

WEAK-STRONG UNIQUENESS FOR THE GENERAL ERICKSEN–LESLIE SYSTEM IN THREE DIMENSIONS

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ABSTRACT. We study the Ericksen–Leslie system equipped with a quadratic free energy functional. The norm restriction of the director is incorporated by a standard relaxation technique using a double-well potential. We use the relative energy concept, often applied in the context of compressible Euler- or related systems of fluid dynamics, to prove weak-strong uniqueness of solutions. A main novelty, not only in the context of the Ericksen–Leslie model, is that the relative energy inequality is proved for a system with a nonconvex energy.

1. Introduction. This paper is devoted to the weak-strong uniqueness of weak solutions to the three-dimensional Ericksen–Leslie model describing liquid crystal flow. The Ericksen–Leslie model (proposed by Ericksen [10] and Leslie [20]) is a very successful model for nematic liquid crystals and agrees with experiments (see [1, Sec. 11.1, p. 463]). The particular model strongly depends on the choice of the free energy.

Recently, global existence of weak solutions was shown in [9] for a very general class of free energies. In this article, we prove weak-strong uniqueness of these solutions for a special, physically relevant free energy as long as the weak solution satisfies a suitable energy inequality. The weak-strong uniqueness property says that the weak solution coincides with a weak solution admitting additional regularity and arising from the same problem data (initial and boundary values, right-hand side) as long as the latter exists.

We use the concept of a relative energy approach (see Feireisl, Jin and Novotný [13]) to prove weak-strong uniqueness. Often this concept is called relative entropy approach. Henceforth, we use the term relative energy since we are rather dealing with energies. In the case of a convex energy functional, this idea goes back to Dafermos [4] in the context of thermodynamical systems. For a convex Gâteaux

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differentiable energy functional $\eta : V \rightarrow \mathbb{R}$ on a Banach space V , the relative energy of two solutions u and \tilde{u} is given by (see [5, Sec. 5.3])

$$\mathcal{E}(u|\tilde{u}) = \eta(u) - \eta(\tilde{u}) - \langle \eta'(\tilde{u}), u - \tilde{u} \rangle. \quad (1)$$

The convexity of η guarantees that \mathcal{E} is nonnegative (see [17, Kapitel III, Lemma 4.10]).

A novelty of the article in hand is that the concept of relative energies is generalized to nonconvex functionals. Therefore, the standard definition (1) of a relative energy has to be adapted (see (24) below). In comparison to Dai [6], this allows to prove the weak-strong uniqueness without assuming the pointwise boundedness $|\mathbf{d}| \leq 1$ of the weak-solution \mathbf{d} , which is stipulated in [6, Theorem 1.5] but is not known to hold in general. Additionally, the relative energy approach allows to handle more involved energies as the general Oseen–Frank energy (see [19]).

For convex energy functionals, the relative energy serves as a general comparing tool and beside the weak-strong uniqueness of solutions (see Feireisl and Novotný [14]), it is also employed to show the stability of an equilibrium state (see Feireisl [11]), the convergence to a singular limit problem (see Breit, Feireisl and Hofmanová [2] as well as Feireisl [12]), or to derive a posteriori estimates of numerical solutions (see Fischer [16]). Another possible application is the definition of a generalized solution concept, the so-called dissipative solutions. The formulation of such a concept relies on an inequality instead of an equation (see Lions [23, Sec. 4.4]). We hope that such applications also arise in the new nonconvex case, which is presented in this article.

1.1. Review of known results. A simplified Ericksen–Leslie model

$$\begin{aligned} \partial_t \mathbf{d} + (\mathbf{v} \cdot \nabla) \mathbf{d} &= \Delta \mathbf{d} + \frac{1}{\varepsilon^2} (|\mathbf{d}|^2 - 1) \mathbf{d}, \\ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p - \Delta \mathbf{d} &= -\nabla \cdot (\nabla \mathbf{d}^T \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{v} &= 0, \end{aligned} \quad (2)$$

was first considered in Lin and Liu [21], where global existence of weak solutions as well as local existence of strong solutions was shown. Later, Lin and Liu [22] showed the same result for a generalized system. Existence of weak solutions to the model considered in the article at hand (see (4) below) equipped with the Dirichlet energy with double-well potential,

$$F(\mathbf{d}, \nabla \mathbf{d}) = \frac{k}{2} |\nabla \mathbf{d}|^2 + \frac{1}{4\varepsilon} (|\mathbf{d}|^2 - 1)^2,$$

was first proved in Cavaterra, Rocca and Wu [3]. In [9], the existence of weak solutions to the model considered here was proved for a more general class of free-energies.

The concept of weak-strong uniqueness was first considered by Prodi in 1959 (see [26]) and Serrin in 1962 (see [28]). Both studied the Navier–Stokes equation and showed weak-strong uniqueness for a class of weak solutions fulfilling additional regularity requirements.

In several works, the weak-strong uniqueness property for different simplifications of the Ericksen–Leslie model is studied. Zhao and Liu [31] established weak-strong uniqueness for the simplified system (2) with different assumptions on the strong solution. Dai [6, 7] established weak-strong uniqueness for a simplified incompressible model and a more general incompressible Ericksen–Leslie model with

additional assumptions on the weak solution, which cannot be shown to hold in general. Yang et al. [29] showed the weak-strong uniqueness for the simplified incompressible Ericksen–Leslie system without nonlinear penalization using ideas of Feireisl et al. [15] based upon a relative energy approach and suitable weak solutions. In the article at hand, we use similar ideas. However, we are able to incorporate the nonlinear part of the free energy in the relative energy and to show the weak-strong uniqueness without further assumptions on the weak solution. This is done by adapting the relative energy to the nonconvex energy of the system. A similar weak-strong uniqueness result for measure-valued solutions to the Ericksen–Leslie system equipped with the nonconvex Oseen–Frank energy (see [18] for the existence of such solutions) was recently proved by the second author [19].

1.2. Notation. Vectors of \mathbb{R}^3 are denoted by bold small Latin letters. Matrices of $\mathbb{R}^{3 \times 3}$ are denoted by bold capital Latin letters. Moreover, numbers are denoted by small Latin or Greek letters, and capital Latin letters are reserved for potentials.

The Euclidean inner product in \mathbb{R}^3 is denoted by a dot, $\mathbf{a} \cdot \mathbf{b} := \mathbf{a}^T \mathbf{b} = \sum_{i=1}^3 \mathbf{a}_i \mathbf{b}_i$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. The Frobenius inner product in the space $\mathbb{R}^{3 \times 3}$ of matrices is denoted by a double dot, $\mathbf{A} : \mathbf{B} := \text{tr}(\mathbf{A}^T \mathbf{B}) = \sum_{i,j=1}^3 \mathbf{A}_{ij} \mathbf{B}_{ij}$ for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3 \times 3}$. We also employ the corresponding Euclidean norm with $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$ for $\mathbf{a} \in \mathbb{R}^3$ and Frobenius norm with $|\mathbf{A}|^2 = \mathbf{A} : \mathbf{A}$ for $\mathbf{A} \in \mathbb{R}^{3 \times 3}$. The product of a fourth order with a second order tensor is defined by

$$\mathbf{\Gamma} : \mathbf{A} := \left[\sum_{k,l=1}^3 \mathbf{\Gamma}_{ijkl} \mathbf{A}_{kl} \right]_{i,j=1}^3, \quad \mathbf{\Gamma} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}, \mathbf{A} \in \mathbb{R}^{3 \times 3}.$$

The standard matrix and matrix-vector multiplication, however, is written without an extra sign for brevity,

$$\mathbf{A}\mathbf{B} = \left[\sum_{j=1}^3 \mathbf{A}_{ij} \mathbf{B}_{jk} \right]_{i,k=1}^3, \quad \mathbf{A}\mathbf{a} = \left[\sum_{j=1}^3 \mathbf{A}_{ij} \mathbf{a}_j \right]_{i=1}^3, \quad \mathbf{A} \in \mathbb{R}^{3 \times 3}, \mathbf{B} \in \mathbb{R}^{3 \times 3}, \mathbf{a} \in \mathbb{R}^3.$$

The outer product is denoted by $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T = [\mathbf{a}_i \mathbf{b}_j]_{i,j=1}^3$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Note that $\text{tr}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$. The symmetric and skew-symmetric part of a matrix are denoted by $\mathbf{A}_{\text{sym}} := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ and $\mathbf{A}_{\text{skw}} := \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$ for $\mathbf{A} \in \mathbb{R}^{3 \times 3}$, respectively.

We use the Nabla symbol ∇ for real-valued functions $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ as well as vector-valued functions $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ denoting

$$\nabla f := \left[\frac{\partial f}{\partial \mathbf{x}_i} \right]_{i=1}^3, \quad \nabla \mathbf{f} := \left[\frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_j} \right]_{i,j=1}^3.$$

For brevity, we write $\nabla \mathbf{f}^T$ instead of $(\nabla \mathbf{f})^T$. The divergence of a vector-valued function $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and a matrix-valued function $\mathbf{A} : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ is defined by

$$\nabla \cdot \mathbf{f} := \sum_{i=1}^3 \frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_i} = \text{tr}(\nabla \mathbf{f}), \quad \nabla \cdot \mathbf{A} := \left[\sum_{j=1}^3 \frac{\partial \mathbf{A}_{ij}}{\partial \mathbf{x}_j} \right]_{i=1}^3.$$

Note that $(\mathbf{v} \cdot \nabla) \mathbf{f} = (\nabla \mathbf{f}) \mathbf{v} = \nabla \mathbf{f} \mathbf{v}$ for vector-valued functions $\mathbf{v}, \mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

Throughout this paper, let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class \mathcal{C}^2 . We rely on the usual notation for spaces of continuous functions, Lebesgue and Sobolev

spaces. Spaces of vector-valued functions are emphasized by bold letters, for example $\mathbf{L}^p(\Omega) := L^p(\Omega; \mathbb{R}^3)$, $\mathbf{W}^{k,p}(\Omega) := W^{k,p}(\Omega; \mathbb{R}^3)$. If it is clear from the context, we also use this bold notation for spaces of matrix-valued functions. For brevity, we often omit calling the domain Ω . The standard inner product in $L^2(\Omega; \mathbb{R}^3)$ is denoted by (\cdot, \cdot) and in $L^2(\Omega; \mathbb{R}^{3 \times 3})$ by $(\cdot; \cdot)$.

The space of smooth solenoidal functions with compact support is denoted by $\mathcal{C}_{c,\sigma}^\infty(\Omega; \mathbb{R}^3)$. By $\mathbf{L}_\sigma^p(\Omega)$, $\mathbf{H}_{0,\sigma}^1(\Omega)$, and $\mathbf{W}_{0,\sigma}^{1,p}(\Omega)$, we denote the closure of $\mathcal{C}_{c,\sigma}^\infty(\Omega; \mathbb{R}^3)$ with respect to the norm of $\mathbf{L}^p(\Omega)$, $\mathbf{H}^1(\Omega)$, and $\mathbf{W}^{1,p}(\Omega)$, respectively ($1 \leq p < \infty$).

The dual space of a Banach space V is always denoted by V^* and equipped with the standard norm; the duality pairing is denoted by $\langle \cdot, \cdot \rangle$. The duality pairing between $\mathbf{L}^p(\Omega)$ and $\mathbf{L}^q(\Omega)$ (with $1/p + 1/q = 1$), however, is denoted by (\cdot, \cdot) or $(\cdot; \cdot)$.

The Banach space of linear bounded operators mapping a Banach space V into itself is denoted by $\mathcal{L}(V)$ and equipped with the usual norm. For a given Banach space V , Bochner–Lebesgue spaces are denoted, as usual, by $L^p(0, T; V)$ for $1 \leq p \leq \infty$. Moreover, $W^{1,p}(0, T; V)$ denotes the Banach space of abstract functions in $L^p(0, T; V)$ whose weak time derivative exists and is again in $L^p(0, T; V)$ (see also Diestel and Uhl [8, Section II.2] or Roubířek [27, Section 1.5] for more details). We often omit the time interval $(0, T)$ and the domain Ω and just write, e.g., $L^p(\mathbf{W}^{k,p})$ for brevity. By $\mathcal{C}_w([0, T]; V)$, we denote the spaces of abstract functions mapping $[0, T]$ into V that are continuous with respect to the weak topology in V .

