in infinite dimensions are compared in detail with those known for the DAEs that arise after semi-discretization in space. Explicit solution formulas are derived in both cases.

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1. Introduction

In this paper we study the solvability of operator differential-algebraic equations (DAEs) (sometimes also called abstract DAEs [25, 41]) of the form

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ p \end{bmatrix} + \begin{bmatrix} A & B \\ -D^T & C \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1}$$

on a time interval $[0, T]$, with linear operators M, A, B, C, D defined on appropriate Hilbert spaces and with appropriate right-hand side functions f, g. Here, the time derivative is usually understood in the distributional sense.

We are interested in solutions to initial value problems with initial condition

$$\begin{bmatrix} v(0) \\ p(0) \end{bmatrix} = \begin{bmatrix} v^0 \\ p^0 \end{bmatrix} \tag{2}$$

where, if the solution does not exist in the classical sense at $t = 0$ then the initial condition is viewed in a generalized sense, see [24, 30].

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Operator DAEs of the form (1) (with $M = I, B = D, C = 0$) arise in the functional analytic formulation of the initial value problem for the *Stokes* as well as for the *linearized Navier–Stokes* or *Oseen* equations [6,38,40], in which v and p, denoting velocity and pressure, respectively, then are abstract functions mapping the time interval into appropriate spatial function spaces.

If one linearizes the incompressible or nearly incompressible *Navier–Stokes equations* describing the flow of a Newtonian fluid,

$$\begin{aligned} \partial_t v - \nu \Delta v + (v \cdot \nabla) v + \nabla p &= f \qquad \text{in } \Omega \times (0,T), \\ \nabla^T v &= 0 \qquad \text{in } \Omega \times (0,T), \end{aligned}$$

around a prescribed vector field v_∞, then one obtains the *linearized Navier–Stokes equations*,

$$\begin{aligned} \partial_t v - \nu \Delta v + (v_\infty \cdot \nabla) v + (v \cdot \nabla) v_\infty + \nabla p &= f \qquad \text{in } \Omega \times (0,T), \\ \nabla^T v &= 0 \qquad \text{in } \Omega \times (0,T). \end{aligned} \tag{3}$$

In what follows, we restrict ourselves to the case that v_∞ is independent of time. Note that if v_∞ is also independent of space then the term $(v \cdot \nabla) v_\infty$ does not appear, and the equations are then called *Oseen equations* (see [36]). Operator DAEs of the form (1) also arise when the Oseen system is semi-discretized in space via the method of lines, see, e.g., [2, 33], using, e.g., a finite element discretization in space [26, 33] and a fixed point iteration to resolve the nonlinearity. Due to the convection term, in general, in the fixed point iteration the resulting coefficient matrix A is nonsymmetric. Furthermore, if the corresponding finite element spaces do not fulfill the discrete Babuška–Brezzi condition [6, 15, 33, 35], a stabilization is needed which then leads to an additional term in (3). Finally, also quasi-compressible fluid flow [29] can be modeled via an additional term in (3). In all described cases the equations have to be supplemented by suitable initial and boundary conditions.

Differential-algebraic (operator) equations are currently the standard modeling concept in many applications such as circuit simulation, multibody dynamics, and chemical process engineering, see [2,4,11,17,19,20,24,31,32] and the references therein. They have a particular advantage for the treatment of multi-physics models arising from modern automatic modeling tools such as Dymola and Simulink[1], and as we have described, they arise in computational fluid dynamics in the special form of the linear operator DAE (1). Nevertheless the analysis of general differential-algebraic equations in the infinite-dimensional case is still in its infancy, and there are yet very few results available on their well-posedness, see [25, 41] and the references cited therein.

In this paper we will carry out the analysis for (1), in particular, we study existence and uniqueness of solutions for the abstract DAE (1). In the infinite-dimensional case our analysis is based on combining methods known for stationary mixed problems with those used for parabolic problems in its weak formulation. The results presented here are new in the general abstract setting although they are partially known for special situations such as the incompressible Stokes problem. The main focus of this work is on a non-zero right-hand side g in (1). It turns out that higher-order time regularity of g (but not of f) and consistency conditions on the problem data are essential ingredients when proving well-posedness.

The paper is organized as follows. In Section 2 we review the classical theory for linear DAEs and apply this theory to the specially structured system given by (1). We present

[1] *Dymola*, Multi-Engineering Modelling and Simulation, 2006, Dynasim AB, Ideon Research Park, 22370 Lund, Sweden. *Simulink*, version 6.4.1, 2005, The MathWorks Inc., Natick, MA, USA.

explicit solution formulas for the finite-dimensional version of (1). In Section 3.1 we prepare our analysis of the infinite-dimensional case by studying the stationary infinite-dimensional case as a mixed variational problem and by recalling a well-known theorem of Lions and Tartar. Section 3.2 then discusses structured operator DAEs of the form (1) in the infinite-dimensional setting for the special case that $M = I$, $B = D$ and $C = 0$ and shows that the explicit solution formulas can be extended.

The cases of general M, B, C, D, the question of regularity, i.e., the smoothing property for non-smooth data, the non-autonomous case with time-dependent operators as well as nonlinear problems will be the issue of forthcoming work.

2. The finite-dimensional case

In this section, we first recall some well-known results on the solvability of the initial value problem for a system of DAEs. Moreover, we provide an explicit representation of the solution, which is a generalization of Duhamel's principle, and discuss the relation between the consistency (compatibility) of the data and the differentiation index of the DAE. We follow [24] in style and notation.

2.1. Well-posedness and explicit solution of DAEs

In this subsection we will discuss explicit solution representations for linear DAEs and thereby also well-posedness results.

Denoting by $C([0,T];\mathbb{R}^n)$ the space of continuous functions from $[0,T]$ to $\mathbb{R}^n$, it is well known, see, e.g., [9], that for the initial value problems associated with the linear ordinary differential equation

$$\dot{x} + \mathcal{A}x = f, \quad x(0) = x^0,$$

where $\mathcal{A} \in \mathbb{R}^{n,n}$ and $f \in C([0,T];\mathbb{R}^n)$, one has the well-known solution formula (Duhamel's principle)

$$x(t) = \mathrm{e}^{-t\mathcal{A}}x^0 + \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\, ds$$

obtained by variation of constants.

The extension of this formula to initial value problems for DAEs of the form

$$\mathcal{E}\dot{x} + \mathcal{A}x = f \tag{4}$$

with $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$ and sufficiently smooth right-hand side f, and initial conditions

$$x(0) = x^0 \in \mathbb{R}^n \tag{5}$$

is also well known, see, e.g., [7, 24].

To present this formula, we need the following preliminary results.

Theorem 2.1 (Weierstrass canonical form, [14])**.** *Let $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$ and suppose that the pair $(\mathcal{E}, \mathcal{A})$ is* regular, *i.e.,* $\det(\lambda\mathcal{E} + \mathcal{A})$ *does not vanish identically for all $\lambda \in \mathbb{C}$. Then, there exist nonsingular matrices $P, Q \in \mathbb{R}^{n,n}$ such that*

$$(P\mathcal{E}Q, P\mathcal{A}Q) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right),$$

where J is a matrix in real Jordan canonical form, N is a nilpotent matrix also in Jordan canonical form and I denotes an identity matrix of appropriate size. Moreover, it is allowed that one or the other block is not present.

If in Theorem 2.1 the index of nilpotency of N is ν, then we say that the pair $(\mathcal{E}, \mathcal{A})$ has *(differentiation) index* ν and denote this by $\nu = \operatorname{ind}(\mathcal{E}, \mathcal{A})$. For a matrix $\mathcal{E} \in \mathbb{R}^{n,n}$ we set $\operatorname{ind} \mathcal{E} = \operatorname{ind}(\mathcal{E}, I)$. We have $\operatorname{ind} \mathcal{E} = 0$ if and only if $\mathcal{E}$ is nonsingular.

The explicit solution formulas require the *Drazin inverse of a matrix.*

Definition 2.1. Let $\mathcal{E} \in \mathbb{R}^{n,n}$ have $\operatorname{ind} \mathcal{E} = \nu$. A matrix $X \in \mathbb{R}^{n,n}$ satisfying

$$\mathcal{E}X = X\mathcal{E}, \tag{6a}$$

$$X\mathcal{E}X = X, \tag{6b}$$

$$X\mathcal{E}^{\nu+1} = \mathcal{E}^{\nu} \tag{6c}$$

is called a *Drazin inverse* of $\mathcal{E}$.

We recall some well-known facts about the Drazin inverse, see [8].

Theorem 2.2. *Let $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$.*

1. *$\mathcal{E}$ has one and only one Drazin inverse $\mathcal{E}^D$.*
2. *If $\mathcal{E}$ is nonsingular then $\mathcal{E}^D = \mathcal{E}^{-1}$.*
3. *If $\mathcal{E}\mathcal{A} = \mathcal{A}\mathcal{E}$ then*

$$\mathcal{E}\mathcal{A}^D = \mathcal{A}^D\mathcal{E}, \quad \mathcal{E}^D\mathcal{A} = \mathcal{A}\mathcal{E}^D, \quad \mathcal{E}^D\mathcal{A}^D = \mathcal{A}^D\mathcal{E}^D.$$

To present the explicit solution representations, we first assume that in (4) the coefficient matrices $\mathcal{E}$ and $\mathcal{A}$ commute, i.e., that

$$\mathcal{E}\mathcal{A} = \mathcal{A}\mathcal{E}. \tag{7}$$

Then we have the following solution formula, see [7, 24].

Theorem 2.3. *Let $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$ form a regular pair satisfying (7). Furthermore, let $f \in C^{\nu}([0,T];\mathbb{R}^n)$ with $\nu = \operatorname{ind}(\mathcal{E}, \mathcal{A})$. Then every solution $x \in C^1([0,T];\mathbb{R}^n)$ of (4) has the form*

$$x(t) = \mathrm{e}^{-t\mathcal{E}^D\mathcal{A}}\mathcal{E}^D\mathcal{E}q + \int_0^t \mathrm{e}^{-(t-s)\mathcal{E}^D\mathcal{A}}\mathcal{E}^D f(s)\,ds + (I - \mathcal{E}^D\mathcal{E})\sum_{i=0}^{\nu-1}(-\mathcal{E}\mathcal{A}^D)^i\mathcal{A}^D f^{(i)}(t) \tag{8}$$

for some $q \in \mathbb{R}^n$.

Evaluating the solution formula at $t = 0$ one immediately gets consistency conditions for initial values.

Corollary 2.1. *Let the assumptions of Theorem 2.3 hold. The initial value problem consisting of (4) and (5) has a solution $x \in C^1([0,T];\mathbb{R}^n)$ if and only if there exists a vector $q \in \mathbb{R}^n$ with*

$$x^0 = \mathcal{E}^D\mathcal{E}q + (I - \mathcal{E}^D\mathcal{E})\sum_{i=0}^{\nu-1}(-\mathcal{E}\mathcal{A}^D)^i\mathcal{A}^D f^{(i)}(0). \tag{9}$$

If this is the case, then for every such q the solution is unique.

