Received: 26 November 2020 | Revised: 30 April 2021 | Accepted: 24 June 2021

DOI: 10.1002/mma.7673

RESEARCH ARTICLE

WILEY

Analysis of a model for the dynamics of microswimmer suspensions

Etienne Emmrich | **Lukas Geuter**

Institut für Mathematik, Technische Universität Berlin, Berlin, Germany

Correspondence
Etienne Emmrich, Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany.
Email: emmrich@math.tu-berlin.de

Communicated by: T. Roubíček

Funding information
Deutsche Forschungsgemeinschaft, Grant/Award Number: SFB 910

In this paper, a model that was recently derived in Reinken et al. to describe the dynamics of microswimmer suspensions is studied. In particular, the global existence of weak solutions, their weak–strong uniqueness, and a connection to a different model that was proposed in Wensink et al. is shown.

KEYWORDS

active fluid, existence, relative energy, weak solution, weak–strong uniqueness

MSC CLASSIFICATION

35Q35; 76A05

1 | INTRODUCTION

Microswimmers are small, self-propelling particles, for example, bacteria like *Bacillus subtilis*, algae like *Chlamydomonas reinhardtii*, or artificial nanorods. Suspended in a liquid, the interaction between these particles themselves and between them and the surrounding fluid gives rise to interesting phenomena such as active turbulence. For a high density of microswimmers, such a suspension can be regarded as an example of an active fluid.

There are several approaches to capture the dynamics of an active fluid starting with adaptions of a model proposed in Vicsek et al.[1] Later approaches include the derivation of continuum limit hydrodynamic equations as in Toner and Tu[2] or adaptions of liquid crystal models as in Thampi and Yeomans.[3] Another model we want to mention is the one suggested in Wensink et al,[4] where a Toner–Tu-like equation is supplemented with a fourth-order Swift–Hohenberg term.

The model that we want to study here was recently derived in Reinken et al[5] with the goal of it being able to incorporate short-range interactions favoring alignment as well as long-range hydrodynamic interactions. The authors start from overdamped Langevin equations for a generic microscopic model and couple them with a Stokes equation supplemented by an ansatz for the stress tensor.

Our paper is arranged as follows. First, we introduce the equations that are the object of our analysis. Then—under a simplifying assumption—we prove the existence of weak solutions via a Galerkin approximation. The main part of the paper is then devoted to prove that such weak solutions obey a relative energy inequality. Such a relative energy inequality can be employed to a variety of uses, such as showing the stability of equilibria (e.g., in Feireisl[6]), deriving a posteriori estimates for modeling errors (see Fischer[7]), or as the basis of a generalized concept of solution (e.g., in Lasarzik[8]).

In our case, we will apply the relative energy inequality to prove the weak–strong uniqueness of weak solutions as well as the convergence of those to strong solutions of another problem as one parameter tends to zero.

Throughout this paper, we denote by $c > 0$ a generic constant that does not depend on any changing quantity. Furthermore, let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathscr{C}^2$ and $T > 0$ some fixed time.

Spaces of vector- or matrix-valued functions are denoted by bold letters, for example, $\boldsymbol{L}^p(\Omega) := L^p(\Omega;\mathbb{R}^3)$ for the spaces of integrable functions, and $\boldsymbol{W}^{k,p}(\Omega) := W^{k,p}(\Omega;\mathbb{R}^3)$ as well as $\boldsymbol{H}^k(\Omega) := W^{k,2}(\Omega;\mathbb{R}^3)$ for Sobolev spaces. The Sobolev space associated with homogeneous Dirichlet boundary conditions is denoted by $\boldsymbol{H}_0^1(\Omega) := \mathrm{clos}_{\boldsymbol{H}^1}\mathscr{C}_c^\infty(\Omega;\mathbb{R}^3)$, where $\mathscr{C}_c^\infty(\Omega;\mathbb{R}^3)$ denotes the set of infinitely many times differentiable functions that are compactly supported in Ω. We write $\mathscr{C}_{c,\sigma}^\infty(\Omega;\mathbb{R}^3)$ for all such functions that are in addition solenoidal and then denote by $\boldsymbol{L}_\sigma^p(\Omega)$ and $\boldsymbol{H}_{0,\sigma}^1(\Omega)$ the closure of $\mathscr{C}_{c,\sigma}^\infty(\Omega;\mathbb{R}^3)$ with respect to the standard norms of $\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{H}^1(\Omega)$, respectively.