By $\mathbf{\Lambda}$, we denote a constant tensor of order 4 that is symmetric, i.e., $\mathbf{\Lambda}_{ijkl} = \mathbf{\Lambda}_{klij}$, $i, j, k, l \in \{1, 2, 3\}$, and obeys the strong ellipticity condition (see McLean [24]), i.e., there exists $\eta > 0$ such that

$$(\mathbf{a} \otimes \mathbf{b}) : \mathbf{\Lambda} : (\mathbf{a} \otimes \mathbf{b}) \geq \eta |\mathbf{a}|^2 |\mathbf{b}|^2 \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^3. \quad (3)$$

We introduce the norm $\|\cdot\|_{\mathbf{\Lambda}} := \|\cdot : \mathbf{\Lambda} : \cdot\|_{L^1}^{1/2}$. The norm $\|\nabla \cdot\|_{\mathbf{\Lambda}}$ is equivalent to the \mathbf{H}^1 -norm on \mathbf{H}_0^1 . We use the abbreviation $\Delta_{\mathbf{\Lambda}} \mathbf{d} = \nabla \cdot \mathbf{\Lambda} : \nabla \mathbf{d}$ for $\mathbf{d} \in \mathbf{H}^2$.

Finally, by $c > 0$, we denote a generic positive constant and by C_δ a constant depending on a given parameter $\delta > 0$.

2. Model and main result. We consider the general Ericksen–Leslie system, which was investigated in [9]. In comparison to the model in [9], we consider a particular free energy function and reformulate the stress tensor. The system is given by

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p - \nabla \mathbf{d}^T \mathbf{q} - \nabla \cdot \mathbf{T}^L = \mathbf{g}, \quad (4a)$$

$$\partial_t \mathbf{d} + (\mathbf{v} \cdot \nabla) \mathbf{d} - (\nabla \mathbf{v})_{\text{skw}} \mathbf{d} + \lambda (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} + \gamma \mathbf{q} = 0, \quad (4b)$$

$$\nabla \cdot \mathbf{v} = 0. \quad (4c)$$

The vector fields $\mathbf{v} : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}^3$ and $\mathbf{d} : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}^3$ represent the velocity field and the director field, respectively. The pressure is denoted by $p : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}$. In what follows, we do not address the problem of existence or uniqueness of the pressure. For the free energy potential, we choose the function

$$F(\mathbf{d}, \nabla \mathbf{d}) := \frac{1}{2} \nabla \mathbf{d} : \mathbf{\Lambda} : \nabla \mathbf{d} + \frac{1}{4\varepsilon} (|\mathbf{d}|^2 - 1)^2. \quad (5)$$

Here $\mathbf{\Lambda}$ is a constant symmetric fourth order tensor fulfilling the strong ellipticity condition (3) (see McLean [24] and Section 1.2). Moreover, $\varepsilon > 0$ denotes the fixed parameter for the relaxation of the requirement $|\mathbf{d}| = 1$. We do not address the

question of the limit $\varepsilon \rightarrow 0$. For such a singular limit analysis in the context of the Ericksen–Leslie model, we refer to [19].

The free energy is the functional induced by the free energy potential,

$$\mathcal{F}(\mathbf{d}) := \int_{\Omega} F(\mathbf{d}, \nabla \mathbf{d}) \, d\mathbf{x} = \frac{1}{2} \|\nabla \mathbf{d}\|_{\Lambda}^2 + \frac{1}{4\varepsilon} \|\|\mathbf{d}\|^2 - 1\|_{L^2}^2. \quad (6)$$

The vector field \mathbf{q} in (4) is the variational derivative of the free energy (6),

$$\mathbf{q} := \frac{\delta \mathcal{F}}{\delta \mathbf{d}} = -\Delta_{\Lambda} \mathbf{d} + \frac{1}{\varepsilon} (|\mathbf{d}|^2 - 1) \mathbf{d}. \quad (7)$$

For the definition of the operator Δ_{Λ} , we refer to Section 1.2. In comparison to the system studied in [9], the divergence of the Ericksen stress given by $\nabla \cdot \mathbf{T}^E = \nabla \cdot (\nabla \mathbf{d}^T (\partial F / \partial \nabla \mathbf{d}))$ is replaced by $-\nabla \mathbf{d}^T \mathbf{q}$. This reformulation is valid due to the integration-by-parts formula

$$(\nabla \cdot \mathbf{T}^E, \boldsymbol{\varphi}) = -(\nabla \mathbf{d}^T \mathbf{q}, \boldsymbol{\varphi}) + (\nabla F, \boldsymbol{\varphi})$$

derived in [9, Section 3.3] that holds for every test function $\boldsymbol{\varphi} \in \mathcal{C}_c^{\infty}(\Omega; \mathbb{R}^3)$. Hence, via a reformulation, the term F can be incorporated in the pressure and one ends up with the formulation (4). The Leslie stress tensor is given by

$$\begin{aligned} \mathbf{T}^L := & \mu_1 (\mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d}) \mathbf{d} \otimes \mathbf{d} + \mu_4 (\nabla \mathbf{v})_{\text{sym}} - \gamma (\mu_2 + \mu_3) (\mathbf{d} \otimes \mathbf{q})_{\text{sym}} + (\mathbf{d} \otimes \mathbf{q})_{\text{skw}} \\ & + ((\mu_5 + \mu_6) - \lambda (\mu_2 + \mu_3)) (\mathbf{d} \otimes (\nabla \mathbf{v})_{\text{sym}} \mathbf{d})_{\text{sym}}. \end{aligned} \quad (8)$$

Note that in view of (4b), the formulation (8) is equivalent to the formulation of the Leslie stress in [9]. In order to assure the dissipative character of the system, we assume that the parameters λ , γ , μ_1 , μ_2 , μ_3 , μ_4 , μ_5 , and μ_6 satisfy

$$\begin{aligned} \mu_1 > 0, \quad \mu_4 > 0, \quad \gamma > 0, \quad (\mu_5 + \mu_6) - \lambda (\mu_2 + \mu_3) > 0 \\ 4\gamma ((\mu_5 + \mu_6) - \lambda (\mu_2 + \mu_3)) > (\gamma (\mu_2 + \mu_3) - \lambda)^2. \end{aligned} \quad (9)$$

Finally, we assume that $\mathbf{g} \in L^2(0, T; (\mathbf{H}_{0, \sigma}^1)^*)$. We equip the system with initial conditions and Dirichlet boundary conditions such that

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega, \quad \mathbf{v}(\mathbf{x}, t) = \mathbf{0} \quad \text{for } (t, \mathbf{x}) \in [0, T] \times \partial\Omega, \quad (10a)$$

$$\mathbf{d}(\mathbf{x}, 0) = \mathbf{d}_0(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega, \quad \mathbf{d}(\mathbf{x}, t) = \mathbf{d}_1 \quad \text{for } (t, \mathbf{x}) \in [0, T] \times \partial\Omega. \quad (10b)$$

We always assume that $\mathbf{d}_0 = \mathbf{d}_1$ on $\partial\Omega$, which is a compatibility condition providing regularity. For the initial and boundary values, we assume the regularity

$$\mathbf{v}_0 \in \mathbf{L}_{\sigma}^2, \quad \mathbf{d}_0 \in \mathbf{H}^1, \quad \text{and} \quad \mathbf{d}_1 \in \mathbf{H}^{3/2}(\partial\Omega). \quad (11)$$

Definition 2.1. The pair (\mathbf{v}, \mathbf{d}) is said to be a *weak solution* to system (4)–(11) if

$$\begin{aligned} \mathbf{v} \in L^{\infty}(0, T; \mathbf{L}_{\sigma}^2) \cap L^2(0, T; \mathbf{H}_{0, \sigma}^1) \cap W^{1,2}(0, T; (\mathbf{W}_{0, \sigma}^{1,6})^*), \\ \mathbf{d} \in L^{\infty}(0, T; \mathbf{H}^1) \cap L^2(0, T; \mathbf{H}^2) \cap W^{1,2}(0, T; \mathbf{L}^{3/2}), \end{aligned} \quad (12)$$

and

$$\begin{aligned} \int_0^T \langle \partial_t \mathbf{v}, \boldsymbol{\varphi} \rangle \, ds + \int_0^T ((\mathbf{v} \cdot \nabla) \mathbf{v}, \boldsymbol{\varphi}) \, ds \\ - \int_0^T \langle \nabla \mathbf{d}^T \mathbf{q}, \boldsymbol{\varphi} \rangle \, dt + \int_0^T (\mathbf{T}^L : \nabla \boldsymbol{\varphi}) \, dt - \int_0^T \langle \mathbf{g}, \boldsymbol{\varphi} \rangle \, dt = 0, \end{aligned} \quad (13a)$$

$$\begin{aligned} & \int_0^T (\partial_t \mathbf{d}, \boldsymbol{\psi}(t)) \, dt \\ & + \int_0^T ((\mathbf{v} \cdot \nabla) \mathbf{d}, \boldsymbol{\psi}) - ((\nabla \mathbf{v})_{\text{skw}} \mathbf{d}, \boldsymbol{\psi}) + \lambda ((\nabla \mathbf{v})_{\text{sym}} \mathbf{d}, \boldsymbol{\psi}) + \gamma (\mathbf{q}, \boldsymbol{\psi}) \, dt = 0 \end{aligned} \quad (13b)$$

for all test functions $\boldsymbol{\varphi} \in L^2(0, T; \mathbf{W}_{0,\sigma}^{1,6})$ and $\boldsymbol{\psi} \in L^2(0, T; \mathbf{L}^3)$.

The global existence of weak solutions was proved under the given assumptions (4)–(11) in [9, Theorem 3.1] for a domain of class \mathcal{C}^2 .

Definition 2.2. A weak solution (\mathbf{v}, \mathbf{d}) (see Definition 2.1) is said to be a *suitable weak solution* to system (4)–(11) if it is a weak solution and additionally satisfies the energy inequality

$$\begin{aligned} & \frac{1}{2} \|\mathbf{v}(t)\|_{\mathbf{L}^2}^2 + \mathcal{F}(\mathbf{d}(t)) + \int_0^t (\mu_1 \|\mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d}\|_{\mathbf{L}^2}^2 + \mu_4 \|(\nabla \mathbf{v})_{\text{sym}}\|_{\mathbf{L}^2}^2) \, ds \\ & + \int_0^t ((\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) \|(\nabla \mathbf{v})_{\text{sym}} \mathbf{d}\|_{\mathbf{L}^2}^2 + \gamma \|\mathbf{q}\|_{\mathbf{L}^2}^2) \, ds \\ & \leq \frac{1}{2} \|\mathbf{v}_0\|_{\mathbf{L}^2}^2 + \mathcal{F}(\mathbf{d}_0) + \int_0^t (\langle \mathbf{g}, \mathbf{v} \rangle + (\gamma(\mu_2 + \mu_3) - \lambda) (\mathbf{q}, (\nabla \mathbf{v})_{\text{sym}} \mathbf{d})) \, ds \end{aligned} \quad (14)$$

for almost all $t \in (0, T)$.

Definition 2.3. A weak solution $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$ (see Definition 2.1) is said to be a *strong solution* to (4)–(11) if it admits the additional regularity

$$\tilde{\mathbf{v}} \in L^2(0, T; \mathbf{W}^{1,6}), \quad \tilde{\mathbf{d}} \in L^2(0, T; \mathbf{W}^{2,3}), \quad \tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}} \in L^2(0, T; \mathbf{L}^6). \quad (15)$$

Remark 1. For $\mu_1 = 0$ it would be sufficient to assume the regularity $\tilde{\mathbf{v}} \in L^2(0, T; \mathbf{W}^{1,3} \cap \mathbf{L}^\infty)$ and $(\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}} \in L^2(0, T; \mathbf{L}^3)$ instead of $\tilde{\mathbf{v}} \in L^2(0, T; \mathbf{W}^{1,6})$ and $\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}} \in L^2(0, T; \mathbf{L}^6)$.

A strong solution is also a suitable weak solution (see Definition 2.2) since it fulfills an energy equation (see Lemma 3.1).

We can now state the main theorem of this paper.

Theorem 2.4. Let $\Omega \subset \mathbb{R}^3$ be a domain of class \mathcal{C}^2 . Let (\mathbf{v}, \mathbf{d}) be a suitable weak solution (see Definition 2.2) to the Ericksen–Leslie system (4)–(11) and $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$ a strong solution (see Definition 2.3) to the same initial and boundary conditions (10)–(11).