Remark 2.1. Corollary 2.1 gives consistency conditions for classical continuously differentiable solutions. By going over to weaker smoothness requirements for the solutions, these consistency conditions may be partially weakened, see [24, 30].

The commutativity requirement (7) is not really a restriction, since if $(\mathcal{E}, \mathcal{A})$ is regular and $\hat{\lambda} \in \mathbb{R}$ is chosen such that $\hat{\lambda}\mathcal{E} + \mathcal{A}$ is nonsingular, then

$$\hat{\mathcal{E}} = (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}\mathcal{E}, \quad \hat{\mathcal{A}} = (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}\mathcal{A}$$

commute. Since the factor $(\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}$ represents a simple scaling of (4) from the left by a nonsingular matrix, results analogous to Theorem 2.3 and Corollary 2.1 hold for the general case by setting

$$\mathcal{E} \leftarrow (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}\mathcal{E}, \quad \mathcal{A} \leftarrow (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}\mathcal{A}, \quad f \leftarrow (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1}f$$

in (8) and (9). It should also be noted that none of the solution formulas depends on the choice of the value $\hat{\lambda}$, see [24].

2.2. Well-posedness and explicit solution for DAEs of type (1)

In this subsection we specialize the results of Section 2.1 to the finite-dimensional version of the operator DAE (1). The associated matrix pair is then

$$(\mathcal{E}, \mathcal{A}) = \left(\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ -D^T & C \end{bmatrix} \right). \tag{10}$$

We assume that the pair arises from a reasonable discretization, i.e., a discretization which preserves the structure of the problem and which leads to a regular pair with the mass matrix M and the matrix $\mathcal{A}$ being invertible.

The latter condition, which will usually be satisfied in practice, is not really necessary for the analysis, but if this is not the case, then the presentation becomes rather technical. Since $\mathcal{A}$ is nonsingular, it follows that $\mathrm{ind}(\mathcal{E}) = \mathrm{ind}(\mathcal{E}, \mathcal{A})$ as well.

Typically, the matrix C in (10) is singular or even 0 depending on the discretization. If C were invertible, then we would immediately have that $\mathrm{ind}(\mathcal{E}, \mathcal{A}) = 1$, see, e.g., [24]. We also assume that B and D have full column rank. For the latter condition it is usually necessary to remove the freedom in the pressure by an extra condition or a factorization of the underlying function space [15, 21].

If C is singular, then let P_1 and P_2 be matrices, such that their columns span the nullspace of C and C^T, respectively. It is another reasonable assumption that $P_2^T D^T B P_1$ is square and nonsingular. Under this assumption we have that $\mathrm{ind}(\mathcal{E}, \mathcal{A}) = 2$, see, e.g., [24]. This holds for example in the particular case that $D = B$ has full column rank and $C = 0$, that we will study below in the infinite-dimensional case. We thus restrict our considerations to the case $\mathrm{ind}(\mathcal{E}) = \mathrm{ind}(\mathcal{E}, \mathcal{A}) \in \{1, 2\}$.

To apply the explicit solution formula to (1), we first need to pick a value $\hat{\lambda}$ such that $\hat{\lambda}\mathcal{E} + \mathcal{A}$ is invertible. Under the given assumptions, it is sufficient to pick $\hat{\lambda} \in \mathbb{R}$ such that $\hat{\lambda}M + A$ is nonsingular, which means that $\hat{\lambda}$ is not an eigenvalue of the (discretized) Laplace operator.

Introducing the Schur complement $S := C + D^T(\hat{\lambda}M + A)^{-1}B$, we obtain

$$\hat{\mathcal{E}} = \begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix} := \begin{bmatrix} \hat{\lambda}M + A & B \\ -D^T & C \end{bmatrix}^{-1} \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \tag{11}$$

and

$$\hat{\mathcal{A}} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix} := \begin{bmatrix} \hat{\lambda}M + A & B \\ -D^T & C \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ -D^T & C \end{bmatrix}, \tag{12}$$

with the following formula for the block inverse:

$$\begin{aligned} (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1} &= \begin{bmatrix} \hat{\lambda}M + A & B \\ -D^T & C \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (\hat{\lambda}M + A)^{-1} - (\hat{\lambda}M + A)^{-1}BS^{-1}D^T(\hat{\lambda}M + A)^{-1} & -(\hat{\lambda}M + A)^{-1}BS^{-1} \\ S^{-1}D^T(\hat{\lambda}M + A)^{-1} & S^{-1} \end{bmatrix}. \end{aligned}$$

We thus find

$$\begin{aligned} E_{11} &= (\hat{\lambda}M + A)^{-1}M - (\hat{\lambda}M + A)^{-1}BS^{-1}D^T(\hat{\lambda}M + A)^{-1}M \\ &= [I - (\hat{\lambda}M + A)^{-1}BS^{-1}D^T](\hat{\lambda}M + A)^{-1}M, \\ E_{21} &= S^{-1}D^T(\hat{\lambda}M + A)^{-1}M, \\ A_{11} &= (\hat{\lambda}M + A)^{-1}\big(A + BS^{-1}D^T[I - (\hat{\lambda}M + A)^{-1}A]\big), \\ A_{21} &= -S^{-1}D^T[I - (\hat{\lambda}M + A)^{-1}A]. \end{aligned}$$

Note that since both $\hat{\mathcal{E}}, \hat{\mathcal{A}}$ are block lower triangular and commute, also the blocks E_{11} and A_{11} commute. Note further that the state vector $[v^T, p^T]^T$ remains unchanged by this operation, while the inhomogeneity transforms to

$$\begin{bmatrix} \hat{f} \\ \hat{g} \end{bmatrix} := (\hat{\lambda}\mathcal{E} + \mathcal{A})^{-1} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \hat{\lambda}M + A & B \\ -D^T & C \end{bmatrix}^{-1} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} E_{11}M^{-1} & V_1 \\ E_{21}M^{-1} & V_2 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}, \tag{13}$$

with $V_1 = -(\hat{\lambda}M + A)^{-1}BS^{-1}$ and $V_2 = S^{-1}$.

We will now determine the Drazin inverse

$$\hat{\mathcal{E}}^D = X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

of $\hat{\mathcal{E}}$ and assume that it is partitioned analogous to $\hat{\mathcal{E}}$.

Since we have assumed that M is invertible, we automatically have that $\operatorname{rank}\hat{\mathcal{E}} = \operatorname{rank}\mathcal{E} = \operatorname{rank} M$. By (6a), we have that

$$\begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} X_{12} = 0.$$

This implies $X_{12} = 0$. From (6b) we then obtain immediately that $X_{22} = 0$ and thus $X_{11}E_{11}X_{11} = X_{11}$. We now make use of (6c) and use, for $j \geqslant 1$, the fact that we have

$$\begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix}^j = \begin{bmatrix} E_{11}^j & 0 \\ E_{21}E_{11}^{j-1} & 0 \end{bmatrix}.$$

Therefore, since $\nu = \operatorname{ind}\hat{\mathcal{E}} \geqslant 1$ ($\mathcal{E}$ and thus $\hat{\mathcal{E}}$ is singular) we have the following equations for X_{11} and X_{21}:

$$E_{11}X_{11} = X_{11}E_{11}, \tag{14a}$$
$$X_{11} = X_{11}E_{11}X_{11}, \tag{14b}$$
$$X_{11}E_{11}^{\nu+1} = E_{11}^{\nu}, \tag{14c}$$
$$X_{21}E_{11}^{\nu+1} = E_{21}E_{11}^{\nu-1}, \tag{14d}$$
$$E_{21}X_{11} = X_{21}E_{11}, \tag{14e}$$
$$X_{21}E_{11}X_{11} = X_{21}. \tag{14f}$$

From (14), we have immediately that $X_{21} = E_{21}X_{11}^2$ and thus it is sufficient to determine X_{11}. For this we need information on the index of E_{11}, which is given by the following lemma.

Lemma 2.1. *Consider the pair* (10) *and the transformed pair* $(\hat{\mathcal{E}}, \hat{\mathcal{A}})$ *of* (11)–(12). *Then either*

$$\operatorname{ind}(\hat{\mathcal{E}}) = \operatorname{ind}(\hat{\mathcal{E}}, \hat{\mathcal{A}}) = \operatorname{ind}(E_{11}, A_{11})$$

or

$$\operatorname{ind}(\hat{\mathcal{E}}) = \operatorname{ind}(\hat{\mathcal{E}}, \hat{\mathcal{A}}) = \operatorname{ind}(E_{11}, A_{11}) + 1.$$

Proof. Since $\hat{\mathcal{A}}$ and hence also A_{11} is invertible, we have that $\operatorname{ind}\hat{\mathcal{E}} = \operatorname{ind}(\hat{\mathcal{E}}, \hat{\mathcal{A}})$ and $\operatorname{ind} E_{11} = \operatorname{ind}(E_{11}, A_{11})$. Using that

$$(\hat{\mathcal{E}}, \hat{\mathcal{A}}) = \left(\begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix} \right),$$

and that the Weierstrass canonical form (see Theorem 2.1) of the regular pair (E_{11}, A_{11}) is given by

$$\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$$

with a nilpotent matrix N of nilpotency index $\operatorname{ind} E_{11}$, by simple algebraic manipulations we obtain that

$$\left(\begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix} \right)$$

is equivalent to

$$\left(\begin{bmatrix} I & 0 & 0 \\ 0 & N & 0 \\ 0 & \tilde{E}_{32} & 0 \end{bmatrix}, \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right).$$

Since

$$\begin{bmatrix} N & 0 \\ \tilde{E}_{32} & 0 \end{bmatrix}^j = \begin{bmatrix} N^j & 0 \\ \tilde{E}_{32}N^{j-1} & 0 \end{bmatrix},$$

the nilpotency index of

$$\begin{bmatrix} N & 0 \\ \tilde{E}_{32} & 0 \end{bmatrix}$$

and hence the index of the pair (E_{11}, A_{11}) can increase at most by 1. □

Since for a reasonable discretization $\operatorname{ind}(\mathcal{E}) = \operatorname{ind}(\mathcal{E}, \mathcal{A}) \in \{1, 2\}$, we can assume that $1 \leqslant \operatorname{ind}(E_{11}) \leqslant 2$. Thus, we obtain the following explicit formulas for $\mathcal{E}^D$ in terms of E_{11}^D.