The spaces for time-dependent continuous and continuously differentiable functions mapping $[0,T]$ into a Banach space X are denoted by $\mathscr{C}([0,T];X)$ and $\mathscr{C}^1([0,T];X)$, respectively, while the corresponding spaces of Bochner-integrable and weakly differentiable functions are denoted by $L^p(0,T;X)$ and $W^{1,p}(0,T;X)$, respectively. We often omit the time interval $(0,T)$ and the domain Ω in this notation and write, for example, $L^p(\boldsymbol{W}^{1,q})$. Also, for the sake of brevity, we generally do not write out the time dependence of functions under the integral.

For an arbitrary Banach space X, we denote its dual by X^* and the dual pairing between X^* and X by $\langle\cdot,\cdot\rangle$. With a slight abuse of notation, however, the dual pairing between the spaces $\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{L}^q(\Omega)$, where q is the conjugate exponent to p, is denoted by $(\cdot,\cdot)$. We use the same notation for the inner product on the Hilbert space $\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)$.

The space $\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_{0,\sigma}^1(\Omega)$ is equipped with the norm $\|\Delta\cdot\|_{\boldsymbol{L}^2}$ (cf. Boyer & Fabrie[9, Proposition IV.5.9]).

2 | MODEL

We want to study the following equations derived in Reinken et al.[5] Consider the initial-boundary value problem

$$\begin{aligned} -\Delta\boldsymbol{u} + \mu_1\Delta^2\boldsymbol{p} - \gamma_1\Delta\boldsymbol{p} + \lambda_1(\boldsymbol{p}\cdot\nabla)\boldsymbol{p} + \nabla\pi_1 &= 0,\\ \partial_t\boldsymbol{p} + \mu_2\Delta^2\boldsymbol{p} - \gamma_2\Delta\boldsymbol{p} + \lambda_2(\boldsymbol{p}\cdot\nabla)\boldsymbol{p} + \alpha|\boldsymbol{p}|^2\boldsymbol{p} + \beta\boldsymbol{p} &\\ +(\boldsymbol{u}\cdot\nabla)\boldsymbol{p} + \kappa(\nabla\boldsymbol{u})_{\mathrm{sym}}\boldsymbol{p} - (\nabla\boldsymbol{u})_{\mathrm{skw}}\boldsymbol{p} + \nabla\pi_2 &= 0,\\ \nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{p} &= 0, \end{aligned} \tag{1a}$$

on $\Omega\times(0,T)$ and

$$\begin{aligned} \boldsymbol{u} = \boldsymbol{p} = \Delta\boldsymbol{p} = 0 \quad &\text{on } \partial\Omega\times(0,T),\\ \boldsymbol{p}(\cdot,0) = \boldsymbol{p}_0 \quad &\text{in } \Omega, \end{aligned} \tag{1b}$$

where $\alpha,\lambda_1,\lambda_2,\mu_1,\mu_2>0$, and $\beta,\gamma_1,\gamma_2,\kappa\in\mathbb{R}$. In contrast to other models, both the influence of the polar ordering of the microswimmers themselves, described by the vector field $\boldsymbol{p} : \bar{\Omega}\times[0,T]\to\mathbb{R}^3$, as well as the one from the velocity field $\boldsymbol{u} : \bar{\Omega}\times[0,T]\to\mathbb{R}^3$ of the suspension fluid is taken into account. The latter is incorporated into the model by coupling the Stokes equation with the equation for the polar ordering parameter $\boldsymbol{p}$. The strength of the coupling is described by a dimensionless parameter ε, meaning there exist $\varepsilon,\tilde{\lambda}_1,\tilde{\mu}_1>0$, and $\tilde{\gamma}_1\in\mathbb{R}$ such that

$$\gamma_1 = \varepsilon\tilde{\gamma}_1,\ \lambda_1 = \varepsilon\tilde{\lambda}_1,\ \mu_1 = \varepsilon\tilde{\mu}_1. \tag{2}$$

Then for $\varepsilon = 0$, both equations decouple (see Section 5).