Then

$$\mathbf{v} \equiv \tilde{\mathbf{v}}, \quad \mathbf{d} \equiv \tilde{\mathbf{d}}.$$

Remark 2. Theorem 2.4 is a direct consequence of Lemma 5.1 below. In Lemma 5.1, even the continuous dependence on the initial values is shown as long as a strong solution exists.

Before we present the proof of the main result, we give an important remark on the existence of suitable weak solutions.

Remark 3 (Existence of suitable weak solutions). In our recent work [9], we proved global existence of weak solutions to the system (4)–(11) in the sense of Definition 2.1. This is done by establishing a Galerkin approximation leading to an

approximate system whose solutions $\{(\mathbf{v}_n, \mathbf{d}_n)\}_{n \in \mathbb{N}}$ obey the energy equation as in (17). This allows us to show a priori estimates for the sequence of solutions to the approximate system and extract weakly- and weakly*-converging subsequences. Finally, it is possible to identify the limit of these subsequences with the solution (\mathbf{v}, \mathbf{d}) to (13). It turns out that the energy inequality (14) cannot be shown to hold for the limit. The a priori estimates for the approximate system (see [9]) imply the following weak convergences as $n \rightarrow \infty$:

$$\begin{aligned} \mathbf{v}_n &\overset{*}{\rightharpoonup} \mathbf{v} && \text{in } L^\infty(0, T; \mathbf{L}_\sigma^2) \cap L^2(0, T; \mathbf{H}_{0, \sigma}^1) \cap W^{1,2}(0, T; (\mathbf{W}_{0, \sigma}^{1,6})^*), \\ \mathbf{q}_n &\rightharpoonup \mathbf{q} && \text{in } L^2(0, T; \mathbf{L}^2), \\ \mathbf{d}_n &\overset{*}{\rightharpoonup} \mathbf{d} && \text{in } L^\infty(0, T; \mathbf{H}^1) \cap L^2(0, T; \mathbf{H}^2) \cap W^{1,2}(0, T; \mathbf{L}^{3/2}) \end{aligned}$$

as well as

$$\begin{aligned} (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n &\rightharpoonup (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} && \text{in } L^2(0, T; \mathbf{L}^2), \\ \mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n &\rightharpoonup \mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} && \text{in } L^2(0, T; \mathbf{L}^2). \end{aligned}$$

Due to the weakly lower semi-continuity of the appearing norms, one can deduce that

$$\begin{aligned} &\liminf_{n \rightarrow \infty} \left(\frac{1}{2} \|\mathbf{v}_n(t)\|_{\mathbf{L}^2}^2 \right. \\ &\quad + \mathcal{F}(\mathbf{d}_n(t)) + \int_0^t (\mu_1 \|\mathbf{d}_n \cdot (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{\mathbf{L}^2}^2 + \mu_4 \|(\nabla \mathbf{v}_n)_{\text{sym}}\|_{\mathbf{L}^2}^2) \, ds \\ &\quad + \int_0^t ((\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) \|(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n\|_{\mathbf{L}^2}^2 + \gamma \|\mathbf{q}_n\|_{\mathbf{L}^2}^2 - \langle \mathbf{g}, \mathbf{v}_n \rangle) \, ds \Big) \\ &\geq \left(\frac{1}{2} \|\mathbf{v}(t)\|_{\mathbf{L}^2}^2 + \mathcal{F}(\mathbf{d}(t)) + \int_0^t (\mu_1 \|\mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d}\|_{\mathbf{L}^2}^2 + \mu_4 \|(\nabla \mathbf{v})_{\text{sym}}\|_{\mathbf{L}^2}^2) \, ds \right. \\ &\quad \left. + \int_0^t ((\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) \|(\nabla \mathbf{v})_{\text{sym}} \mathbf{d}\|_{\mathbf{L}^2}^2 + \gamma \|\mathbf{q}\|_{\mathbf{L}^2}^2 - \langle \mathbf{g}, \mathbf{v} \rangle) \, ds \right). \quad (16) \end{aligned}$$

Note that $\mathbf{v}_n \in C_w([0, T]; \mathbf{L}_\sigma^2)$ and $\mathbf{d}_n \in C_w([0, T]; \mathbf{H}^1)$. However, we are not able to identify the limit of the remaining term $(\mathbf{q}_n, (\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n)$ since \mathbf{q}_n and $(\nabla \mathbf{v}_n)_{\text{sym}} \mathbf{d}_n$ only converge weakly. Thus, it remains open whether a suitable weak solution in the sense of Definition 2.2 exists.

Nevertheless, the existence of a suitable weak solution (see Definition 2.2) can be proved when Parodi's relation $\gamma(\mu_2 + \mu_3) = \lambda$ is assumed. Then the last term in the energy inequality (14) vanishes, and with (16) the energy inequality also holds for the limit of the approximate solutions, which is the weak solution.

3. Properties of the strong solution.

Lemma 3.1 (Energy equation). *A strong solution $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$ (see Definition 2.3) of the system (4)–(11) fulfills the energy equation*

$$\begin{aligned} &\frac{1}{2} \|\tilde{\mathbf{v}}(t)\|_{\mathbf{L}^2}^2 + \mathcal{F}(\tilde{\mathbf{d}}(t)) + \int_0^t \left(\mu_1 \|\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{\mathbf{L}^2}^2 + \mu_4 \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^2}^2 \right) \, ds \\ &\quad + \int_0^t \left((\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) \|(\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{\mathbf{L}^2}^2 + \gamma \|\tilde{\mathbf{q}}\|_{\mathbf{L}^2}^2 \right) \, ds \\ &= \frac{1}{2} \|\tilde{\mathbf{v}}_0\|_{\mathbf{L}^2}^2 + \mathcal{F}(\tilde{\mathbf{d}}_0) + \int_0^t (\langle \mathbf{g}, \tilde{\mathbf{v}} \rangle + (\gamma(\mu_2 + \mu_3) - \lambda) (\tilde{\mathbf{q}}, (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}})) \, ds \end{aligned} \quad (17)$$

for $t \in [0, T]$.

Proof. Due to the regularity assumptions (15) on the strong solution, we can take $(\tilde{\mathbf{v}}, \tilde{\mathbf{q}})$ as test functions in (13) and obtain the energy equation in the same way as in [9, Proposition 4.1]. \square

Lemma 3.2 (Regularity of the strong solution). *A strong solution $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$ (see Definition 2.3) admits the regularity*

$$\partial_t \tilde{\mathbf{v}} \in L^2(0, T; (\mathbf{H}_{0, \sigma}^1)^*), \quad \partial_t \tilde{\mathbf{d}} \in L^1(0, T; \mathbf{L}^3). \quad (18)$$

Proof. First, we estimate the time derivative of $\tilde{\mathbf{v}}$. Let $\varphi \in L^2(0, T; \mathbf{H}_{0, \sigma}^1)$ be a test function in (13a). We estimate the terms individually. Because of the continuous embedding $\mathbf{H}_{0, \sigma}^1 \hookrightarrow \mathbf{L}^6$, we obtain for the convection term

$$\int_0^T |(\tilde{\mathbf{v}}(t) \cdot \nabla) \tilde{\mathbf{v}}(t), \varphi(t)| dt \leq \|\tilde{\mathbf{v}}\|_{L^\infty(\mathbf{L}^2)} \|\nabla \tilde{\mathbf{v}}\|_{L^2(\mathbf{L}^3)} \|\varphi\|_{L^2(\mathbf{H}_{0, \sigma}^1)}.$$

Similarly, the Ericksen stress can be estimated as

$$\int_0^T |(\nabla \tilde{\mathbf{d}}(t)^T \tilde{\mathbf{q}}(t), \varphi(t))| dt \leq \|\nabla \tilde{\mathbf{d}}\|_{L^\infty(\mathbf{L}^2)} \|\tilde{\mathbf{q}}\|_{L^2(\mathbf{L}^3)} \|\varphi\|_{L^2(\mathbf{H}_{0, \sigma}^1)}.$$

For the right-hand side, we find that

$$\int_0^T |(\mathbf{g}(t), \varphi(t))| dt \leq \|\mathbf{g}\|_{L^2((\mathbf{H}_{0, \sigma}^1)^*)} \|\varphi\|_{L^2(\mathbf{H}_0^1)}.$$

The definition of the Leslie stress tensor (see (8)) yields

$$\begin{aligned} & \int_0^T |(\mathbf{T}^L(t); \nabla \varphi(t))| dt \\ & \leq \left(\mu_1 \|\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{L^2(\mathbf{L}^6)} \|\tilde{\mathbf{d}}\|_{L^\infty(\mathbf{L}^6)}^2 + \mu_4 \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{L^2(\mathbf{L}^2)} \right. \\ & \quad + (|\gamma(\mu_2 + \mu_3)| + 1) \|\tilde{\mathbf{d}}\|_{L^\infty(\mathbf{L}^6)} \|\tilde{\mathbf{q}}\|_{L^2(\mathbf{L}^3)} \\ & \quad \left. + ((\mu_5 + \mu_6) - \lambda(\mu_2 + \mu_3)) \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{L^2(\mathbf{L}^6)} \|\tilde{\mathbf{d}}\|_{L^\infty(\mathbf{L}^6)}^2 \right) \|\varphi\|_{L^2(\mathbf{H}_{0, \sigma}^1)} \\ & \leq c \left(\|\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{L^2(\mathbf{L}^6)} + \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{L^2(\mathbf{L}^6)} + \|\tilde{\mathbf{q}}\|_{L^\infty(\mathbf{L}^3)} \right) \left(\|\tilde{\mathbf{d}}\|_{L^\infty(\mathbf{L}^6)}^2 + 1 \right) \\ & \quad \times \|\varphi\|_{L^2(\mathbf{H}_{0, \sigma}^1)}. \end{aligned}$$

Due to the regularity assumptions on the strong solution (see Definition 2.3), the variational derivative (7) of the free energy can be estimated in terms of the $L^2(0, T; \mathbf{L}^3)$ -norm by standard embeddings and the Gagliardo–Nirenberg inequality [30, Section 21.19],

$$\begin{aligned} \|\tilde{\mathbf{q}}\|_{L^2(\mathbf{L}^3)} & \leq |\mathbf{A}| \|\tilde{\mathbf{d}}\|_{L^2(\mathbf{W}^{2,3})} + \frac{1}{\varepsilon} \left(\|\tilde{\mathbf{d}}\|_{L^6(\mathbf{L}^9)}^3 + \|\tilde{\mathbf{d}}\|_{L^2(\mathbf{L}^3)} \right) \\ & \leq c \left(\|\tilde{\mathbf{d}}\|_{L^2(\mathbf{W}^{2,3})} + \|\tilde{\mathbf{d}}\|_{L^2(\mathbf{W}^{2,3})}^{1/3} \|\tilde{\mathbf{d}}\|_{L^\infty(\mathbf{L}^6)}^{8/3} + 1 \right). \end{aligned} \quad (19)$$

Note that ε is a constant parameter. Altogether, we observe that $\partial_t \tilde{\mathbf{v}} \in L^2(0, T; (\mathbf{H}_{0, \sigma}^1)^*)$ and

$$\begin{aligned} & \|\partial_t \tilde{\mathbf{v}}\|_{L^2((\mathbf{H}_{0, \sigma}^1)^*)} \\ & \leq \|\tilde{\mathbf{v}}\|_{L^\infty(\mathbf{L}^2)} \|\nabla \tilde{\mathbf{v}}\|_{L^2(\mathbf{L}^3)} + \|\nabla \tilde{\mathbf{d}}\|_{L^\infty(\mathbf{L}^2)} \|\tilde{\mathbf{q}}\|_{L^2(\mathbf{L}^3)} + \|\mathbf{g}\|_{L^2((\mathbf{H}_{0, \sigma}^1)^*)} \end{aligned}$$

$$+ c \left(\left(\|\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{L^2(\mathbf{L}^6)} + \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{L^2(\mathbf{L}^6)} + \|\tilde{\mathbf{q}}\|_{L^\infty(\mathbf{L}^3)} \right) \left(\|\tilde{\mathbf{d}}\|_{L^\infty(\mathbf{L}^6)}^2 + 1 \right) \right)$$

with $\tilde{\mathbf{q}}$ estimated in (19).