Lemma 2.2. *Consider the pair* (10) *with* $\operatorname{ind}(\mathcal{E}, \mathcal{A}) \leqslant 2$ *and the transformed pair* $(\hat{\mathcal{E}}, \hat{\mathcal{A}})$ *of* (11)–(12)*. Then the following solution formulas hold.*

$$\hat{\mathcal{E}}^D = \begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix}^D = \begin{bmatrix} E_{11}^D & 0 \\ E_{21}(E_{11}^D)^2 & 0 \end{bmatrix}, \tag{15a}$$

$$\hat{\mathcal{A}}^D = \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}, \tag{15b}$$

$$\hat{\mathcal{E}}^D \hat{\mathcal{A}} = \begin{bmatrix} E_{11}^D A_{11} & 0 \\ E_{21}(E_{11}^D)^2 A_{11} & 0 \end{bmatrix}, \tag{15c}$$

$$\hat{\mathcal{E}}^D \hat{\mathcal{E}} = \begin{bmatrix} E_{11}^D E_{11} & 0 \\ E_{21} E_{11}^D & 0 \end{bmatrix}, \tag{15d}$$

$$I - \hat{\mathcal{E}}^D \hat{\mathcal{E}} = \begin{bmatrix} I - E_{11}^D E_{11} & 0 \\ -E_{21} E_{11}^D & I \end{bmatrix} \tag{15e}$$

$$\hat{\mathcal{E}} \hat{\mathcal{A}}^D = \begin{bmatrix} E_{11} A_{11}^{-1} & 0 \\ E_{21} A_{11}^{-1} & 0 \end{bmatrix}. \tag{15f}$$

Proof. To prove (15a), let $\alpha = \operatorname{ind}(E_{11})$, then by Lemma 2.1 we have that $\alpha = \nu$ or $\alpha = \nu - 1$. If $\alpha = \nu$, then by (14a)–(14c) it follows that $X_{11} = E_{11}^D$ and then $X_{21} = E_{21} X_{11}^2 = E_{21}(E_{11}^D)^2$.

If $\alpha = \nu - 1$ then choosing $X_{11} = E_{11}^D$ and $X_{21} = E_{21}(E_{11}^D)^2$, the relations (14a)–(14b) hold automatically and from $X_{11} E_{11}^{\alpha+1} = E_{11}^{\alpha}$, then (14c) follows by multiplying with E_{11} from the right.

Equations (14d)–(14f) read as

$$\begin{aligned} E_{21}(E_{11}^D)^2 E_{11}^{\nu+1} &= E_{21} E_{11}^{\nu-1}, \\ E_{21} E_{11}^D &= E_{21}(E_{11}^D)^2 E_{11}, \\ E_{21}(E_{11}^D)^2 E_{11} E_{11}^D &= E_{21}(E_{11}^D)^2. \end{aligned}$$

Since $1 \leqslant \nu = \operatorname{ind}(\hat{\mathcal{E}}) \leqslant 2$, and since E_{11}^D and E_{11} commute, these equations are satisfied.

The assertion then follows by the uniqueness of the Drazin inverse. The other parts follow trivially. □

Note that whenever $\nu = \operatorname{ind}(\mathcal{E}, \mathcal{A}) \leqslant 2$, it follows from (14d) that

$$E_{21} E_{11}^D E_{11}^{\nu} = E_{21} E_{11}^{\nu-1},$$

i.e., in the range of E_{21}, the matrix E_{11} behaves like it is of index $\nu - 1$.

An immediate consequence is the following result on the well-posedness and explicit representation of the solution.

Theorem 2.4. *Consider the differential-algebraic equation* (1)*, correspondingly* (10)*, with an invertible mass matrix* M*, an invertible matrix* $\mathcal{A}$ *and sufficiently smooth inhomogeneities* $\hat{f}, \hat{g}$ *as in* (13)*. Let* $\nu = \operatorname{ind}(\mathcal{E}, \mathcal{A}) \in \{1, 2\}$*. Then for any consistent initial condition*

$$\begin{bmatrix} v(0) \\ p(0) \end{bmatrix} = \begin{bmatrix} v^0 \\ p^0 \end{bmatrix},$$

there exists a unique classical solution to the initial value problem (1)–(2) *given by*

$$\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \exp(-tE_{11}^D A_{11})\tilde{q} \\ E_{21}E_{11}^D \exp(-tE_{11}^D A_{11})\tilde{q} \end{bmatrix} + \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^D A_{11})E_{11}^D \hat{f}(s) \\ E_{21}E_{11}^D \exp(-(t-s)E_{11}^D A_{11})E_{11}^D \hat{f}(s) \end{bmatrix} ds$$
$$+ \begin{bmatrix} (I - E_{11}^D E_{11})A_{11}^{-1}\hat{f}(t) \\ (-E_{21}E_{11}^D A_{11}^{-1} - A_{21}A_{11}^{-1})\hat{f}(t) + \hat{g}(t) \end{bmatrix}$$
$$+ \sum_{i=1}^{\nu-1}(-1)^i \begin{bmatrix} (I - E_{11}^D E_{11})(E_{11}A_{11}^{-1})^i A_{11}^{-1}\hat{f}^{(i)}(t) \\ (-E_{21}E_{11}^D (E_{11}A_{11}^{-1})^i A_{11}^{-1} + E_{21}A_{11}^{-1}(E_{11}A_{11}^{-1})^{i-1}A_{11}^{-1})\hat{f}^{(i)}(t) \end{bmatrix},$$

where $\tilde{q}$ *is a constant vector. An initial condition is consistent if the linear system*

$$\begin{bmatrix} I \\ E_{21}E_{11}^D \end{bmatrix} \tilde{q} = \begin{bmatrix} v^0 \\ p^0 \end{bmatrix} - \begin{bmatrix} I - E_{11}E_{11}^D & 0 \\ -E_{21}E_{11}^D & I \end{bmatrix} \sum_{i=0}^{\nu-1}(-1)^i \begin{bmatrix} E_{11}A_{11}^{-1} & 0 \\ E_{21}A_{11}^{-1} & 0 \end{bmatrix}^i \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{f}^{(i)}(0) \\ \hat{g}^{(i)}(0) \end{bmatrix}$$

has a solution.

Proof. According to (9) let

$$\begin{bmatrix} \tilde{q} \\ \tilde{r} \end{bmatrix} := \hat{\mathcal{E}}^D \hat{\mathcal{E}} q.$$

Applying Theorem 2.3, we obtain the following explicit solution of (1):

$$\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} = \exp\left(-t \begin{bmatrix} E_{11}^D A_{11} & 0 \\ E_{21}(E_{11}^D)^2 A_{11} & 0 \end{bmatrix}\right) \begin{bmatrix} \tilde{q} \\ \tilde{r} \end{bmatrix}$$
$$+ \int_0^t \exp\left(-(t-s) \begin{bmatrix} E_{11}^D A_{11} & 0 \\ E_{21}(E_{11}^D)^2 A_{11} & 0 \end{bmatrix}\right) \begin{bmatrix} E_{11}^D & 0 \\ E_{21}(E_{11}^D)^2 & 0 \end{bmatrix} \begin{bmatrix} \hat{f}(s) \\ \hat{g}(s) \end{bmatrix} ds$$
$$+ \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} E_{11}^D E_{11} & 0 \\ E_{21}E_{11}^D & 0 \end{bmatrix}\right) \sum_{i=0}^{\nu-1}\left(-\begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}\right)^i$$
$$\times \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{f}^{(i)}(t) \\ \hat{g}^{(i)}(t) \end{bmatrix}$$
$$= \exp\left(-t \begin{bmatrix} E_{11}^D A_{11} & 0 \\ E_{21}(E_{11}^D)^2 A_{11} & 0 \end{bmatrix}\right) \begin{bmatrix} \tilde{q} \\ \tilde{r} \end{bmatrix}$$
$$+ \int_0^t \exp\left(-(t-s) \begin{bmatrix} E_{11}^D A_{11} & 0 \\ E_{21}(E_{11}^D)^2 A_{11} & 0 \end{bmatrix}\right) \begin{bmatrix} E_{11}^D \hat{f}(s) \\ E_{21}(E_{11}^D)^2 \hat{f}(s) \end{bmatrix} ds$$
$$+ \begin{bmatrix} (I - E_{11}^D E_{11})A_{11}^{-1}\hat{f}(t) \\ (-E_{21}E_{11}^D A_{11}^{-1} - A_{21}A_{11}^{-1})\hat{f}(t) + \hat{g}(t) \end{bmatrix}$$
$$+ \sum_{i=1}^{\nu-1}(-1)^i \begin{bmatrix} (I - E_{11}^D E_{11})(E_{11}A_{11}^{-1})^i A_{11}^{-1}\hat{f}^{(i)}(t) \\ (-E_{21}E_{11}^D (E_{11}A_{11}^{-1})^i A_{11}^{-1} + E_{21}A_{11}^{-1}(E_{11}A_{11}^{-1})^{i-1}A_{11}^{-1})\hat{f}^{(i)}(t) \end{bmatrix}.$$

Since with $Z = -E_{11}^D A_{11}$, $Y = E_{21}E_{11}^D$, we have

$$\exp\left(t \begin{bmatrix} Z & 0 \\ YZ & 0 \end{bmatrix}\right) = \sum_{i=0}^{\infty} \frac{1}{i!}\left(t \begin{bmatrix} Z & 0 \\ YZ & 0 \end{bmatrix}\right)^i = \sum_{i=0}^{\infty} \frac{t^i}{i!} \begin{bmatrix} Z^i & 0 \\ YZ^i & 0 \end{bmatrix} = \begin{bmatrix} \exp(tZ) & 0 \\ Y\exp(tZ) & 0 \end{bmatrix},$$

we can simplify the exponential functions, and since we have from the commutativity of $\hat{\mathcal{E}}$ and $\hat{\mathcal{A}}$ that also E_{11}^D, E_{11}, and A_{11} commute, the result follows. The consistency condition follows by inserting $t = 0$. □

2.3. Some special cases

In the solution formula we may consider several simplifying cases.

If E_{11} is invertible, which by Lemma 2.1 can only happen if $\nu = 1$, then $I - E_{11}^D E_{11} = 0$, and thus we obtain

$$\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \exp(-tE_{11}^{-1}A_{11})\tilde{q} \\ E_{21}E_{11}^{-1}\exp(-tE_{11}^{-1}A_{11})\tilde{q} \end{bmatrix} + \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^{-1}A_{11})E_{11}^{-1}\hat{f}(s) \\ E_{21}E_{11}^{-1}\exp(-(t-s)E_{11}^{-1}A_{11})E_{11}^{-1}\hat{f}(s) \end{bmatrix} ds$$
$$+ \begin{bmatrix} 0 \\ (-E_{21}E_{11}^{-1}A_{11}^{-1} - A_{21}A_{11}^{-1})\hat{f}(t) + \hat{g}(t) \end{bmatrix}. \tag{16}$$

In this case from the first equation we obtain $\tilde{q} = v^0$ and thus the consistency condition is

$$p^0 = E_{21}E_{11}^{-1}v^0 + (-E_{21}E_{11}^{-1}A_{11}^{-1} - A_{21}A_{11}^{-1})\hat{f}(0) + \hat{g}(0). \tag{17}$$

This means that if v^0 is given, then p^0 is fixed in an easy way, but we could also fix p^0 and then both equations together give a consistency condition for v^0.

If we have (as will typically be the case) that $\nu = 2$ and $\operatorname{ind}(E_{11}) = 1$, then by (6c) we have that $E_{11}^D E_{11}^2 = E_{11}$ and hence the formulas simplify to

$$\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \exp(-tE_{11}^{D}A_{11})\tilde{q} \\ E_{21}E_{11}^{D}\exp(-tE_{11}^{D}A_{11})\tilde{q} \end{bmatrix} + \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^{D}A_{11})E_{11}^{D}\hat{f}(s) \\ E_{21}E_{11}^{D}\exp(-(t-s)E_{11}^{D}A_{11})E_{11}^{D}\hat{f}(s) \end{bmatrix} ds$$
$$+ \begin{bmatrix} (I - E_{11}^D E_{11})A_{11}^{-1}\hat{f}(t) \\ (-E_{21}E_{11}^{D}A_{11}^{-1} - A_{21}A_{11}^{-1})\hat{f}(t) + \hat{g}(t) \end{bmatrix}$$
$$- \begin{bmatrix} 0 \\ (-E_{21}E_{11}^{D}E_{11}A_{11}^{-2} + E_{21}A_{11}^{-2})\dot{\hat{f}}(t) \end{bmatrix}. \tag{18}$$

This again gives an algebraic relationship between v^0 and p^0 and again by choosing v^0 we obtain

$$\tilde{q} = v^0 - (I - E_{11}^D E_{11})A_{11}^{-1}\hat{f}(0) \tag{19}$$

and this then fixes p^0 uniquely. We could also again fix p^0 and then both equations together give a consistency condition for v^0.

We emphasize that the derivative of $\hat{f}$ but not that of $\hat{g}$ appears. Looking in detail at the last term and using (13), which implies that

$$\dot{\hat{f}} = E_{11}M^{-1}\dot{f} + V_1\dot{g},$$

we see by (6c) that the factor of $\dot{f}$ in (18) vanishes, while the factor of $\dot{g}$ may not be zero if E_{11} is not invertible.

If, however, there is no inhomogeneity g, then it follows that whenever E_{11} is of index 1, then the last term vanishes, thus despite the fact that $\nu = 2$, no derivative of f occurs, i.e., the system behaves somewhat like a system with $\nu = 1$.

Note that an analogous argument holds for systems of this form with $\nu > 2$ and $\operatorname{ind}(E_{11}) = \nu - 1$, because also then the coefficient of the highest derivative $f^{(\nu-1)}$ is zero.

Let us now consider the even more special case that $M = I$, A is invertible, $B = D$ has full column rank and $C = 0$ and that we choose $\hat{\lambda} = 0$. This case will be studied in the infinite-dimensional setting.

In this situation, the Schur complement is given by $S = B^T A^{-1} B$, and we have

$$E_{11} = A^{-1} - A^{-1}BS^{-1}B^TA^{-1}, \quad E_{21} = S^{-1}B^TA^{-1}, \quad A_{11} = I, \quad A_{21} = 0.$$

Then it is well known that $\nu = 2$, and for $\operatorname{ind} E_{11} = 1$ we obtain the solution

$$\begin{aligned}\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} &= \begin{bmatrix} \exp(-tE_{11}^D)\tilde{q} \\ E_{21}E_{11}^D \exp(-tE_{11}^D)\tilde{q} \end{bmatrix} \\ &+ \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^D)E_{11}^D(E_{11}f(s) - A^{-1}BS^{-1}g(s)) \\ E_{21}E_{11}^D \exp(-(t-s)E_{11}^D)E_{11}^D(E_{11}f(s) - A^{-1}BS^{-1}g(s)) \end{bmatrix} ds \\ &+ \begin{bmatrix} -(I - E_{11}^D E_{11})A^{-1}BS^{-1}g(t) \\ E_{21}(I - E_{11}^D E_{11})f(t) + (I + E_{21}E_{11}^D A^{-1}B)S^{-1}g(t)) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ E_{21}(I - E_{11}^D E_{11})A^{-1}BS^{-1}\dot{g}(t) \end{bmatrix}, \end{aligned} \tag{20}$$

where we have used again that $(E_{11}^D)^2 E_{11} = E_{11}^D$, $E_{11}^D E_{11}^3 = E_{11}^2$, as well as $E_{21}E_{11}^D E_{11}^2 = E_{21}E_{11}$.

In this case it is very easy to determine consistent initial conditions. Rather than going through the solution formula, the second equation of (1) immediately gives the consistency condition

$$0 = B^T v^0 + g(0). \tag{21}$$

This is exactly the consistency condition that will also appear in the infinite-dimensional case, while directly in the system no consistency condition for p^0 arises. However, a differentiation with respect to t of the second equation of (1) and insertion of the first equation gives

$$B^T B p(t) = -B^T A v(t) + B^T f(t) + \dot{g}(t), \tag{22}$$

which corresponds to the Poisson problem for the pressure that is typically used to solve for the pressure.

Evaluating (22) at $t = 0$ we get the consistency system for the initial values

$$\begin{bmatrix} B^T A & B^T B \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} v^0 \\ p^0 \end{bmatrix} = \begin{bmatrix} B^T f(0) + \dot{g}(0) \\ g(0) \end{bmatrix}. \tag{23}$$

Note that the invertibility of $B^T B$ allows to solve for p^0 in terms of v^0.

The initial condition is indeed consistent (see Theorem 2.4) if we can find a vector $\tilde{q}$ such that (20) is satisfied at $t = 0$. By taking

$$\tilde{q} = v^0 + (I - E_{11}^D E_{11})A^{-1}BS^{-1}g(0), \tag{24}$$

the first equation of (20) at $t = 0$ is automatically satisfied. Inserting (24) into the second equation of (20) at $t = 0$ and employing (21) as well as (22) at $t = 0$, a straightforward calculation shows that also this second equation is fulfilled.

This, finally, proves that (23) is a sufficient as well as necessary condition for the solvability, in the classical sense, of the initial value problem under consideration.

Besides the presented explicit solution formula, there are many different routes that one takes in the study of operator DAEs. One approach for index reduction that is closely related to the treatment in the infinite-dimensional case is to introduce a minimal number of new variables [23, 24].

Let us introduce the primitive of p as a new variable w and introduce as further equation $\dot{w} = p$. Inserting this into the operator DAE (1), we get

$$\begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} A & 0 \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{25}$$

Using the fact that $B^T B$ is invertible, a simple calculation shows that system (25) has differentiation index $\nu = 1$. This follows by the change of variables

$$\begin{bmatrix} v_r \\ w_r \end{bmatrix} = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$

which gives the system

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_r \\ \dot{w}_r \end{bmatrix} + \begin{bmatrix} A & -AB \\ -B^T & B^T B \end{bmatrix} \begin{bmatrix} v_r \\ w_r \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}.$$

The only consistency condition for the initial value is

$$-B^T v(0) = g(0), \tag{26}$$

which is indeed the same as (21).

Finally we also briefly discuss a variable splitting approach. Let $P = P^T$ be a projector onto the nullspace of B^T, i.e., satisfying $B^T P = 0$. This can be easily obtained from a singular value decomposition of B^T. Split v as $v_0 + v_g := Pv + (I - P)v$.

Multiplying the first equation of (1) from the left by $P = P^T$, and using that $PB = P^T B = (B^T P)^T = 0$, we obtain the system

$$P\dot{v}_0 + P\dot{v}_g + PAv_0 + PAv_g = Pf,$$

and hence

$$\dot{v}_0 + PAv_0 = Pf - P\dot{v}_g - PAv_g, \tag{27}$$

which corresponds to (43) below. Here the term $P\dot{v}_g$ vanishes since $P(I - P) = 0$. The second equation of (1) gives the constraint

$$g = -B^T v = -B^T (I - P)v = -B^T (I - P)(I - P)v = -B^T (I - P)v_g.$$

Since $-B^T(I - P)$ is invertible on the subspace, we get $v_g = -(B^T(I - P))^{-1} g$. This implies a consistency condition for the initial value and can be inserted in the first equation to obtain a differential equation

$$\dot{v}_0 + PAv_0 = Pf + PA(B^T(I - P))^{-1} g$$

for v_0. Note that the derivative of g again occurs in the equation for p as in (22).

In this section we have derived explicit formulas for the finite-dimensional operator DAE (1), which includes the cases of semi-discretized (in space) Oseen, Stokes and linearized Navier–Stokes equations. We have considered the cases shown in Table 1.

Situation	Solution given by	Consistency condition given by
$\operatorname{ind}(E_{11}) = 0$, $\nu = 1$	(16) no derivative of f or g occurs	(17)
$\operatorname{ind}(E_{11}) = 1$, $\nu = 2$	(18) no derivative of f occurs, but of g (in the equation for p)	(19)
$\operatorname{ind}(E_{11}) = 1$, $\nu = 2$, $M = I$, $\operatorname{ind}(A) = 0$, $C = 0$, $B = D$ has full column rank	(20) no derivative of f occurs, but of g (in the equation for p)	(23)
system (25)		(26)

Table 1. Some special cases.

2.4. The symmetric case

As already noted, in general we do not know the index of E_{11}. However, if the discretization is such that M, A, C are symmetric and $D = B$, then we have the following lemma.

Lemma 2.3. *Consider the coefficient matrices in* (1) *and the transformed coefficients in* (11)–(12). *If A, C, M are symmetric, if $D = B$, and if $\hat{\lambda}$ is chosen so that $W = \hat{\lambda}M + A$ is positive (or negative) definite, then* $\operatorname{ind}(E_{11}) \leqslant 1$.

Proof. Under the given assumptions we have that

$$E_{11} = W^{-1} - W^{-1}BS^{-1}B^TW^{-1}.$$

If W is positive definite then let $Z = W^{-1/2}W^{-1/2}$, where $W^{-1/2}$ denotes the positive definite square root of W^{-1}. Then it follows that

$$E_{11} = W^{-1/2}Z^{-1/2}[Z - Z^{1/2}W^{-1/2}BS^{-1}B^TW^{-1/2}Z^{1/2}]Z^{1/2}W^{1/2},$$

i.e., E_{11} is similar to a symmetric matrix, which has index less than or equal to 1 and hence $\operatorname{ind}(E_{11}) \leqslant 1$. If W is negative definite then the same proof follows by replacing W by $-W$. □

We can directly apply Lemma 2.3 if A is definite and if we choose $\hat{\lambda} = 0$. Then we obtain the simplified formulas

$$A_{11} = I, \quad A_{21} = 0, \quad E_{11} = I - A^{-1}BS^{-1}B^TA^{-1}M, \quad E_{21} = S^{-1}B^TA^{-1}M.$$

Let us assume that A is positive definite, the same result follows if A is negative definite by replacing A with $-A$.