Recalling equation (13b) and estimate (19), standard embeddings show that $\partial_t \tilde{\mathbf{d}} \in L^1(0, T; \mathbf{L}^3)$ with

$$\begin{aligned} \|\partial_t \tilde{\mathbf{d}}\|_{L^1(\mathbf{L}^3)} &\leq \|\tilde{\mathbf{v}}\|_{L^2(\mathbf{L}^\infty)} \|\nabla \tilde{\mathbf{d}}\|_{L^2(\mathbf{L}^3)} + \|(\nabla \tilde{\mathbf{v}})_{\text{skw}}\|_{L^2(\mathbf{L}^3)} \|\tilde{\mathbf{d}}\|_{L^2(\mathbf{L}^\infty)} \\ &\quad + |\lambda| \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{L^2(\mathbf{L}^3)} \|\tilde{\mathbf{d}}\|_{L^2(\mathbf{L}^\infty)} + \|\tilde{\mathbf{q}}\|_{L^1(\mathbf{L}^3)} \\ &\leq c \left(\|\tilde{\mathbf{v}}\|_{L^2(\mathbf{W}^{1,6})} \|\tilde{\mathbf{d}}\|_{L^2(\mathbf{W}^{2,3})} + \|\tilde{\mathbf{q}}\|_{L^2(\mathbf{L}^3)} \right). \end{aligned}$$

□

4. Integration-by-parts formulae. In the course of the proof of Theorem 2.4, we shall employ the following three integration-by-parts formulae.

Lemma 4.1. *For functions (\mathbf{v}, \mathbf{d}) and $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$ fulfilling (12) and (18), respectively, the integration-by-parts formulae*

$$\begin{aligned} (\mathbf{v}(t), \tilde{\mathbf{v}}(t)) - (\mathbf{v}(s), \tilde{\mathbf{v}}(s)) &= \int_s^t (\langle \mathbf{v}(\tau), \partial_t \tilde{\mathbf{v}}(\tau) \rangle + \langle \partial_t \mathbf{v}(\tau), \tilde{\mathbf{v}}(\tau) \rangle) d\tau, \\ (\nabla \mathbf{d}(t); \mathbf{\Lambda} : \nabla \tilde{\mathbf{d}}(t)) - (\nabla \mathbf{d}(s); \mathbf{\Lambda} : \nabla \tilde{\mathbf{d}}(s)) \\ &= - \int_s^t \left((\partial_t \mathbf{d}(\tau), \Delta_{\mathbf{\Lambda}} \tilde{\mathbf{d}}(\tau)) + (\Delta_{\mathbf{\Lambda}} \mathbf{d}(\tau), \partial_t \tilde{\mathbf{d}}(\tau)) \right) d\tau, \\ (|\mathbf{d}(t)|^2, |\tilde{\mathbf{d}}(t)|^2) - (|\mathbf{d}(s)|^2, |\tilde{\mathbf{d}}(s)|^2) \\ &= 2 \int_s^t \left((\partial_t \mathbf{d}(\tau) \cdot \mathbf{d}(\tau), |\tilde{\mathbf{d}}(\tau)|^2) + (|\mathbf{d}(\tau)|^2, \partial_t \tilde{\mathbf{d}}(\tau) \cdot \tilde{\mathbf{d}}(\tau)) \right) d\tau \end{aligned} \quad (20)$$

hold true for every $s, t \in [0, T]$.

Proof. We choose two approximate sequences $\{\mathbf{v}_n\}$ and $\{\tilde{\mathbf{v}}_n\}$ with $\mathbf{v}_n \in \mathcal{C}^1([0, T]; \mathbf{H}_{0,\sigma}^1)$ and $\tilde{\mathbf{v}}_n \in \mathcal{C}^1([0, T]; \mathbf{W}_{0,\sigma}^{1,6})$ such that

$$\begin{aligned} \mathbf{v}_n &\rightarrow \mathbf{v} \in L^2(0, T; \mathbf{H}_{0,\sigma}^1) \cap W^{1,2}(0, T; (\mathbf{W}_{0,\sigma}^{1,6})^*), \\ \tilde{\mathbf{v}}_n &\rightarrow \tilde{\mathbf{v}} \in L^2(0, T; \mathbf{W}_{0,\sigma}^{1,6}) \cap W^{1,2}(0, T; (\mathbf{H}_{0,\sigma}^1)^*), \end{aligned} \quad (21)$$

which is possible in view of density. For the approximate sequences, the integration-by-parts formula

$$(\mathbf{v}_n(t), \tilde{\mathbf{v}}_n(t)) - (\mathbf{v}_n(s), \tilde{\mathbf{v}}_n(s)) = \int_s^t ((\mathbf{v}_n(\tau), \partial_t \tilde{\mathbf{v}}_n(\tau)) + (\partial_t \mathbf{v}_n(\tau), \tilde{\mathbf{v}}_n(\tau))) d\tau \quad (22)$$

obviously holds true for all $s, t \in [0, T]$. In the following, we derive estimates for the terms on the left-hand side of (22). Let us define a partition of the unity via a function $\phi \in \mathcal{C}^1([0, T])$ with

$$|\phi(t)| \leq 1 \quad \text{for all } t \in [0, T], \quad \phi(0) = 0 \text{ and } \phi(T) = 1.$$

Let $\mathbf{u} \in \mathcal{C}^1([0, T]; \mathbf{H}_{0,\sigma}^1)$ and $\tilde{\mathbf{u}} \in \mathcal{C}^1([0, T]; \mathbf{W}_{0,\sigma}^{1,6})$. We have that

$$\mathbf{u}(t) = \phi(t)\mathbf{u}(t) + (1 - \phi(t))\mathbf{u}(t), \quad \tilde{\mathbf{u}}(t) = \phi(t)\tilde{\mathbf{u}}(t) + (1 - \phi(t))\tilde{\mathbf{u}}(t).$$

We abbreviate $\bar{\phi}(t) := 1 - \phi(t)$ for all $t \in [0, T]$ such that $\bar{\phi}(T) = 0$. It can easily be seen that for $t \in [0, T]$

$$\begin{aligned} \phi(t)(\mathbf{u}(t), \tilde{\mathbf{u}}(t)) &= \phi(0)(\mathbf{u}(0), \tilde{\mathbf{u}}(0)) + \int_0^t \phi'(\tau)(\mathbf{u}(\tau), \tilde{\mathbf{u}}(\tau)) \, d\tau \\ &\quad + \int_0^t \phi(\tau)((\partial_t \mathbf{u}(\tau), \tilde{\mathbf{u}}(\tau)) + (\mathbf{u}(\tau), \partial_t \tilde{\mathbf{u}}(\tau))) \, d\tau \\ \bar{\phi}(t)(\mathbf{u}(t), \tilde{\mathbf{u}}(t)) &= \bar{\phi}(T)(\mathbf{u}(T), \tilde{\mathbf{u}}(T)) + \int_t^T \phi'(\tau)(\mathbf{u}(\tau), \tilde{\mathbf{u}}(\tau)) \, d\tau \\ &\quad - \int_t^T \bar{\phi}(\tau)((\partial_t \mathbf{u}(\tau), \tilde{\mathbf{u}}(\tau)) + (\mathbf{u}(\tau), \partial_t \tilde{\mathbf{u}}(\tau))) \, d\tau. \end{aligned}$$

Summing up the two previous equations, we find that for all $t \in [0, T]$

$$\begin{aligned} (\mathbf{u}(t), \tilde{\mathbf{u}}(t)) &= \int_0^T \phi'(\tau)(\mathbf{u}(\tau), \tilde{\mathbf{u}}(\tau)) \, d\tau + \int_0^T \phi(\tau)((\partial_t \mathbf{u}(\tau), \tilde{\mathbf{u}}(\tau)) + (\mathbf{u}(\tau), \partial_t \tilde{\mathbf{u}}(\tau))) \, d\tau \\ &\quad - \int_t^T ((\partial_t \mathbf{u}(\tau), \tilde{\mathbf{u}}(\tau)) + (\mathbf{u}(\tau), \partial_t \tilde{\mathbf{u}}(\tau))) \, d\tau \\ &\leq \max_{t \in [0, T]} |\phi'(t)| \|\mathbf{u}\|_{L^2(\mathbf{L}^2)} \|\tilde{\mathbf{u}}\|_{L^2(\mathbf{L}^2)} + 2\|\partial_t \mathbf{u}\|_{L^2((\mathbf{W}_{0,\sigma}^{1,6})^*)} \|\tilde{\mathbf{u}}\|_{L^2(\mathbf{W}_{0,\sigma}^{1,6})} \\ &\quad + 2\|\mathbf{u}\|_{L^2(\mathbf{H}_{0,\sigma}^1)} \|\partial_t \tilde{\mathbf{u}}\|_{L^2((\mathbf{H}_{0,\sigma}^1)^*)}. \end{aligned} \tag{23}$$

The above estimate is now applied to the left-hand side of (22). Since $\{\mathbf{v}_n\}$ and $\{\tilde{\mathbf{v}}_n\}$ are Cauchy sequences in the spaces indicated in (21), we observe with

$$\begin{aligned} (\mathbf{v}_n(t), \tilde{\mathbf{v}}_n(t)) - (\mathbf{v}_m(t), \tilde{\mathbf{v}}_m(t)) &= (\mathbf{v}_n(t) - \mathbf{v}_m(t), \tilde{\mathbf{v}}_n(t)) + (\mathbf{v}_m(t), \tilde{\mathbf{v}}_n(t) - \tilde{\mathbf{v}}_m(t)) \\ &\quad \text{for } t \in [0, T] \text{ and } m, n \in \mathbb{N} \end{aligned}$$

and the estimate (23) that $\{(\mathbf{v}_n, \tilde{\mathbf{v}}_n)\}$ is a Cauchy sequence in $\mathcal{C}([0, T])$. Since $\mathcal{C}([0, T])$ is complete and since the limit is unique, the sequence $\{(\mathbf{v}_n, \tilde{\mathbf{v}}_n)\}$ converges to $(\mathbf{v}, \tilde{\mathbf{v}})$ in $\mathcal{C}([0, T])$. For the approximation of the terms on the right-hand side of the identity (22), we observe that the difference of the approximation and the limit can be estimated by

$$\begin{aligned} &\left| \int_s^t \langle \mathbf{v}(\tau), \partial_t \tilde{\mathbf{v}}(\tau) \rangle - (\mathbf{v}_n(\tau), \partial_t \tilde{\mathbf{v}}_n(\tau)) \, d\tau \right| \\ &\leq \|\mathbf{v}\|_{L^2(\mathbf{H}_{0,\sigma}^1)} \|\partial_t \tilde{\mathbf{v}} - \partial_t \tilde{\mathbf{v}}_n\|_{L^2((\mathbf{H}_{0,\sigma}^1)^*)} + \|\mathbf{v} - \mathbf{v}_n\|_{L^2(\mathbf{H}_{0,\sigma}^1)} \|\partial_t \tilde{\mathbf{v}}_n\|_{L^2((\mathbf{H}_{0,\sigma}^1)^*)}, \\ &\left| \int_s^t \langle \partial_t \mathbf{v}(\tau), \tilde{\mathbf{v}}(\tau) \rangle - (\partial_t \mathbf{v}_n(\tau), \tilde{\mathbf{v}}_n(\tau)) \, d\tau \right| \\ &\leq \|\partial_t \mathbf{v} - \partial_t \mathbf{v}_n\|_{L^2((\mathbf{W}_{0,\sigma}^{1,6})^*)} \|\tilde{\mathbf{v}}\|_{L^2(\mathbf{W}_{0,\sigma}^{1,6})} + \|\partial_t \mathbf{v}_n\|_{L^2(\mathbf{W}_{0,\sigma}^{1,6})^*} \|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_n\|_{L^2(\mathbf{W}_{0,\sigma}^{1,6})}. \end{aligned}$$

The right-hand sides of the above estimates converge to zero as $n \rightarrow \infty$ since $\{\mathbf{v}_n\}$ and $\{\tilde{\mathbf{v}}_n\}$ converge to \mathbf{v} and $\tilde{\mathbf{v}}$ in the sense of (21). This proves that the integration-by-parts formula (22) holds for \mathbf{v} and $\tilde{\mathbf{v}}$.