Let $A = LL^T$ be the Cholesky decomposition (we could also take the positive square root $L = A^{1/2}$), let $\tilde{B} = L^{-1}B$ and let

$$\begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q^T\tilde{B}$$

be a QR decomposition [16], with Q orthogonal and R_1 upper triangular. By the assumption that B has full column rank it follows that R_1 is invertible. We then have

$$E_{11} = L^{-T}Q\left(I - \begin{bmatrix} R_1 \\ 0 \end{bmatrix}(C + R_1^TR_1)^{-1}\begin{bmatrix} R_1^T & 0 \end{bmatrix}\right)Q^TL^{-1}M = L^{-T}Q\begin{bmatrix} Z_1 & 0 \\ 0 & I \end{bmatrix}Q^TL^{-1}M,$$

with $Z_1 = I - R_1(C + R_1^T R_1)^{-1} R_1^T$. (In the special case that $C = 0$, we then have $Z_1 = 0$). In the same way it follows that

$$E_{21} = R_1^{-1} \begin{bmatrix} I - Z_1 & 0 \end{bmatrix} Q^T L^{-1} M.$$

Checking the conditions for E_{11}^D, we immediately have that

$$\begin{aligned} E_{11}^D &= M^{-1} L Q \begin{bmatrix} Z_1^D & 0 \\ 0 & I \end{bmatrix} Q^T L^T, \\ E_{11} E_{11}^D &= L^{-T} Q \begin{bmatrix} Z_1 Z_1^D & 0 \\ 0 & I \end{bmatrix} Q^T L^T, \\ E_{21} E_{11}^D &= R_1^{-1} \begin{bmatrix} (I - Z_1) Z_1^D & 0 \end{bmatrix} Q^T L^T, \end{aligned}$$

and we can insert this in (18) and obtain with $\tilde{q} = v^0$ that

$$\begin{aligned} \begin{bmatrix} v(t) \\ p(t) \end{bmatrix} &= \begin{bmatrix} \exp(-tE_{11}^D)\tilde{q} \\ E_{21} E_{11}^D \exp(-tE_{11}^D)\tilde{q} \end{bmatrix} \\ &+ \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^D) E_{11}^D (E_{11} M^{-1} f(s) - A^{-1} B S^{-1} g(s))\, ds \\ E_{21} E_{11}^D \exp(-(t-s)E_{11}^D) E_{11}^D (E_{11} M^{-1} f(s) - A^{-1} B S^{-1} g(s)) \end{bmatrix} ds \\ &+ \begin{bmatrix} (I - E_{11}^D E_{11})(E_{11} M^{-1} f(t) - A^{-1} B S^{-1} g(t)) \\ E_{21}(I - E_{11}^D E_{11}) M^{-1} f(t) + (S^{-1} + E_{21} E_{11}^D A^{-1} B S^{-1}) g(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ E_{21}(I - E_{11}^D E_{11}) A^{-1} B S^{-1} \dot{g}(t) \end{bmatrix}. \end{aligned}$$

If $C = 0$, then the formulas become even more simple, since then

$$\begin{aligned} E_{11} &= L^{-T} Q \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Q^T L^{-1} M, \\ E_{11}^D &= M^{-1} L Q \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Q^T L^T, \\ E_{21} &= R_1^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q^T L^{-1} M, \end{aligned}$$

and from this we see immediately that $E_{21} E_{11}^D = 0$ and hence

$$\begin{aligned} \begin{bmatrix} v(t) \\ p(t) \end{bmatrix} &= \begin{bmatrix} \exp(-tE_{11}^D)\tilde{q} \\ 0 \end{bmatrix} \\ &+ \int_0^t \begin{bmatrix} \exp(-(t-s)E_{11}^D) E_{11}^D (E_{11} M^{-1} f(s) - A^{-1} B S^{-1} g(s)) \\ 0 \end{bmatrix} ds \\ &+ \begin{bmatrix} (I - E_{11}^D E_{11})(E_{11} M^{-1} f(t) - A^{-1} B S^{-1} g(t)) \\ E_{21} M^{-1} f(t) + S^{-1} g(t) + E_{21} A^{-1} B S^{-1} \dot{g}(t) \end{bmatrix}. \end{aligned}$$

In particular we have an explicit formula for the pressure as

$$p(t) = S^{-1} B^T A^{-1} f(t) + S^{-1} g(t) + S^{-1} B^T A^{-1} M A^{-1} B S^{-1} \dot{g}(t).$$

As a summary of the findings, we see that derivatives of the inhomogeneity f do not occur, whenever we have $\operatorname{ind}(\mathcal{E}, \mathcal{A}) \leqslant 2$ but an inhomogeneity g will have to be at least differentiable.

3. The infinite-dimensional case

The aim of this section is to provide results on the well-posedness of (1) in the infinite-dimensional case where A, B, C, D, M are suitable linear operators acting on a Hilbert space. We are not going to deal with the problem in its full generality but rather concentrate on some simplifying cases and we consider weak solutions, i.e., we interpret the time derivative in the generalized or distributional sense. For the discussed special cases, we again derive an explicit solution formula that is of the same form as in the finite-dimensional setting.

Results on the well-posedness of abstract DAEs have recently been obtained in [25]. In particular, existence of weak solutions is shown for a class of linear abstract ODEs. The class of equations considered in [25], however, does not include the class of equations we study in this paper. A different approach to a class of evolution equations which contain block operators as in (1) and which are reduced to closed subspaces can be found in [28].

3.1. Preliminaries

In what follows, we recall known results on stationary and non-stationary problems.

3.1.1. Stationary problems In this subsection we prepare our study of operator DAEs by recalling the results known for *stationary mixed variational problems*. Let X and Y be real Banach spaces. By $\mathcal{L}(X, Y)$ we denote the space of bounded linear operators mapping X into Y. The dual space of X is denoted by $X^* = \mathcal{L}(X, \mathbb{R})$ and equipped with the standard norm. The application of $z \in X^*$ on $w \in X$ is always denoted by $\langle z, w\rangle$. By $\|\cdot\|_X$ we denote the norm in X.

Let V and Q be real Hilbert spaces and let

$$a : V \times V \to \mathbb{R}, \quad b, d : Q \times V \to \mathbb{R}, \quad c : Q \times Q \to \mathbb{R}$$

be bounded bilinear forms with norms $\|a\|$, $\|b\|$, $\|d\|$, $\|c\|$, and with the associated operators

$$A \in \mathcal{L}(V, V^*), \quad B, D \in \mathcal{L}(Q, V^*), \quad C \in \mathcal{L}(Q, Q^*).$$

We consider the following mixed variational (generalized saddle-point) problem:

For given $(f, g) \in V^* \times Q^*$ find $(v, p) \in V \times Q$ such that

$$\begin{aligned} a(v, w) + b(p, w) &= \langle f, w\rangle \quad \text{for all } w \in V, \\ -d(q, v) + c(p, q) &= \langle g, q\rangle \quad \text{for all } q \in Q. \end{aligned} \tag{28}$$

This problem is equivalent to finding a solution to the operator equation

$$\begin{bmatrix} A & B \\ -D^T & C \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{in } V^* \times Q^*,$$

where $D^T \in \mathcal{L}(V, Q^*)$ denotes the dual operator of D. Such problems and their approximation have been studied extensively in the literature; for an overview, we refer to [3, 6, 13, 15, 18, 34].

Let us consider the case with $d = b$ and $c = 0$. For $g \in Q^*$, we introduce the affine subspace

$$V_g := \{w \in V : -b(q, w) = \langle g, q\rangle \text{ for all } q \in Q\}.$$

So $w \in V_g$ if and only if $-B^T w = g$ in Q^*. Note that $V_0 = \operatorname{kernel} B^T$ is closed, since B^T is continuous. We then decompose V as $V = V_0 \oplus V_\perp$, where $V_\perp := (V_0)^\perp$ denotes the orthogonal complement of V_0 in V. Let $j_V : V \to V^*$ be the Riesz isomorphism [5]. Then (see, e.g., [18])

$$\begin{aligned} j_V(V_0) = V_0^* &:= \big\{z \in V^* : \langle z, w\rangle = 0 \text{ for all } w \in V_\perp\big\}, \\ j_V(V_\perp) = V_\perp^* &:= \big\{z \in V^* : \langle z, w\rangle = 0 \text{ for all } w \in V_0\big\}, \\ V^* &= V_0^* \oplus V_\perp^*, \end{aligned}$$

and V_0^* is orthogonal to $V_\perp^*$ in V^*. Note that one has to distinguish between the annihilator V_0^*, which is a subspace of V^* equipped with the norm $\|\cdot\|_{V^*}$, and the dual space $(V_0)^*$, which has a norm weaker than $\|\cdot\|_{V^*}$, since $V_0 \subset V$. The same applies to $V_\perp^*$ and $(V_\perp)^*$.

Any linear functional that is bounded on V is also bounded on V_0, but there is no injection of V^* into $(V_0)^*$, since V_0 is not dense in V (except if $B = 0$ and $V_0 = V$). This is a consequence of the Hahn–Banach theorem, see, e.g., [5]. However, V_0^* and $(V_0)^*$ are isometrically isomorphic, since, in particular, with each $z \in (V_0)^*$ we can associate a functional $\tilde{z} \in V^*$ by means of

$$\langle \tilde{z}, w\rangle = \langle z, w_0\rangle \quad \text{for all } w \in V, \tag{29}$$

where w_0 denotes the orthogonal projection of $w \in V$ onto V_0. We find that $\tilde{z}$ is an element of V_0^*. It can easily be shown that

$$\|\tilde{z}\|_{V^*} = \|z\|_{(V_0)^*}.$$

Conversely, with each $\tilde{z} \in V_0^* \subset V^*$ (note that $\tilde{z}$ vanishes on $V_\perp$) we associate the restriction $z = \tilde{z}|_{V_0} \in (V_0)^*$.

If $\pi : V \to V_0$ denotes the orthogonal projection of V onto V_0 then the mapping $(V_0)^* \ni z \mapsto \tilde{z} \in V_0^* \subset V^*$ given by (29) is just the dual operator $\pi^T : (V_0)^* \to V^*$, i.e., $\tilde{z} = \pi^T z$. Moreover, π^T has range $\pi^T((V_0)^*) = V_0^*$ and the mapping $V_0^* \ni \tilde{z} \mapsto z = \tilde{z}|_{V_0} \in (V_0)^*$ is the inverse of $\pi^T : (V_0)^* \to V_0^*$. So $\pi^T : (V_0)^* \to V_0^*$ becomes an isometric isomorphism.

Similarly, one can show that $V_\perp^*$ and $(V_\perp)^*$ are isometrically isomorphic, see, e.g., [15].

The existence theorem for the stationary problem (28) is then as follows, see, e.g., [15,18].

Theorem 3.1. *Let $a : V \times V \to \mathbb{R}$ and $b : Q \times V \to \mathbb{R}$ be bounded bilinear forms. For any $(f, g) \in V^* \times Q^*$ there exists a unique solution $(v, p) \in V \times Q$ to (28) if and only if*

(i) *there exists a constant $\mu > 0$ such that*

$$\inf_{v \in V_0\setminus\{0\}} \sup_{w \in V_0\setminus\{0\}} \frac{|a(v,w)|}{\|v\|_V \|w\|_V} \geqslant \mu, \qquad \inf_{w \in V_0\setminus\{0\}} \sup_{v \in V_0\setminus\{0\}} \frac{|a(v,w)|}{\|v\|_V \|w\|_V} \geqslant \mu, \tag{30}$$

(ii) *there exists a constant $\gamma > 0$ such that*

$$\inf_{q \in Q\setminus\{0\}} \sup_{w \in V\setminus\{0\}} \frac{b(q,w)}{\|q\|_Q \|w\|_V} \geqslant \gamma. \tag{31}$$

The solution (v, p) then satisfies the a priori estimate

$$\|v\|_V + \|p\|_Q \leqslant C\big(\mu^{-1}, \gamma^{-1}, \|a\|\big)\big(\|f\|_{V^*} + \|g\|_{Q^*}\big),$$

where $C(\cdot)$ is bounded on bounded subsets.

The inf-sup conditions (30) for the bilinear form a ensure that the operator A is an isomorphism as a mapping of V_0 into $(V_0)^*$, i.e., for each $z \in (V_0)^*$ there is a unique $u_z \in V_0$ such that $Au_z = z$ in $(V_0)^*$ (and not necessarily in V^* as $A : V \to V^*$ may not be invertible). To be precise, we may introduce the operator $A_0 : V_0 \to (V_0)^*$ associated with the restriction of $a : V \times V \to \mathbb{R}$ to $V_0 \times V_0$. Then (30) is equivalent to the condition that $A_0 : V_0 \to (V_0)^*$ is bijective.

The conditions (30) are satisfied if the form a is strongly positive on V_0, i.e., if there exists $\mu > 0$ such that

$$a(w, w) \geqslant \mu \|w\|_V^2 \quad \text{for all } w \in V_0.$$

Let us consider the following particular case in which we automatically get the appropriate inf-sup condition: Let H be a real Hilbert space and let H_0 be a closed subspace such that V_0 is dense and compactly embedded in H_0 ($V_0 \subset H_0 = (H_0)^* \subset (V_0)^*$ forms a Gelfand triple, see, e.g., [42], with compact embeddings). Moreover, let the form a satisfy a Gårding inequality on V_0, see, e.g., [42], i.e., there exist constants $\mu > 0$ and $\kappa \geqslant 0$ such that

$$a(w, w) \geqslant \mu \|w\|_V^2 - \kappa \|w\|_H^2 \quad \text{for all } w \in V_0. \tag{32}$$

If in addition $a(v, w) = 0$ for all $w \in V_0$ implies that $v = 0$ then a satisfies the inf-sup conditions (30), see, e.g., [18].

The inf-sup condition (31) for b is equivalent to each of the following two properties, see, e.g., [15]:

(i) The operator $B \in \mathcal{L}(Q, V^*)$ has range $B(Q) = V_\perp^*$ and $B : Q \to V_\perp^*$ is bijective with

$$\|Bq\|_{V^*} \geqslant \gamma \|q\|_Q \quad \text{for all } q \in Q. \tag{33}$$

(ii) The restriction of $B^T \in \mathcal{L}(V, Q^*)$ to $V_\perp$ has range $B^T(V_\perp) = Q^*$ and $B^T : V_\perp \to Q^*$ is bijective with

$$\|B^T w\|_{Q^*} \geqslant \gamma \|w\|_V \quad \text{for all } w \in V_\perp, \tag{34}$$

where again γ is a positive constant.

An example that fits into the above abstract setting is the stationary Stokes problem (i.e., (3) with $v_\infty = 0$) in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ ($d \in \{2, 3\}$) with homogeneous Dirichlet boundary conditions, where we have (see, e.g., [18, 40])

$$\begin{gathered} V = H_0^1(\Omega)^d, \quad V_0 = \{v \in V : \operatorname{div} v = 0\}, \quad Q = L^2(\Omega)/\mathbb{R}, \\ a(v, w) = \nu \int_\Omega \nabla v \cdot \nabla w \, dx, \quad b(q, w) = -\int_\Omega q \operatorname{div} w \, dx. \end{gathered}$$

For a definition of the underlying function spaces, see also [5, 42].

Note that (31) implies that $B : Q \to V^*$ is injective ($\operatorname{kernel} B = \{0\}$). In [6], Brezzi and Fortin replace Q in (31) by the factor space $Q/\operatorname{kernel} B$ and get unique solvability of the mixed variational problem in $V \times Q/\operatorname{kernel} B$ for all $(f, g) \in V^* \times B^T(V)$. For the Stokes problem, this corresponds to the choice $Q = L^2(\Omega)$ with $\operatorname{kernel} B = \mathbb{R}$.

3.1.2. Non-stationary problems and the theorem of Lions–Tartar For completeness, we recall the following well-known theorem of Lions–Tartar, see [27, 37, 39, 42], where we restrict ourselves to a bilinear form that does not explicitly depend on time. Moreover, following [37], we avoid to identify the pivot space of the underlying Gelfand triple with its dual, see also [5, 42].

For a Banach space X, we employ the usual notation (and standard norm) for Bochner–Lebesgue spaces $L^r(0,T;X)$ ($r \in [1,\infty]$), the spaces $\mathcal{C}([0,T];X)$ and $\mathcal{AC}([0,T];X)$ of continuous and absolutely continuous functions, and the Sobolev spaces $W^{1,1}(0,T;X)$ and $W^{1,2}(0,T;X) \equiv H^1(0,T;X)$ of functions mapping $[0,T]$ into X, see, e.g., [10, 12, 38, 40, 42]. Finally, $\mathcal{C}_c^\infty(0,T)$ denotes the space of infinitely times differentiable functions with compact support in $(0,T)$.

Theorem 3.2. *Let X and Y be Hilbert spaces and let X be dense and continuously embedded in Y. Let $a : X \times X \to \mathbb{R}$ be a bounded bilinear form that satisfies a Gårding inequality, i.e., there exist constants $\mu > 0$ and $\kappa \geqslant 0$ such that*

$$a(w,w) \geqslant \mu \|w\|_X^2 - \kappa \|w\|_Y^2 \quad \text{for all } w \in X.$$

Then for every $(f, v^0) \in (L^1(0,T;Y^*) + L^2(0,T;X^*)) \times Y$ there exists a unique solution

$$v \in L^2(0,T;X) \cap \mathcal{C}([0,T];Y)$$

to the initial value problem

$$\frac{d}{dt}(v(\cdot), w)_Y + a(v(\cdot), w) = \langle f(\cdot), w\rangle \quad \text{for all } w \in X, \quad \text{in } (0,T), \tag{35}$$

$$v(0) = v^0. \tag{36}$$

Moreover, the mapping

$$\big(L^1(0,T;Y^*) + L^2(0,T;X^*)\big) \times Y \ni (f, v^0) \mapsto v \in L^2(0,T;X) \cap \mathcal{C}([0,T];Y)$$

is continuous.

The initial condition (36) makes sense as an equation in Y, since one can show that $v \in \mathcal{C}([0,T];Y)$. The equation (35) has to be understood in the sense of distributions, i.e., as

$$\int_0^T \big(- (v(t), w)_Y \dot{\varphi}(t) + a(v(t), w)\varphi(t)\big)\, dt = \int_0^T \langle f(t), w\rangle \varphi(t)\, dt$$

for all $w \in X$ and all $\varphi \in \mathcal{C}_c^\infty(0,T)$. It can, however, also be interpreted as an equation that holds almost everywhere in $(0,T)$, since $t \mapsto (v(t), w)_Y$ is in $W^{1,1}(0,T)$ for every $w \in X$, and thus absolutely continuous on $[0,T]$.

We may also write

$$\frac{d}{dt} j_Y(v) + Av = f \quad \text{in } (0,T)$$

instead of (35), where again j_Y denotes the Riesz isomorphism mapping Y onto Y^*. To be precise, j_Y is the composition of the continuous embedding operator of X into Y, the Riesz isomorphism, and the continuous embedding operator of Y^* into X^*. This allows to consider the solution $v : [0,T] \to X$ as a function taking values in X^*. The time derivative of $j_Y(v)$ has to be understood in the generalized or distributional sense: A locally Bochner-integrable function $z : [0,T] \to X^*$ is said to be the generalized derivative of a locally Bochner-integrable function $w : [0,T] \to X^*$, and we write $z = \dot{w} = \frac{d}{dt}w$, if

$$\int_0^T w(t)\dot{\varphi}(t)\,dt = -\int_0^T z(t)\varphi(t)\,dt \quad \text{in } X^* \text{ for all } \varphi \in \mathcal{C}_c^\infty(0,T).$$

Moreover, we denote the extension of the operator $A \in \mathcal{L}(X, X^*)$ associated with the bounded bilinear form a to a mapping of $L^2(0,T;X)$ into $L^2(0,T;X^*)$ by A again. Such an extension exists in view of the linearity and continuity of $A \in \mathcal{L}(X,X^*)$ and is defined via $(Aw)(t) := Aw(t)$ for a function $w : [0,T] \to X$.

In addition, one can then show that $\frac{d}{dt}j_Y(v) \in L^1(0,T;Y^*) + L^2(0,T;X^*)$ and that the mapping

$$\big(L^1(0,T;Y^*) + L^2(0,T;X^*)\big) \times Y \ni (f, v^0) \mapsto \frac{d}{dt}j_Y(v) \in L^1(0,T;Y^*) + L^2(0,T;X^*)$$

is also continuous.

Identifying Y with its dual Y^* such that $X \subset Y = Y^* \subset X^*$ forms a Gelfand triple, the above result amounts to the existence and uniqueness of a solution

$$v \in \mathcal{W}_1(0,T;X,Y) := \big\{w \in L^2(0,T;X) : \dot{w} \in L^1(0,T;Y^*) + L^2(0,T;X^*)\big\}$$

to the initial value problem

$$\dot{v} + Av = f \quad \text{in } (0,T), \qquad v(0) = v^0$$

that depends continuously on (f, v^0), i.e., the mapping

$$\big(L^1(0,T;Y^*) + L^2(0,T;X^*)\big) \times Y \ni (f, v^0) \mapsto v \in \mathcal{W}_1(0,T;X,Y)$$

is continuous.

We recall that the space $\mathcal{W}_1(0,T;X,Y)$ is continuously embedded in $W^{1,1}(0,T;X^*) \subset \mathcal{AC}([0,T];X^*)$ as well as in $\mathcal{C}([0,T];Y)$.

3.2. Operator DAEs with $M = I$, $B = D$ and $C = 0$

The aim of this subsection is to study non-stationary mixed variational problems that can be interpreted as operator DAEs of the type (1) in the special situation where $M = I$, $B = D$ and $C = 0$, i.e.,

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ p \end{bmatrix} + \begin{bmatrix} A & B \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{37}$$

However, as in the previous subsections, we will start with a formulation based on bounded bilinear forms.

3.2.1. Well-posedness of operator DAEs In what follows, we rely on the notation of Section 3.1.1. In addition to the Hilbert spaces V and Q, let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ such that V is dense and continuously embedded in H. In particular, there exists a constant $\alpha > 0$ such that

$$\|v\|_H \leqslant \alpha \|v\|_V \quad \text{for all } v \in V, \tag{38}$$

which, in applications, will be the Poincaré–Friedrichs inequality [5, 42].