In order to prove the second and third formula in (20), \mathbf{d} and $\tilde{\mathbf{d}}$ are approximated by sequences of smooth functions \mathbf{d}_n and $\tilde{\mathbf{d}}_n$ in $L^2(0, T; \mathbf{H}^2) \cap W^{1,2}(0, T; \mathbf{L}^{3/2})$ and $L^2(0, T; \mathbf{W}^{2,3}) \cap W^{1,2}(0, T; \mathbf{L}^2)$, respectively. The approximate sequences can be chosen such that every element, i.e., \mathbf{d}_n and $\tilde{\mathbf{d}}_n$, of these sequences fulfills the same boundary conditions as \mathbf{d} and $\tilde{\mathbf{d}}$, respectively. Since the boundary values of \mathbf{d}_n and $\tilde{\mathbf{d}}_n$ are constant in time, their time derivative vanishes on the boundary. Hence, there are no boundary terms in the integration-by-parts formula

$$\begin{aligned}
& \left(\nabla \mathbf{d}_n(t); \boldsymbol{\Lambda} : \nabla \tilde{\mathbf{d}}_n(t) \right) - \left(\nabla \mathbf{d}_n(s); \boldsymbol{\Lambda} : \nabla \tilde{\mathbf{d}}_n(s) \right) \\
&= \int_s^t \left(\left(\nabla \partial_t \mathbf{d}_n(\tau); \boldsymbol{\Lambda} : \nabla \tilde{\mathbf{d}}_n(\tau) \right) + \left(\nabla \mathbf{d}_n(\tau); \boldsymbol{\Lambda} : \nabla \partial_t \tilde{\mathbf{d}}_n(\tau) \right) \right) d\tau \\
&= - \int_s^t \left(\left(\partial_t \mathbf{d}_n(\tau), \Delta_{\boldsymbol{\Lambda}} \tilde{\mathbf{d}}_n(\tau) \right) + \left(\Delta_{\boldsymbol{\Lambda}} \mathbf{d}_n(\tau), \partial_t \tilde{\mathbf{d}}_n(\tau) \right) \right) d\tau.
\end{aligned}$$

Passing to the limit as $n \rightarrow \infty$ shows the second integration-by-parts formula in (20). The third integration-by-parts formula in (20) is proved by the same approximation. \square

5. Proof of the main result. We define the relative energy for two solutions (\mathbf{v}, \mathbf{d}) and $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$ to system (4) by

$$\mathcal{E}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) := \frac{1}{2} \|\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbf{L}^2}^2 + \frac{1}{2} \|\nabla \mathbf{d} - \nabla \tilde{\mathbf{d}}\|_{\boldsymbol{\Lambda}}^2 + \frac{1}{4\varepsilon} \left\| (|\mathbf{d}|^2 - 1) - (|\tilde{\mathbf{d}}|^2 - 1) \right\|_{\mathbf{L}^2}^2, \quad (24)$$

and the relative dissipation by

$$\begin{aligned}
\mathcal{W}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) &:= \mu_1 \left\| \mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - \tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}} \right\|_{\mathbf{L}^2}^2 + \mu_4 \left\| (\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}} \right\|_{\mathbf{L}^2}^2 \\
&\quad + (\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) \left\| (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}} \right\|_{\mathbf{L}^2}^2 + \gamma \|\mathbf{q} - \tilde{\mathbf{q}}\|_{\mathbf{L}^2}^2, \quad (25)
\end{aligned}$$

where \mathbf{q} and $\tilde{\mathbf{q}}$ are given by (7) for (\mathbf{v}, \mathbf{d}) and $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$, respectively. Note that $\mathcal{E}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}})$ and $\mathcal{W}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}})$ are still functions in $t \in [0, T]$. We abbreviate $\mathcal{E}(\mathbf{v}(t), \mathbf{d}(t) | \tilde{\mathbf{v}}(t), \tilde{\mathbf{d}}(t))$ and $\mathcal{W}(\mathbf{v}(t), \mathbf{d}(t) | \tilde{\mathbf{v}}(t), \tilde{\mathbf{d}}(t))$ by $\mathcal{E}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}})(t)$ and $\mathcal{W}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}})(t)$, respectively

Lemma 5.1. *Let (\mathbf{v}, \mathbf{d}) be a suitable weak solution (see Definition 2.2) to system (4)–(11) for given initial values $(\mathbf{v}_0, \mathbf{d}_0)$. Let $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$ be a strong solution (see Definition 2.3) to system (4)–(11) for given initial values $(\tilde{\mathbf{v}}_0, \tilde{\mathbf{d}}_0)$. Then for almost all $t \in [0, T]$*

$$\begin{aligned}
\mathcal{E}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}})(t) &+ \frac{1}{2} \int_0^t \mathcal{W}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}})(s) e^{\int_s^t \mathcal{K}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}})(\tau) d\tau} ds \\
&\leq \mathcal{E}(\mathbf{v}_0, \mathbf{d}_0 | \tilde{\mathbf{v}}_0, \tilde{\mathbf{d}}_0) e^{\int_0^t \mathcal{K}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}})(s) ds}, \quad (26)
\end{aligned}$$

where \mathcal{K} is given by

$$\begin{aligned}
\mathcal{K}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}})(t) &= c \left(1 + \|\mathbf{d}\|_{\mathbf{L}^\infty(\mathbf{L}^6)}^2 + \|\tilde{\mathbf{d}}\|_{\mathbf{L}^\infty(\mathbf{L}^6)}^2 \right) \\
&\quad \times \left(\|\tilde{\mathbf{v}}(t)\|_{\mathbf{W}^{1,6}}^2 + \|\tilde{\mathbf{q}}(t)\|_{\mathbf{L}^3}^2 + \|\tilde{\mathbf{d}}(t) \cdot (\nabla \tilde{\mathbf{v}}(t))_{\text{sym}} \tilde{\mathbf{d}}(t)\|_{\mathbf{L}^6}^2 \right. \\
&\quad \left. + \|\partial_t \tilde{\mathbf{d}}(t)\|_{\mathbf{L}^3} + \|\tilde{\mathbf{d}}(t)\|^2 - 1\|_{\mathbf{L}^6}^2 + \|\mathbf{v}(t)\|_{\mathbf{L}^6}^2 + \|\nabla \tilde{\mathbf{d}}(t)\|_{\mathbf{L}^2}^2 \right) \quad (27)
\end{aligned}$$

and c is a possibly large constant.

Remark 4. The function (27) only depends on the two norms $\|\mathbf{v}\|_{\mathbf{L}^2(\mathbf{L}^6)}$ and $\|\mathbf{d}\|_{\mathbf{L}^\infty(\mathbf{L}^6)}$ of the suitable weak solution, which are known to be finite. Additionally, it depends on several norms of the strong solution $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$. Due to the regularity assumptions (18) and estimate (19), the function \mathcal{K} is in $L^1(0, T)$. For the relative energy and the relative dissipation, note that $\mathcal{E}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) \in L^\infty(0, T)$ and $\mathcal{W}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) \in L^1(0, T)$ due to (12) and (14), respectively.

Proof. Consider the relative energy

$$\begin{aligned} \mathcal{E}(\mathbf{v}, \mathbf{d}|\tilde{\mathbf{v}}, \tilde{\mathbf{d}}) &= \frac{1}{2}\|\mathbf{v}\|_{\mathbf{L}^2}^2 + \frac{1}{2}\|\nabla\mathbf{d}\|_{\mathbf{\Lambda}}^2 + \frac{1}{4\varepsilon}\|\mathbf{d}\|^2 - 1\|_{L^2}^2 + \frac{1}{2}\|\tilde{\mathbf{v}}\|_{\mathbf{L}^2}^2 + \frac{1}{2}\|\nabla\tilde{\mathbf{d}}\|_{\mathbf{\Lambda}}^2 \\ &\quad + \frac{1}{4\varepsilon}\|\tilde{\mathbf{d}}\|^2 - 1\|_{L^2}^2 - (\mathbf{v}, \tilde{\mathbf{v}}) - (\nabla\mathbf{d}; \mathbf{\Lambda} : \nabla\tilde{\mathbf{d}}) - \frac{1}{2\varepsilon}(|\mathbf{d}|^2 - 1, |\tilde{\mathbf{d}}|^2 - 1). \end{aligned}$$

We insert the energy inequality (14) for the suitable weak solution (\mathbf{v}, \mathbf{d}) and the energy equation (17) for the strong solution $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$. This leads to

$$\begin{aligned} \mathcal{E}(\mathbf{v}, \mathbf{d}|\tilde{\mathbf{v}}, \tilde{\mathbf{d}}) &\leq \frac{1}{2}\|\mathbf{v}_0\|_{\mathbf{L}^2}^2 + \frac{1}{2}\|\tilde{\mathbf{v}}_0\|_{\mathbf{L}^2}^2 + \mathcal{F}(\mathbf{d}_0) + \mathcal{F}(\tilde{\mathbf{d}}_0) \\ &\quad - \mu_1 \int_0^t \left(\|\mathbf{d} \cdot (\nabla\mathbf{v})_{\text{sym}}\mathbf{d}\|_{L^2}^2 + \|\tilde{\mathbf{d}} \cdot (\nabla\tilde{\mathbf{v}})_{\text{sym}}\tilde{\mathbf{d}}\|_{L^2}^2 \right) ds \\ &\quad - \mu_4 \int_0^t \left(\|(\nabla\mathbf{v})_{\text{sym}}\|_{L^2}^2 + \|(\nabla\tilde{\mathbf{v}})_{\text{sym}}\|_{L^2}^2 \right) ds \\ &\quad - (\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) \int_0^t \left(\|(\nabla\mathbf{v})_{\text{sym}}\mathbf{d}\|_{L^2}^2 + \|(\nabla\tilde{\mathbf{v}})_{\text{sym}}\tilde{\mathbf{d}}\|_{L^2}^2 \right) ds \\ &\quad - \gamma \int_0^t \left(\|\mathbf{q}\|_{L^2}^2 + \|\tilde{\mathbf{q}}\|_{L^2}^2 \right) ds + \int_0^t \langle \mathbf{g}, \mathbf{v} + \tilde{\mathbf{v}} \rangle ds \\ &\quad + (\gamma(\mu_2 + \mu_3) - \lambda) \int_0^t \left(\langle \mathbf{q}, (\nabla\mathbf{v})_{\text{sym}}\mathbf{d} \rangle + \langle \tilde{\mathbf{q}}, (\nabla\tilde{\mathbf{v}})_{\text{sym}}\tilde{\mathbf{d}} \rangle \right) ds \\ &\quad - (\mathbf{v}, \tilde{\mathbf{v}}) - (\nabla\mathbf{d}; \mathbf{\Lambda} : \nabla\tilde{\mathbf{d}}) - \frac{1}{2\varepsilon}(|\mathbf{d}|^2 - 1, |\tilde{\mathbf{d}}|^2 - 1). \end{aligned}$$

Adding the integral over the relative dissipation yields

$$\begin{aligned} \mathcal{E}(\mathbf{v}, \mathbf{d}|\tilde{\mathbf{v}}, \tilde{\mathbf{d}}) &+ \int_0^t \mathcal{W}(\mathbf{v}, \mathbf{d}|\tilde{\mathbf{v}}, \tilde{\mathbf{d}}) ds \leq \frac{1}{2}\|\mathbf{v}_0\|_{\mathbf{L}^2}^2 + \frac{1}{2}\|\tilde{\mathbf{v}}_0\|_{\mathbf{L}^2}^2 + \mathcal{F}(\mathbf{d}_0) + \mathcal{F}(\tilde{\mathbf{d}}_0) \\ &\quad - 2\mu_1 \int_0^t \left(\mathbf{d} \cdot (\nabla\mathbf{v})_{\text{sym}}\mathbf{d}, \tilde{\mathbf{d}} \cdot (\nabla\tilde{\mathbf{v}})_{\text{sym}}\tilde{\mathbf{d}} \right) ds - 2\mu_4 \int_0^t \left((\nabla\mathbf{v})_{\text{sym}}; (\nabla\tilde{\mathbf{v}})_{\text{sym}} \right) ds \\ &\quad - 2(\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) \int_0^t \left((\nabla\mathbf{v})_{\text{sym}}\mathbf{d}, (\nabla\tilde{\mathbf{v}})_{\text{sym}}\tilde{\mathbf{d}} \right) ds - 2\gamma \int_0^t \langle \mathbf{q}, \tilde{\mathbf{q}} \rangle ds \\ &\quad + \int_0^t \langle \mathbf{g}, \mathbf{v} + \tilde{\mathbf{v}} \rangle ds + (\gamma(\mu_2 + \mu_3) - \lambda) \int_0^t \left(\langle \mathbf{q}, (\nabla\mathbf{v})_{\text{sym}}\mathbf{d} \rangle + \langle \tilde{\mathbf{q}}, (\nabla\tilde{\mathbf{v}})_{\text{sym}}\tilde{\mathbf{d}} \rangle \right) ds \\ &\quad - (\mathbf{v}, \tilde{\mathbf{v}}) - (\nabla\mathbf{d}; \mathbf{\Lambda} : \nabla\tilde{\mathbf{d}}) - \frac{1}{2\varepsilon}(|\mathbf{d}|^2 - 1, |\tilde{\mathbf{d}}|^2 - 1). \end{aligned} \tag{28}$$