We then consider the following initial value problem:

For given $(f,g) \in (L^1(0,T;H^*) + L^2(0,T;V^*)) \times L^2(0,T;Q^*)$ *and* $v^0 \in H$ *find distributions* v *and* p *with values in* V *and* Q*, respectively, such that*

$$\begin{aligned} \frac{d}{dt}(v(\cdot),w)_H + a(v(\cdot),w) + b(p(\cdot),w) &= \langle f(\cdot),w\rangle \quad \textit{for all } w \in V, \\ -b(q,v(\cdot)) &= \langle g(\cdot),q\rangle \quad \textit{for all } q \in Q \end{aligned} \tag{39}$$

holds on $(0,T)$ *in the distributional sense with* $\lim_{t\to 0+} v(t) = v^0$*.*

The latter condition replaces classical initial conditions in the case of distributional solutions, see, e.g., [30]. The time derivative also has to be understood in the distributional sense and even the variable p turns out to be the distributional time derivative of a function $\hat{p}$. Hence, we look for $(v,\hat{p}) \in L^2(0,T;V) \times L^2(0,T;Q)$ that fulfill the first equation in (39) in the sense that for all $w \in V$ and all $\varphi \in \mathcal{C}_c^\infty(0,T)$

$$\int_0^T \big(-(v(t),w)_H\dot{\varphi}(t) + a(v(t),w)\varphi(t) - b(\hat{p}(t),w)\dot{\varphi}(t)\big)\,dt = \int_0^T \langle f(t),w\rangle\varphi(t)\,dt, \tag{40}$$

where the second equation in (39) is understood in the sense of $L^2(0,T)$.

It has to be clarified later in which sense the initial condition is taken as this depends on the regularity of the solution v.

We may extend the operator $A \in \mathcal{L}(V,V^*)$ associated with the bounded bilinear form a to a mapping that acts on abstract functions $w : [0,T] \to V$ via $(Aw)(t) := Aw(t)$ $(t \in [0,T])$. Analogously, we may extend B and B^T.

Because of linearity and continuity, we can then show that

$$\begin{aligned} A &\in \mathcal{L}(L^2(0,T;V), L^2(0,T;V^*)), \\ B &\in \mathcal{L}(L^2(0,T;Q), L^2(0,T;V^*)), \\ B^T &\in \mathcal{L}(L^2(0,T;V), L^2(0,T;Q^*)). \end{aligned}$$

The extension of $B^T \in \mathcal{L}(V,Q^*)$ coincides with the dual of the extension of $B \in \mathcal{L}(Q,V^*)$. Moreover, the operator $A_0 \in \mathcal{L}(V_0,(V_0)^*)$ associated to the restriction of the bounded bilinear form $a : V \times V \to \mathbb{R}$ to $V_0 \times V_0$ via $\langle A_0 v, w\rangle = a(v,w)$ $(v,w \in V_0)$ extends to an operator $A_0 \in \mathcal{L}(L^2(0,T;V_0), L^2(0,T;(V_0)^*))$.

The equations in (39) can then, at least formally, be written as the operator DAE (37). It remains, however, to answer the question in which sense the time derivative should be understood. The problem here is that the identification of H with its dual and of $H_0 := \operatorname{clos}_{\|\cdot\|_H} V_0$ with its dual is not compatible with each other, see [37, 39]. In the course of the proof of Theorem 3.3, both Riesz isomorphisms j_H and j_{H_0} will appear.

Moreover, we shall emphasize that we perform our analysis on (40), which, at least formally, can be written as

$$\begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ \hat{p} \end{bmatrix} + \begin{bmatrix} A & 0 \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} v \\ \hat{p} \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}.$$

This is nothing else than the infinite-dimensional version of (25).

It is clear that well-posedness can only be expected for consistent data v^0, g (see Corollary 2.1 and [24] for the finite-dimensional case), and as in the finite-dimensional case, for a given consistent initial value v^0 there is no freedom in choosing an initial condition for p.

With respect to the solvability, we have the following main result. Note that in this result we do not need that the form a is symmetric.

Theorem 3.3. *Let the real Hilbert space V be dense and continuously embedded in the real Hilbert space H, let $j_H : H \to H^*$ denote the Riesz isomorphism, let $V_0 = \operatorname{kernel} B^T \subset V$ and let H_0 be the closure of V_0 with respect to the norm of H. Assume that the bounded bilinear form $a : V \times V \to \mathbb{R}$ satisfies the Gårding inequality (32) on V_0 and that the bounded bilinear form $b : Q \times V \to \mathbb{R}$ satisfies the inf-sup condition (31).*

Then for any data $(f, g) \in (L^1(0,T;H^*)+L^2(0,T;V^*)) \times H^1(0,T;Q^*)$ and $v^0 \in H$, which are consistent such that

$$v^0 + (B^T|_{V_\perp})^{-1} g(0) \in H_0, \tag{41}$$

there exists a unique distributional solution (v, p) to (39) with

$$v \in L^2(0,T;V) \cap \mathcal{C}([0,T];H), \quad \frac{d}{dt} j_H(v) + Bp \in L^1(0,T;H^*) + L^2(0,T;V^*),$$
$$p = \dot{\hat{p}} \quad \text{for } \hat{p} \in \mathcal{C}([0,T];Q),$$

for which the initial condition is fulfilled in the sense of $v(0) = v^0$ in H, i.e.,

$$\lim_{t\to 0+} \|v(t) - v^0\|_H = 0.$$

Moreover, the mapping

$$(f, g, v^0) \mapsto (v, \hat{p}) : \big(L^1(0,T;H^*) + L^2(0,T;V^*)\big) \times H^1(0,T;Q^*) \times H$$
$$\to L^2(0,T;V) \cap \mathcal{C}([0,T];H) \times \mathcal{C}([0,T];Q)$$

is continuous.

Proof. In what follows, we assume that $f = f_1 + f_2$ with $f_1 \in L^1(0,T;H^*)$ and $f_2 \in L^2(0,T;V^*)$.

Since the inf-sup condition (31) holds for b, there exists $(B^T|_{V_\perp})^{-1} \in \mathcal{L}(Q^*, V_\perp)$ and it extends to $(B^T|_{V_\perp})^{-1} \in \mathcal{L}(L^2(0,T;Q^*), L^2(0,T;V_\perp))$. Note that the extension of the inverse of the restriction of B^T to $V_\perp$ coincides with the inverse of the restriction to $L^2(0,T;V_\perp)$ of the extension of B^T.

We thus obtain a unique element $v_g := -(B^T|_{V_\perp})^{-1} g \in L^2(0,T;V_\perp)$ as well as a unique element $w_g := -(B^T|_{V_\perp})^{-1} \dot{g} \in L^2(0,T;V_\perp)$, and because of the continuity of the time-independent operator $B^T|_{V_\perp}$ it follows that w_g is the generalized time derivative of v_g such that

$$v_g = -(B^T|_{V_\perp})^{-1} g \in H^1(0,T;V_\perp) \subset \mathcal{AC}([0,T];V_\perp). \tag{42}$$

Let us now consider the initial value problem

$$\frac{d}{dt}(v_0(\cdot), w)_H + a(v_0(\cdot), w)$$
$$= \langle f(\cdot), w\rangle - \frac{d}{dt}(v_g(\cdot), w)_H - a(v_g(\cdot), w) \quad \text{for all } w \in V_0, \text{ in } (0,T), \tag{43}$$
$$v_0(0) = v^0 - v_g(0). \tag{44}$$

Here, (43) is the infinite-dimensional version of (27), in which, however, the term with the derivative of $(v_g(\cdot), w)_H$ does not appear (but the derivative of g then enters the equation for p).

First of all, we note that $v_g \in \mathcal{AC}([0,T];V_\perp)$ can be evaluated at $t = 0$ with $v_g(0) \in V_\perp$. In view of the consistency condition (41) we have that $v^0 - v_g(0) \in H_0$.

With respect to the right-hand side in (43), we observe the following: For almost all $t \in (0,T)$, $f_1(t)$ is a bounded linear functional on H. The restriction of $f_1(t)$ to H_0 remains bounded and can thus be seen as an element of $(H_0)^*$ with

$$\langle f_1(t)|_{H_0}, w\rangle = \langle f_1(t), w\rangle \quad \text{for all } w \in H_0$$

and

$$\|f_1(t)|_{H_0}\|_{(H_0)^*} \leqslant \|f_1(t)\|_{H^*}.$$

Moreover, the Bochner measurability of $f_1 : [0,T] \to H^*$ implies the Bochner measurability of $f_1|_{H_0} : [0,T] \to (H_0)^*$, where $(f_1|_{H_0})(t) := f_1(t)|_{H_0}$ for $t \in [0,T]$. Hence we have that $f_1|_{H_0} \in L^1(0,T;(H_0)^*)$. An analogous argument shows that $f_2|_{V_0} \in L^2(0,T;(V_0)^*)$. The first term on the right-hand side of (43) is thus given by

$$\langle f(\cdot), w\rangle = \langle (f_1|_{H_0})(\cdot), w\rangle + \langle (f_2|_{V_0})(\cdot), w\rangle,$$

where $f_1|_{H_0} \in L^1(0,T;(H_0)^*)$ and $f_2|_{V_0} \in L^2(0,T;(V_0)^*)$.

We also observe that for all $w \in V_0$, we have

$$a(v_g(\cdot), w) = \langle Av_g(\cdot), w\rangle = \langle (Av_g|_{V_0})(\cdot), w\rangle,$$

where $Av_g \in L^2(0,T;V^*)$ and thus $Av_g|_{V_0} \in L^2(0,T;(V_0)^*)$. Finally, since v_g is sufficiently smooth, we observe that for all $w \in V_0$

$$\frac{d}{dt}(v_g(\cdot), w)_H = (\dot{v}_g(\cdot), w)_H,$$

where $\dot{v}_g \in L^2(0,T;V_\perp)$. Since V is continuously embedded into H, it follows that

$$z \mapsto \int_0^T (\dot{v}_g(t), z(t))_H \, dt$$

is a bounded linear mapping on $L^2(0,T;V_0)$.

Altogether, we see that the right-hand side in (43) is given by

$$\langle f(\cdot), w\rangle - \frac{d}{dt}(v_g(\cdot), w)_H - a(v_g(\cdot), w) = \langle f_0(\cdot), w\rangle \tag{45}$$

for all $w \in V_0$, where $f_0 \in L^1(0,T;(H_0)^*) + L^2(0,T;(V_0)^*)$.

Theorem 3.2 (with $X = V_0$ and $Y = H_0$) hence provides existence of a unique solution

$$v_0 \in L^2(0,T;V_0) \cap \mathcal{C}([0,T];H_0)$$

to (43), (44) with

$$\frac{d}{dt}(v_0(\cdot), w)_H \in W^{1,1}(0,T) \quad \text{for all } w \in V_0$$

and $\frac{d}{dt} j_{H_0}(v_0) \in L^1(0,T;(H_0)^*) + L^2(0,T;(V_0)^*)$, where $j_{H_0} : H_0 \to H_0^*$ denotes the Riesz isomorphism.