The last term can be written via Lemma 4.1 as

$$\begin{aligned} &-\frac{1}{2\varepsilon}(|\mathbf{d}|^2 - 1, |\tilde{\mathbf{d}}|^2 - 1)(t) + \frac{1}{2\varepsilon}(|\mathbf{d}_0|^2 - 1, |\tilde{\mathbf{d}}_0|^2 - 1) \\ &= -\frac{1}{2\varepsilon} \int_0^t \left((|\mathbf{d}|^2 - 1, \partial_t |\tilde{\mathbf{d}}|^2) + (\partial_t |\mathbf{d}|^2, |\tilde{\mathbf{d}}|^2 - 1) \right) ds \\ &= -\frac{1}{2\varepsilon} \int_0^t \left(((|\mathbf{d}|^2 - 1) - (|\tilde{\mathbf{d}}|^2 - 1), \partial_t |\tilde{\mathbf{d}}|^2) + (\partial_t |\mathbf{d}|^2 + \partial_t |\tilde{\mathbf{d}}|^2, |\tilde{\mathbf{d}}|^2 - 1) \right) ds \\ &= -\frac{1}{2\varepsilon} \int_0^t \left(((|\mathbf{d}|^2 - 1) - (|\tilde{\mathbf{d}}|^2 - 1), 2\partial_t \tilde{\mathbf{d}} \cdot (\tilde{\mathbf{d}} - \mathbf{d})) + (\partial_t |\mathbf{d} - \tilde{\mathbf{d}}|^2, |\tilde{\mathbf{d}}|^2 - 1) \right) ds \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2\varepsilon} \int_0^t \left((|\mathbf{d}|^2 - 1) - (|\tilde{\mathbf{d}}|^2 - 1), 2\partial_t \tilde{\mathbf{d}} \cdot \mathbf{d} + (2\partial_t(\mathbf{d} \cdot \tilde{\mathbf{d}}), |\tilde{\mathbf{d}}|^2 - 1) \right) ds \\
&= -\frac{1}{2\varepsilon} \int_0^t \left((|\mathbf{d}|^2 - 1) - (|\tilde{\mathbf{d}}|^2 - 1), 2\partial_t \tilde{\mathbf{d}} \cdot (\tilde{\mathbf{d}} - \mathbf{d}) + (\partial_t |\mathbf{d} - \tilde{\mathbf{d}}|^2, |\tilde{\mathbf{d}}|^2 - 1) \right) ds \\
& \quad - \frac{1}{\varepsilon} \int_0^t \left((|\mathbf{d}|^2 - 1)\mathbf{d}, \partial_t \tilde{\mathbf{d}} + (\partial_t \mathbf{d}, (|\tilde{\mathbf{d}}|^2 - 1)\tilde{\mathbf{d}}) \right) ds
\end{aligned}$$

Recall the definition of the variational derivative of the free energy (see (7)),

$$\begin{aligned}
& -\frac{1}{\varepsilon} \int_0^t \left((|\mathbf{d}|^2 - 1)\mathbf{d}, \partial_t \tilde{\mathbf{d}} + (\partial_t \mathbf{d}, (|\tilde{\mathbf{d}}|^2 - 1)\tilde{\mathbf{d}}) \right) ds \\
& \quad + \int_0^t \left((\partial_t \tilde{\mathbf{d}}, \Delta_{\Lambda} \mathbf{d}) + (\partial_t \mathbf{d}, \Delta_{\Lambda} \tilde{\mathbf{d}}) \right) ds = - \int_0^t \left(\mathbf{q}, \partial_t \tilde{\mathbf{d}} + (\partial_t \mathbf{d}, \tilde{\mathbf{q}}) \right) ds.
\end{aligned}$$

Due to the integration-by-parts formulae (20) and the two previous equations, the last line in (28) can be reformulated as

$$\begin{aligned}
& -(\mathbf{v}, \tilde{\mathbf{v}}) - (\nabla \mathbf{d}; \mathbf{\Lambda} : \nabla \tilde{\mathbf{d}}) - \frac{1}{2\varepsilon} (|\mathbf{d}|^2 - 1, |\tilde{\mathbf{d}}|^2 - 1) \\
&= -(\mathbf{v}_0, \tilde{\mathbf{v}}_0) - (\nabla \mathbf{d}_0; \mathbf{\Lambda} : \nabla \tilde{\mathbf{d}}_0) - \frac{1}{2\varepsilon} (|\mathbf{d}_0|^2 - 1, |\tilde{\mathbf{d}}_0|^2 - 1) \\
& \quad - \frac{1}{2\varepsilon} \int_0^t \left((|\mathbf{d}|^2 - 1) - (|\tilde{\mathbf{d}}|^2 - 1), 2\partial_t \tilde{\mathbf{d}} \cdot (\tilde{\mathbf{d}} - \mathbf{d}) + (\partial_t |\mathbf{d} - \tilde{\mathbf{d}}|^2, |\tilde{\mathbf{d}}|^2 - 1) \right) ds \\
& \quad - \int_0^t \left(\langle \mathbf{v}, \partial_t \tilde{\mathbf{v}} \rangle + \langle \partial_t \mathbf{v}, \tilde{\mathbf{v}} \rangle + \langle \mathbf{q}, \partial_t \tilde{\mathbf{d}} \rangle + \langle \partial_t \mathbf{d}, \tilde{\mathbf{q}} \rangle \right) ds.
\end{aligned} \tag{29}$$

Note that the sum of terms with the initial conditions $(\mathbf{v}_0, \mathbf{d}_0)$ and $(\tilde{\mathbf{v}}_0, \tilde{\mathbf{d}}_0)$ appearing in (28) and (29) is the relative energy of the initial values,

$$\begin{aligned}
& \frac{1}{2} \|\mathbf{v}_0\|_{L^2}^2 + \frac{1}{2} \|\tilde{\mathbf{v}}_0\|_{L^2}^2 + \mathcal{F}(\mathbf{d}_0) + \mathcal{F}(\tilde{\mathbf{d}}_0) - (\mathbf{v}_0, \tilde{\mathbf{v}}_0) - (\nabla \mathbf{d}_0; \mathbf{\Lambda} : \nabla \tilde{\mathbf{d}}_0) \\
& \quad - \frac{1}{2\varepsilon} (|\mathbf{d}_0|^2 - 1, |\tilde{\mathbf{d}}_0|^2 - 1) = \mathcal{E}(\mathbf{v}_0, \mathbf{d}_0 | \tilde{\mathbf{v}}_0, \tilde{\mathbf{d}}_0).
\end{aligned}$$

In order to calculate the last line in (29) explicitly, we use the fact that (\mathbf{v}, \mathbf{d}) and $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$ are solutions to (4)–(11). This shows with (28) that

$$\begin{aligned}
& \mathcal{E}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) + \int_0^t \mathcal{W}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) ds \\
& \leq \mathcal{E}(\mathbf{v}_0, \mathbf{d}_0 | \tilde{\mathbf{v}}_0, \tilde{\mathbf{d}}_0) + \int_0^t \left(((\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}}, \mathbf{v}) + ((\mathbf{v} \cdot \nabla) \mathbf{v}, \tilde{\mathbf{v}}) \right) ds \\
& \quad + \int_0^t \left(((\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{d}}, \mathbf{q}) + ((\mathbf{v} \cdot \nabla) \mathbf{d}, \tilde{\mathbf{q}}) - (\nabla \tilde{\mathbf{d}}^T \tilde{\mathbf{q}}, \mathbf{v}) - (\nabla \mathbf{d}^T \mathbf{q}, \tilde{\mathbf{v}}) \right) ds \\
& \quad + \mu_1 \int_0^t \left(\left(\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, (\nabla \mathbf{v})_{\text{sym}} : (\tilde{\mathbf{d}} \otimes \tilde{\mathbf{d}} - \mathbf{d} \otimes \mathbf{d}) \right) \right. \\
& \quad \quad \left. + \left(\mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d}, (\nabla \tilde{\mathbf{v}})_{\text{sym}} : (\mathbf{d} \otimes \mathbf{d} - \tilde{\mathbf{d}} \otimes \tilde{\mathbf{d}}) \right) \right) ds \\
& \quad + (\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) \int_0^t \left(\left((\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, (\nabla \mathbf{v})_{\text{sym}} (\tilde{\mathbf{d}} - \mathbf{d}) \right) \right. \\
& \quad \quad \left. + \left((\nabla \mathbf{v})_{\text{sym}} \mathbf{d}, (\nabla \tilde{\mathbf{v}})_{\text{sym}} (\mathbf{d} - \tilde{\mathbf{d}}) \right) \right) ds
\end{aligned}$$

$$\begin{aligned}
& -\gamma(\mu_2 + \mu_3) \int_0^t \left((\tilde{\mathbf{d}} \otimes \tilde{\mathbf{q}}; (\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}}) + (\mathbf{d} \otimes \mathbf{q}; (\nabla \tilde{\mathbf{v}})_{\text{sym}} - (\nabla \mathbf{v})_{\text{sym}}) \right) ds \\
& - \int_0^t \left((\tilde{\mathbf{d}} \otimes \tilde{\mathbf{q}}; (\nabla \mathbf{v})_{\text{skw}}) + (\mathbf{d} \otimes \mathbf{q}; (\nabla \tilde{\mathbf{v}})_{\text{skw}}) - ((\nabla \tilde{\mathbf{v}})_{\text{skw}} \tilde{\mathbf{d}}, \mathbf{q}) - ((\nabla \mathbf{v})_{\text{skw}} \mathbf{d}, \tilde{\mathbf{q}}) \right) ds \\
& + \lambda \int_0^t \left(((\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, \mathbf{q} - \tilde{\mathbf{q}}) + ((\nabla \mathbf{v})_{\text{sym}} \mathbf{d}, \tilde{\mathbf{q}} - \mathbf{q}) \right) ds \\
& - \frac{1}{2\varepsilon} \int_0^t \left((|\mathbf{d}|^2 - 1) - (|\tilde{\mathbf{d}}|^2 - 1), 2\partial_t \tilde{\mathbf{d}} \cdot (\tilde{\mathbf{d}} - \mathbf{d}) + (\partial_t |\mathbf{d} - \tilde{\mathbf{d}}|^2, |\tilde{\mathbf{d}}|^2 - 1) \right) ds \\
& = \mathcal{E}(\mathbf{v}_0, \mathbf{d}_0 | \tilde{\mathbf{v}}_0, \tilde{\mathbf{d}}_0) + I_1 + I_2 + \mu_1 I_3 + (\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) I_4 \\
& - \gamma(\mu_2 + \mu_3) I_5 + I_6 + \lambda I_7 - \frac{1}{2\varepsilon} I_8.
\end{aligned}$$

In the above equation, we have employed the weak formulation for the solutions (\mathbf{v}, \mathbf{d}) and $(\tilde{\mathbf{v}}, \tilde{\mathbf{d}})$ tested with $(\tilde{\mathbf{v}}, \tilde{\mathbf{q}})$ and (\mathbf{v}, \mathbf{q}) , respectively. Note that the choice of test functions is justified due to the additional regularity (see Lemma 3.2). We calculate and estimate the terms on the right-hand side of the above inequality individually. In the following let $\delta > 0$. For the first term I_1 , we observe that

$$\begin{aligned}
I_1 &= \int_0^t \left(((\mathbf{v} \cdot \nabla(\mathbf{v} - \tilde{\mathbf{v}}), \tilde{\mathbf{v}} - \mathbf{v}) + (((\mathbf{v} - \tilde{\mathbf{v}}) \cdot \nabla) \tilde{\mathbf{v}}, \tilde{\mathbf{v}} - \mathbf{v})) \right) ds \\
&= \int_0^t \left((\mathbf{v} - \tilde{\mathbf{v}}) \otimes (\tilde{\mathbf{v}} - \mathbf{v}); (\nabla \tilde{\mathbf{v}})_{\text{sym}} \right) ds \\
&\leq C_\delta \int_0^t \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^3}^2 \|\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbf{L}^2}^2 ds + \delta \int_0^t \|\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbf{L}^6}^2 ds.
\end{aligned}$$

We recall that \mathbf{v} and $\tilde{\mathbf{v}}$ are solenoidal such that $((\mathbf{v} \cdot \nabla) \mathbf{w}, \mathbf{w}) = ((\tilde{\mathbf{v}} \cdot \nabla) \mathbf{w}, \mathbf{w}) = 0$ for all $\mathbf{w} \in \mathbf{H}_{0,\sigma}^1$.

The term I_2 can be estimated by

$$\begin{aligned}
I_2 &= \int_0^t \left(((\mathbf{v} \cdot \nabla)(\mathbf{d} - \tilde{\mathbf{d}}), \tilde{\mathbf{q}}) + ((\tilde{\mathbf{v}} \cdot \nabla)(\tilde{\mathbf{d}} - \mathbf{d}), \mathbf{q}) \right) ds \\
&= \int_0^t \left(((\mathbf{v} - \tilde{\mathbf{v}}), (\nabla \mathbf{d} - \nabla \tilde{\mathbf{d}})^T \tilde{\mathbf{q}}) + ((\tilde{\mathbf{v}} \cdot \nabla)(\tilde{\mathbf{d}} - \mathbf{d}), \mathbf{q} - \tilde{\mathbf{q}}) \right) ds \\
&\leq C_\delta \int_0^t \left(\|\tilde{\mathbf{q}}\|_{\mathbf{L}^3}^2 \|\nabla \mathbf{d} - \nabla \tilde{\mathbf{d}}\|_{\mathbf{L}^2}^2 \right) ds + \delta \int_0^t \|\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbf{L}^6}^2 ds \\
&\quad + C_\delta \int_0^t \|\tilde{\mathbf{v}}\|_{\mathbf{L}^\infty}^2 \|\nabla \tilde{\mathbf{d}} - \nabla \mathbf{d}\|_{\mathbf{L}^2}^2 ds + \delta \int_0^t \|\mathbf{q} - \tilde{\mathbf{q}}\|_{\mathbf{L}^2}^2 ds.
\end{aligned}$$

With the standard embedding $\mathbf{H}_{0,\sigma}^1 \hookrightarrow \mathbf{L}^6$ and Korn's inequality [24, Theorem 10.1], we find that there exists a constant c such that $\|\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbf{L}^6} \leq c \|(\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^2}$. We rearrange and estimate the term I_3 by

$$\begin{aligned}
I_3 &= \int_0^t \left(\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, ((\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}}) : (\tilde{\mathbf{d}} \otimes \tilde{\mathbf{d}} - \mathbf{d} \otimes \mathbf{d}) \right) ds \\
&\quad + \int_0^t \left(\mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - \tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, (\nabla \tilde{\mathbf{v}})_{\text{sym}} : (\mathbf{d} \otimes \mathbf{d} - \tilde{\mathbf{d}} \otimes \tilde{\mathbf{d}}) \right) ds \\
&= \int_0^t \left(\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, ((\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}}) : (\tilde{\mathbf{d}} \otimes (\tilde{\mathbf{d}} - \mathbf{d})) \right) ds
\end{aligned}$$

$$\begin{aligned}
& + \int_0^t \left(\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, \left((\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}} \right) \cdot (\tilde{\mathbf{d}} - \mathbf{d}) \right) ds \\
& + \int_0^t \left(\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, (\nabla \tilde{\mathbf{v}})_{\text{sym}} : (\tilde{\mathbf{d}} - \mathbf{d}) \otimes (\tilde{\mathbf{d}} - \mathbf{d}) \right) ds \\
& + \int_0^t \left(\mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - \tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, (\nabla \tilde{\mathbf{v}})_{\text{sym}} : (\mathbf{d} - \tilde{\mathbf{d}}) \otimes \tilde{\mathbf{d}} \right) ds \\
& + \int_0^t \left(\mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - \tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, (\nabla \tilde{\mathbf{v}})_{\text{sym}} : \mathbf{d} \otimes (\mathbf{d} - \tilde{\mathbf{d}}) \right) ds \\
\leq & C_\delta \|\tilde{\mathbf{d}}\|_{L^\infty(\mathbf{L}^6)}^2 \int_0^t \|\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 \|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 ds \\
& + \delta \int_0^t \|(\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^2}^2 ds \\
& + C_\delta \int_0^t \|\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{\mathbf{L}^3}^2 \|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 ds + \delta \int_0^t \|(\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{\mathbf{L}^2}^2 ds \\
& + \int_0^t \left(\|\tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{\mathbf{L}^3}^2 + \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^3}^2 \right) \|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 ds \\
& + C_\delta \|\tilde{\mathbf{d}}\|_{L^\infty(\mathbf{L}^6)}^2 \int_0^t \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^6}^2 \|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 ds \\
& + \delta \int_0^t \|\mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - \tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{\mathbf{L}^2}^2 ds \\
& + C_\delta \|\mathbf{d}\|_{L^\infty(\mathbf{L}^6)}^2 \int_0^t \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^6}^2 \|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 ds \\
& + \delta \int_0^t \|\mathbf{d} \cdot (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - \tilde{\mathbf{d}} \cdot (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{\mathbf{L}^2}^2 ds.
\end{aligned}$$

The embedding $\mathbf{H}_0^1 \hookrightarrow \mathbf{L}^6$ together with Poincaré's inequality (see Morrey [25, Thm. 6.5.6.]) assures that

$$\|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6} \leq c \|\nabla \mathbf{d} - \nabla \tilde{\mathbf{d}}\|_{\Lambda}.$$

We continue with the term I_4 ,

$$\begin{aligned}
I_4 & = \int_0^t \left((\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, ((\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}}) (\tilde{\mathbf{d}} - \mathbf{d}) \right) ds \\
& + \int_0^t \left((\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}, (\nabla \tilde{\mathbf{v}})_{\text{sym}} (\mathbf{d} - \tilde{\mathbf{d}}) \right) ds \\
& \leq C_\delta \int_0^t \|(\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{\mathbf{L}^3}^2 \|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 ds + \delta \int_0^t \|(\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^2}^2 ds \\
& + C_\delta \int_0^t \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^3}^2 \|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 ds + \delta \int_0^t \|(\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}\|_{\mathbf{L}^2}^2 ds.
\end{aligned}$$

The term I_5 can be rearranged as

$$\begin{aligned}
& I_5 \\
& = \int_0^t \left(((\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}}), (\tilde{\mathbf{d}} - \mathbf{d}) \otimes \tilde{\mathbf{q}} \right) + \left((\nabla \tilde{\mathbf{v}})_{\text{sym}} - (\nabla \mathbf{v})_{\text{sym}}, \mathbf{d} \otimes (\mathbf{q} - \tilde{\mathbf{q}}) \right) ds
\end{aligned}$$

$$\begin{aligned}
 &= \int_0^t \left(((\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}}, (\tilde{\mathbf{d}} - \mathbf{d}) \otimes \tilde{\mathbf{q}}) + ((\nabla \tilde{\mathbf{v}})_{\text{sym}}(\mathbf{d} - \tilde{\mathbf{d}}), \mathbf{q} - \tilde{\mathbf{q}}) \right) \, ds \\
 &\quad + \int_0^t \left((\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}} - (\nabla \mathbf{v})_{\text{sym}} \mathbf{d}, \mathbf{q} - \tilde{\mathbf{q}} \right) \, ds
 \end{aligned}$$

and thus be estimated by

$$\begin{aligned}
 I_5 &\leq \delta \int_0^t \|(\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^2}^2 \, ds + \delta \int_0^t \|\mathbf{q} - \tilde{\mathbf{q}}\|_{\mathbf{L}^2}^2 \, ds \\
 &\quad + C_\delta \int_0^t (\|\tilde{\mathbf{q}}\|_{\mathbf{L}^3}^2 + \|(\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^3}^2) \|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 \, ds \\
 &\quad + \int_0^t \left((\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}} - (\nabla \mathbf{v})_{\text{sym}} \mathbf{d}, \mathbf{q} - \tilde{\mathbf{q}} \right) \, ds.
 \end{aligned}$$

The term I_6 is bounded by

$$\begin{aligned}
 I_6 &= \int_0^t \left(((\nabla \mathbf{v})_{\text{skw}}(\tilde{\mathbf{d}} - \mathbf{d}), \tilde{\mathbf{q}}) + ((\nabla \tilde{\mathbf{v}})_{\text{skw}}(\mathbf{d} - \tilde{\mathbf{d}}), \mathbf{q}) \right) \, ds \\
 &= \int_0^t \left(((\nabla \mathbf{v})_{\text{skw}} - (\nabla \tilde{\mathbf{v}})_{\text{skw}})(\tilde{\mathbf{d}} - \mathbf{d}), \tilde{\mathbf{q}} \right) + \left((\nabla \tilde{\mathbf{v}})_{\text{skw}}(\mathbf{d} - \tilde{\mathbf{d}}), \mathbf{q} - \tilde{\mathbf{q}} \right) \, ds \\
 &\leq C_\delta \int_0^t \|\tilde{\mathbf{q}}\|_{\mathbf{L}^3}^2 \|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 \, ds + \delta \int_0^t \|(\nabla \mathbf{v})_{\text{skw}} - (\nabla \tilde{\mathbf{v}})_{\text{skw}}\|_{\mathbf{L}^2}^2 \, ds \\
 &\quad + C_\delta \int_0^t \|(\nabla \tilde{\mathbf{v}})_{\text{skw}}\|_{\mathbf{L}^3}^2 \|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 \, ds + \delta \int_0^t \|\mathbf{q} - \tilde{\mathbf{q}}\|_{\mathbf{L}^2}^2 \, ds.
 \end{aligned}$$

Note that due to Korn’s and Poincaré’s inequality, we find that

$$\|(\nabla \mathbf{v})_{\text{skw}} - (\nabla \tilde{\mathbf{v}})_{\text{skw}}\|_{\mathbf{L}^2} \leq \|\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbf{H}_0^1} \leq c \|(\nabla \mathbf{v})_{\text{sym}} - (\nabla \tilde{\mathbf{v}})_{\text{sym}}\|_{\mathbf{L}^2}.$$

The term I_7 is already in the form desired. Finally, we estimate I_8 . Starting with the first term, we observe that

$$\begin{aligned}
 &\int_0^t \left((|\mathbf{d}|^2 - 1) - (|\tilde{\mathbf{d}}|^2 - 1) \right) (\tilde{\mathbf{d}} - \mathbf{d}), \partial_t \tilde{\mathbf{d}} \, ds \\
 &\leq \int_0^t \|\partial_t \tilde{\mathbf{d}}\|_{\mathbf{L}^3} \left(\|(|\mathbf{d}|^2 - 1) - (|\tilde{\mathbf{d}}|^2 - 1)\|_{\mathbf{L}^2}^2 + \|\mathbf{d} - \tilde{\mathbf{d}}\|_{\mathbf{L}^6}^2 \right) \, ds.
 \end{aligned}$$

Since $\tilde{\mathbf{d}}$ is a strong solution (see Definition 2.3), $\partial_t \tilde{\mathbf{d}}$ is in $L^1(0, T; \mathbf{L}^3)$ due to Lemma 3.2.