Let

$$v := v_0 + v_g.$$

In view of the properties of v_0 and v_g, we immediately see that

$$v \in L^2(0,T;V) \cap \mathcal{C}([0,T];H)$$

with

$$v(0) = v_0(0) + v_g(0) = v^0.$$

To prove existence of p, we follow an approach similar to [40]. Let

$$\hat{v}(t) := \int_0^t v(s)ds, \quad \hat{f}(t) := \int_0^t f(s)ds, \quad t \in [0,T].$$

Then $\hat{v} \in \mathcal{AC}([0,T];V)$, $\hat{f} \in \mathcal{AC}([0,T];V^*)$. With (43) and, since A is linear and continuous, the following identity holds for all $w \in V_0$ and $\varphi \in \mathcal{C}_c^\infty(0,T)$:

$$\begin{aligned}
-\int_0^T (v(t),w)_H \dot{\varphi}(t)\,dt &= \int_0^T \langle f(t),w\rangle \varphi(t)\,dt - \int_0^T \langle Av(t),w\rangle \varphi(t)\,dt \\
&= -\int_0^T \langle \hat{f}(t),w\rangle \dot{\varphi}(t)\,dt + \int_0^T \Big\langle \int_0^t Av(s)ds, w\Big\rangle \dot{\varphi}(t)\,dt \\
&= -\int_0^T \langle \hat{f}(t) - A\hat{v}(t),w\rangle \dot{\varphi}(t)\,dt.
\end{aligned}$$

This shows that $(v(t),w)$ is, up to an additive constant, equal to $\langle \hat{f}(t) - A\hat{v}(t), w\rangle$ for almost all $t \in (0,T)$ and all $w \in V_0$. The constant is obviously determined by v^0 such that

$$(v(t) - v^0, w)_H = \langle \hat{f}(t) - A\hat{v}(t), w\rangle.$$

With the Riesz isomorphism $j_H : H \to H^*$, we then have $j_H(v(t) - v^0) \in H^* \subset V^*$ and

$$\langle j_H(v(t) - v^0) + A\hat{v}(t) - \hat{f}(t), w\rangle = 0$$

for all $w \in V_0$ and almost all $t \in (0,T)$. This shows that $j_H(v(t) - v^0) + A\hat{v}(t) - \hat{f}(t) \in V_\perp^*$. Since $v \in \mathcal{C}([0,T];H)$, $\hat{v} \in \mathcal{C}([0,T];V)$, and $\hat{f} \in \mathcal{C}([0,T];V^*)$, we have

$$j_H(v - v^0) + A\hat{v} - \hat{f} \in \mathcal{C}([0,T];V^*)$$

and thus

$$j_H(v - v^0) + A\hat{v} - \hat{f} \in \mathcal{C}([0,T];V_\perp^*).$$

If the bilinear form b satisfies the inf-sup condition (31), then $B : \mathcal{C}([0,T];Q) \to \mathcal{C}([0,T];V_\perp^*)$ is bijective and there exists a unique $\hat{p} \in \mathcal{C}([0,T];Q)$ such that for all $t \in [0,T]$

$$B\hat{p}(t) = -j_H(v(t) - v^0) - A\hat{v}(t) + \hat{f}(t) \quad \text{in } V^*, \tag{46}$$

which means that

$$(v(t) - v^0, w)_H + a(\hat{v}(t), w) + b(\hat{p}(t), w) = \langle \hat{f}(t), w\rangle$$

for all $w \in V$ and all $t \in [0,T]$. Taking the derivative shows that (v,p), with $p := \dot{\hat{p}}$ being the derivative of $\hat{p}$ in the distributional sense, is the unique solution to the original problem (39) in the sense of (40). Moreover, this shows that the distributional derivative of $j_H(v) + B\hat{p}$ equals $f - Av \in L^1(0,T;H^*) + L^2(0,T;V^*)$.

The continuous dependence of the solution on the problem data follows from the corresponding result in Theorem 3.2 together with the continuity of $(B^T|_{V_\perp})^{-1} \in \mathcal{L}(Q^*, V_\perp)$ (see also (34)), $B^{-1}|_{V_\perp^*} \in \mathcal{L}(V_\perp^*, Q)$ (see also (33)), and the continuity of the embedding of V into H (see also (38)). $\square$

Remark 3.1. In the proof of Theorem 3.3 we have seen that it is crucial to assume the consistency condition (41). In the finite-dimensional case, where $V_0 = H_0 = \operatorname{kernel} B^T$, this condition reduces to

$$B^T v^0 + g(0) = 0,$$

which is indeed the same condition as (21), i.e., the second condition of (23), see also (26).

We have also seen that it is sufficient to assume $g \in H^1(0,T;Q^*)$. Also this corresponds to the finite-dimensional case, see (20), where one also has to take into account the first derivative of g, but not of f.

There is, however, no analogue to the first condition of (23). Since the solution p is only a distribution, it does not make sense to consider an evaluation of p at $t = 0$.

With respect to the last comment in Remark 3.1, we emphasize that Theorem 3.3 provides solvability of (39) in the sense of (40) only, which means an integration with respect to p. If we, however, integrate (22), then we obtain

$$B^T B\hat{p}(t) = -B^T A\hat{v}(t) + B^T \hat{f}(t) + g(t) - g(0).$$

Evaluating this relation at $t = 0$ gives the consistency condition

$$B^T B\hat{p}(0) = 0.$$

This condition is automatically fulfilled, since at $t = 0$ equation (46) gives

$$B\hat{p}(0) = 0.$$

Examples that fit into our abstract setting are the Oseen or linearized Navier–Stokes problem (3) in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ ($d \in \{2,3\}$) with homogeneous Dirichlet boundary conditions, where we have

$$\begin{gathered} V = H_0^1(\Omega)^d, \quad V_0 = \big\{v \in V : \operatorname{div} v = 0\big\}, \quad Q = L^2(\Omega)/\mathbb{R}, \\ H = L^2(\Omega), \quad H_0 = \big\{v \in L^2(\Omega)^d : \operatorname{div} v = 0,\ \gamma_n v = 0\big\}, \\ a(v,w) = \int_\Omega \big(\nu \nabla v \cdot \nabla w + (v_\infty \cdot \nabla) v \cdot w + (v \cdot \nabla) v_\infty \cdot w\big) dx, \\ b(p,w) = -\int_\Omega p \operatorname{div} w dx. \end{gathered}$$

Here γ_n denotes the trace in normal direction, see [40].

Let $v_\infty \in L^\infty(\Omega)^d$ be divergence-free. Using integration by parts, as well as the Cauchy–Schwarz and Young inequality [5], we then obtain

$$a(v,v) \geqslant \nu \|v\|_V^2 - c\, \|v_\infty\|_{L^\infty(\Omega)^d} \|v\|_H \|v\|_V \geqslant \frac{\nu}{2} \|v\|_V^2 - c\, \|v_\infty\|_{L^\infty(\Omega)^d}^2 \|v\|_H^2$$

for all $v \in V_0$ (with a generic positive constant c), which shows that a satisfies a Gårding inequality on V_0.

We shall remark that strong solvability of the time-periodic Stokes problem with non-zero divergence of the velocity has been shown in [22]. That result similarly requires $g \in H^1(S^1; H^{-1}(\Omega))$.

3.2.2. An explicit solution formula Under the assumptions of Theorem 3.3 we can define the solution operator corresponding to the initial value problem with zero right-hand side governed by the operator $A_0 : V_0 \to (V_0)^*$, which is the operator associated with the restriction of a to $V_0 \times V_0$. With a slight abuse of notation, we may denote this solution operator by $\{e^{-tA_0}\}_{t\geqslant 0}$, i.e., if $v_0^0 \in H_0$ then $v_0(t) := e^{-tA_0}v_0^0 \in L^2(0,T;V_0) \cap \mathcal{C}([0,T];H_0)$ satisfies

$$\frac{d}{dt}(v_0(\cdot),w) + a(v_0(\cdot),w) = 0 \quad \text{for all } w \in V_0, \text{ in } (0,T),$$
$$v_0(0) = v_0^0.$$

This is just a consequence of Theorem 3.2, since, in particular, $a : V \times V \to \mathbb{R}$ satisfies a Gårding inequality on V_0.

In view of Duhamel's principle we then have that if $v_0^0 \in H_0$ and $f_0 \in L^1(0,T;(H_0)^*)$ then

$$v_0(t) := e^{-tA_0}v_0^0 + \int_0^t e^{-(t-s)A_0} j_{H_0}^{-1} f_0(s)\, ds$$

is the unique solution of

$$\frac{d}{dt}(v_0(\cdot),w) + a(v_0(\cdot),w) = \langle f_0(\cdot),w\rangle \quad \text{for all } w \in V_0, \text{ in } (0,T),$$
$$v_0(0) = v_0^0.$$

This is shown by observing that $j_{H_0}^{-1} f_0 \in L^1(0,T;H_0)$ and by employing the definition and properties of the solution operator.

It is then straightforward to prove the following corollary.

Corollary 3.1. *Suppose that, in addition to the assumptions of Theorem 3.3, there exists*

$$f_0 \in L^1(0,T;(H_0)^*) \tag{47}$$

such that

$$\langle f(\cdot),w\rangle + \big((B^T|_{V_\perp})^{-1}\dot{g}(\cdot),w\big)_H + a\big((B^T|_{V_\perp})^{-1}g(\cdot),w\big) = \langle f_0(\cdot),w\rangle \tag{48}$$

for all $w \in V_0$ in $(0,T)$. Then, for all $t \in [0,T]$, the solution $(v,\hat{p})$ from Theorem 3.3 satisfies the representation

$$\begin{aligned} v(t) &= e^{-tA_0}\big(v^0 + (B^T|_{V_\perp})^{-1}g(0)\big) + \int_0^t e^{-(t-s)A_0} j_{H_0}^{-1} f_0(s)\, ds - (B^T|_{V_\perp})^{-1}g(t), \\ \hat{p}(t) &= B^{-1}\Big(\int_0^t (f(s) - Av(s))\, ds - j_H(v(t) - v^0)\Big). \end{aligned} \tag{49}$$

Proof. We only have to apply Duhamel's principle, which requires to show that the right-hand side in (43) can be represented by a functional in $L^1(0,T;(H_0)^*)$, see also (45). This, however, is a direct consequence of (42) together with (47), (48). □

Remark 3.2. The representation (49) is the infinite-dimensional counterpart of (20). In the finite-dimensional case with $V_0 = H_0 = \operatorname{kernel} B^T$, the additional assumption (47), (48) is always fulfilled if $f, g, \dot{g}$ are integrable.

4. Conclusions and future work

We have analyzed the well-posedness of linear operator differential-algebraic equations (DAE) arising in fluid dynamics in the finite and infinite-dimensional case and also presented explicit solution formulas. While in the finite-dimensional setting the analysis is pretty much complete (except for the analysis, under which conditions the DAE has differentiation index one or two), in the infinite-dimensional setting we have only considered the special case that the coefficient in front of the velocity is the identity, that the operators B and D coincide and that C vanishes. We expect that the presented results can be extended to the more general case but this is forthcoming work. It is also important to study the discretization of the infinite-dimensional problem in such a way that the finite-dimensional problem retains the properties. In the context of special cases of the described operator DAEs, a first step in this direction has been recently done in [1].

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