Now we reformulate the second term of I_8 . Using that equation (13b) is fulfilled by \mathbf{d} and $\tilde{\mathbf{d}}$, respectively, yields

$$\begin{aligned}
 &\frac{1}{2} \int_0^t \left(|\tilde{\mathbf{d}}|^2 - 1, \partial_t |\mathbf{d} - \tilde{\mathbf{d}}|^2 \right) \, ds \\
 &= \int_0^t \left((|\tilde{\mathbf{d}}|^2 - 1)(\mathbf{d} - \tilde{\mathbf{d}}), \partial_t \mathbf{d} - \partial_t \tilde{\mathbf{d}} \right) \, ds \\
 &= \int_0^t \left((|\tilde{\mathbf{d}}|^2 - 1)(\tilde{\mathbf{d}} - \mathbf{d}), (\mathbf{v} \cdot \nabla) \mathbf{d} - (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{d}} - (\nabla \mathbf{v})_{\text{skw}} \mathbf{d} + (\nabla \tilde{\mathbf{v}})_{\text{skw}} \tilde{\mathbf{d}} \right) \, ds \\
 &\quad + \int_0^t \left((|\tilde{\mathbf{d}}|^2 - 1)(\tilde{\mathbf{d}} - \mathbf{d}), \lambda((\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}}) + \gamma(\mathbf{q} - \tilde{\mathbf{q}}) \right) \, ds \\
 &= J_1 + J_2.
 \end{aligned}$$

The term J_1 can be rewritten as

$$\begin{aligned} J_1 &= \int_0^t \left((|\tilde{\mathbf{d}}|^2 - 1)(\tilde{\mathbf{d}} - \mathbf{d}), \nabla \tilde{\mathbf{d}}(\mathbf{v} - \tilde{\mathbf{v}}) \right) ds + \int_0^t (|\tilde{\mathbf{d}}|^2 - 1, \mathbf{v} \cdot (\nabla \mathbf{d} - \nabla \tilde{\mathbf{d}})(\tilde{\mathbf{d}} - \mathbf{d})) ds \\ &\quad - \int_0^t \left((|\tilde{\mathbf{d}}|^2 - 1)(\tilde{\mathbf{d}} - \mathbf{d}), (\nabla \mathbf{v})_{\text{skw}}(\mathbf{d} - \tilde{\mathbf{d}}) \right) ds \\ &\quad - \int_0^t \left((|\tilde{\mathbf{d}}|^2 - 1)(\tilde{\mathbf{d}} - \mathbf{d}), ((\nabla \mathbf{v})_{\text{skw}} - (\nabla \tilde{\mathbf{v}})_{\text{skw}}) \tilde{\mathbf{d}} \right) ds. \end{aligned}$$

We observe that the third term, i.e., $((|\tilde{\mathbf{d}}|^2 - 1)(\tilde{\mathbf{d}} - \mathbf{d}), (\nabla \mathbf{v})_{\text{skw}}(\mathbf{d} - \tilde{\mathbf{d}}))$ vanishes since $(\nabla \mathbf{v})_{\text{skw}}$ is skew-symmetric. Hence, we can estimate J_1 by

$$\begin{aligned} J_1 &\leq C_\delta \int_0^t \left(\| |\tilde{\mathbf{d}}|^2 - 1 \|_{L^6}^2 + \| \nabla \tilde{\mathbf{d}} \|_{L^2}^2 \right) \| \mathbf{d} - \tilde{\mathbf{d}} \|_{L^6}^2 ds + \delta \int_0^t \| \mathbf{v} - \tilde{\mathbf{v}} \|_{L^6}^2 ds \\ &\quad + \int_0^t \left(\| |\tilde{\mathbf{d}}|^2 - 1 \|_{L^6}^2 + \| \mathbf{v} \|_{L^6}^2 \right) \left(\| \nabla \mathbf{d} - \nabla \tilde{\mathbf{d}} \|_{\Lambda}^2 + \| \mathbf{d} - \tilde{\mathbf{d}} \|_{L^6}^2 \right) ds \\ &\quad + C_\delta \| \tilde{\mathbf{d}} \|_{L^\infty(L^6)}^2 \int_0^t \| |\tilde{\mathbf{d}}|^2 - 1 \|_{L^6}^2 \| \mathbf{d} - \tilde{\mathbf{d}} \|_{L^6}^2 ds + \delta \int_0^t \| (\nabla \mathbf{v})_{\text{skw}} - (\nabla \tilde{\mathbf{v}})_{\text{skw}} \|_{L^2}^2 ds. \end{aligned}$$

It remains to estimate the term J_2 :

$$\begin{aligned} J_2 &\leq C_\delta \| \tilde{\mathbf{d}} \|_{L^\infty(L^6)}^2 \int_0^t \| |\tilde{\mathbf{d}}|^2 - 1 \|_{L^6}^2 \| \mathbf{d} - \tilde{\mathbf{d}} \|_{L^6}^2 ds \\ &\quad + \delta \int_0^t \left(\lambda \| (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}} \|_{L^2}^2 + \gamma \| \mathbf{q} - \tilde{\mathbf{q}} \|_{L^2}^2 \right) ds. \end{aligned}$$

Inserting everything back into (28) yields

$$\begin{aligned} \mathcal{E}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}})(t) + \int_0^t \mathcal{W}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) ds &\leq \mathcal{E}(\mathbf{v}_0, \mathbf{d}_0 | \tilde{\mathbf{v}}_0, \tilde{\mathbf{d}}_0) \\ &\quad + (\gamma(\mu_2 + \mu_3) - \lambda) \int_0^t \left(\mathbf{q} - \tilde{\mathbf{q}}, (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}} \right) ds \\ &\quad + \delta c \int_0^t \mathcal{W}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) ds + \int_0^t \mathcal{K}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) \mathcal{E}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) ds. \end{aligned}$$

Since the constants are assumed to fulfill the dissipativity relation (9) we can find a real number $\zeta \in (0, 1)$ such that

$$(\gamma(\mu_2 + \mu_3) - \lambda)^2 \leq \zeta^2 4\gamma(\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)). \quad (30)$$

The relative energy can be estimated further on with Young's and Hölder's inequality, and we arrive at

$$\begin{aligned} \mathcal{E}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}})(t) + \int_0^t \mathcal{W}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) ds &\leq \mathcal{E}(\mathbf{v}_0, \mathbf{d}_0 | \tilde{\mathbf{v}}_0, \tilde{\mathbf{d}}_0) \\ &\quad + \zeta \int_0^t \left(\gamma \| \mathbf{q} - \tilde{\mathbf{q}} \|_{L^2}^2 + (\mu_5 + \mu_6 - \lambda(\mu_2 + \mu_3)) \| (\nabla \mathbf{v})_{\text{sym}} \mathbf{d} - (\nabla \tilde{\mathbf{v}})_{\text{sym}} \tilde{\mathbf{d}} \|_{L^2}^2 \right) ds \\ &\quad + \delta c \int_0^t \mathcal{W}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) ds + \int_0^t \mathcal{K}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) \mathcal{E}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) ds \\ &\leq \mathcal{E}(\mathbf{v}_0, \mathbf{d}_0 | \tilde{\mathbf{v}}_0, \tilde{\mathbf{d}}_0) + (\zeta + \delta c) \int_0^t \mathcal{W}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) ds + \int_0^t \mathcal{K}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) \mathcal{E}(\mathbf{v}, \mathbf{d} | \tilde{\mathbf{v}}, \tilde{\mathbf{d}}) ds. \end{aligned}$$

We now choose δ sufficiently small such that $\delta \leq (1/2 - \zeta)/c$. Thus, the relative dissipation \mathcal{W} can be absorbed into the left-hand side. Note that c does not depend on δ but only on the constants arising from the embeddings and Korn's inequality as well as the constants of the system (see (9)). The assertion (26) immediately follows from Gronwall's lemma (see Remark 4). \square

Proof of Theorem 2.4. The main Theorem 2.4 is a direct consequence of Lemma 5.1. \square

REFERENCES

- [1] A. N. Beris and B. J. Edwards, *Thermodynamics of Flowing Systems with Internal Microstructure*, Oxford University Press, New York, 1994.
- [2] D. Breit, E. Feireisl and M. Hofmanová, [Incompressible limit for compressible fluids with stochastic forcing](#), *Arch. Rational Mech. Anal.*, **222** (2016), 895–926.
- [3] C. Cavaterra, E. Rocca and H. Wu, [Global weak solution and blow-up criterion of the general Ericksen–Leslie system for nematic liquid crystal flows](#), *J. Differential Equations*, **255** (2013), 24–57.
- [4] C. M. Dafermos, [The second law of thermodynamics and stability](#), *Arch. Ration. Mech. Anal.*, **70** (1979), 167–179.
- [5] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer, Berlin, 2016.
- [6] M. Dai, [Existence of regular solutions to an Ericksen–Leslie model of the liquid crystal system](#), *Commun. Math. Sci.*, **13** (2015), 1711–1740.
- [7] M. Dai, J. Qing and M. Schonbek, [Regularity of solutions to the liquid crystals systems in \$\mathbb{R}^2\$ and \$\mathbb{R}^3\$](#) , *Nonlinearity*, **25** (2012), 513–532.
- [8] J. Diestel and J. J. Uhl, Jr. *Vector Measures*, American Mathematical Society, Providence, Rhode Island, 1977.
- [9] E. Emmrich and R. Lasarzik, Existence of weak solutions to the Ericksen–Leslie model for a general class of free energies, ArXiv e-prints, 1711.10277, 2017.
- [10] J. L. Ericksen, [Conservation laws for liquid crystals](#), *J. Rheol.*, **5** (1961), 23–34.
- [11] E. Feireisl, [Relative entropies in thermodynamics of complete fluid systems](#), *Discrete Contin. Dyn. Syst.*, **32** (2012), 3059–3080.
- [12] E. Feireisl, Relative entropies, dissipative solutions, and singular limits of complete fluid systems, In *Hyperbolic Problems: Theory, Numerics, Applications*, volume 8 of AIMS on Applied Mathematics, pages 11–27. AIMS, Springfield, USA, 2014.
- [13] E. Feireisl, B. J. Jin and A. Novotný, [Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier–Stokes system](#), *J. Math. Fluid Mech.*, **14** (2012), 717–730.
- [14] E. Feireisl and A. Novotný, [Weak-strong uniqueness property for the full Navier–Stokes–Fourier system](#), *Arch. Ration. Mech. Anal.*, **204** (2012), 683–706.
- [15] E. Feireisl, A. Novotný and Y. Sun, [Suitable weak solutions to the Navier–Stokes equations of compressible viscous fluids](#), *Indiana Univ. Math. J.*, **60** (2011), 611–631.
- [16] J. Fischer, [A posteriori modeling error estimates for the assumption of perfect incompressibility in the Navier–Stokes equation](#), *SIAM J. Numer. Anal.*, **53** (2015), 2178–2205.
- [17] H. Gajewski, K. Gröger and K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferential-Gleichungen*, Akademie-Verlag, Berlin, 1974.
- [18] R. Lasarzik, Measure-valued solutions to the Ericksen–Leslie model equipped with the Oseen–Frank energy, [arXiv:1711.04638](#), Nov. 2017.
- [19] R. Lasarzik, Weak-strong uniqueness for measure-valued solutions to the Ericksen–Leslie model equipped with the Oseen–Frank free energy, [arXiv:1711.03371](#), Nov. 2017.
- [20] F. M. Leslie, [Some constitutive equations for liquid crystals](#), *Arch. Rational Mech. Anal.*, **28** (1968), 265–283.
- [21] F.-H. Lin and C. Liu, [Nonparabolic dissipative systems modeling the flow of liquid crystals](#), *Comm. Pure Appl. Math.*, **48** (1995), 501–537.
- [22] F.-H. Lin and C. Liu, [Existence of solutions for the Ericksen–Leslie system](#), *Arch. Rational Mech. Anal.*, **154** (2000), 135–156.

- [23] P.-L. Lions, *Mathematical Topics in Fluid Mechanics. Vol. 1*, The Clarendon Press, New York, 1996.
- [24] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.
- [25] C. B. Morrey Jr., *Multiple Integrals in the Calculus of Variations*, Springer, Berlin, 1966.
- [26] G. Prodi, [Un teorema di unicità per le equazioni di Navier–Stokes](#), *Ann. Mat. Pura Appl.*(4), **48** (1959), 173–182.
- [27] T. Roubíček, *Nonlinear Partial Differential Equations with Applications*, Birkhäuser/Springer Basel AG, Basel, 2013.
- [28] J. Serrin, [On the interior regularity of weak solutions of the Navier–Stokes equations](#), *Arch. Rational Mech. Anal.*, **9** (1962), 187–195.
- [29] Y.-F. Yang, C. Dou and Q. Ju. [Weak-strong uniqueness property for the compressible flow of liquid crystals](#), *J. Differential Equations*, **255** (2013), 1233–1253.
- [30] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. II/A*, Springer-Verlag, New York, 1990.
- [31] J.-H. Zhao and Q. Liu, [Weak-strong uniqueness of hydrodynamic flow of nematic liquid crystals](#), *Electron. J. Differential Equations*, **2012** (2012), 1–16.

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