Since both the solvent as well as the whole fluid are assumed to be incompressible, it follows that $\boldsymbol{u}$ and $\boldsymbol{p}$ are supposed to be solenoidal vector fields. The Lagrange multipliers for these divergence-free constraints are denoted by π_1 and π_2, respectively. The velocity field of the whole fluid is then given as the sum $\boldsymbol{u}+\nu\boldsymbol{p}$ of both vector fields, where $\nu>0$ denotes a reference velocity of the swimmers. The unknown so-called pressure $\pi : \bar{\Omega}\times[0,T]\to\mathbb{R}$ enters the equations via the relation $\pi = \pi_1+\nu\pi_2$ and is the Lagrange multiplier for the divergence-free constraint $\nabla\cdot(\boldsymbol{u}+\nu\boldsymbol{p}) = 0$ on the velocity field of the whole fluid. For a certain constant $c_1>0$, one can think of $c_1|\boldsymbol{p}|^2$ as the so-called active pressure and of $\pi_1 - c_1|\boldsymbol{p}|^2$ as the effective pressure of the active fluid (cf. Dunkel et al.[10]). We consider the problem in a variational framework with solenoidal test functions and will not be dealing with the terms π, π_1, and π_2 since they vanish in the weak formulation.

Unfortunately, we are not able to show the appropriate a priori estimates to prove the existence of weak solutions for the general case above. The problematic term here is $\kappa(\nabla\boldsymbol{u})_{\mathrm{sym}}\boldsymbol{p}$. If we proceed in a standard fashion to derive a priori estimates and multiply the second equation by $\boldsymbol{p}$, integrate and perform an integration by parts, we end up with the expression

$$-\kappa\int_0^T\int_\Omega(\boldsymbol{p}\cdot\nabla)\boldsymbol{p}\cdot\boldsymbol{u}\,\mathrm{d}x\ \mathrm{d}t.$$

Since by the first equation $\boldsymbol{u}$ is of the same order as $\Delta \boldsymbol{p}$, we seem not to be able to estimate and absorb this term into the left-hand side. Therefore, we restrict ourselves to the simpler case $\kappa = 0$.

3 | EXISTENCE OF WEAK SOLUTIONS

We start by giving the definition of a weak solution to (1). Note that the term $\Delta^2 \boldsymbol{p}$ occurs in the Stokes equation, meaning that $\Delta \boldsymbol{u}$ has at most the regularity of $\Delta^2 \boldsymbol{p}$. Hence, the weak formulation of the second equation necessitates a very weak formulation for the Stokes equation. For this, by $A^{-1} : \boldsymbol{L}^2_\sigma(\Omega) \to \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$, we denote the inverse of the Stokes operator, meaning that for $\varphi \in \boldsymbol{L}^2_\sigma(\Omega)$, the vector field $A^{-1}\varphi := \boldsymbol{v}$ is the unique solution of

$$\begin{aligned} -\Delta \boldsymbol{v} + \nabla \pi = \varphi \quad & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{v} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{v} = 0 \quad & \text{on } \partial\Omega \end{aligned}$$

in the weak sense. For the existence and properties of this operator, see Temam.[11, Chapter I, § 2.6]

Definition 1 (Weak solution). Let $\boldsymbol{p}_0 \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$ be given. A pair $(\boldsymbol{u}, \boldsymbol{p}) \in L^2(\boldsymbol{L}^2_\sigma) \times (L^\infty(\boldsymbol{L}^2_\sigma) \cap L^4(\boldsymbol{L}^4_\sigma) \cap L^2(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}))$ such that $\partial_t \boldsymbol{p} \in L^{\frac{4}{3}}((\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma})^*)$ is called weak solution to (1) if the equations

$$\begin{aligned} \int_0^T (\boldsymbol{u}, \varphi)\,\mathrm{d}t - \mu_1 \int_0^T (\Delta \boldsymbol{p}, \Delta A^{-1}\varphi)\,\mathrm{d}t + \gamma_1 \int_0^T (\Delta \boldsymbol{p}, A^{-1}\varphi)\,\mathrm{d}t & \\ - \lambda_1 \int_0^T ((\boldsymbol{p} \cdot \nabla)\boldsymbol{p}, A^{-1}\varphi)\,\mathrm{d}t = 0 & \end{aligned} \tag{3}$$

and

$$\begin{aligned} \int_0^T \langle \partial_t \boldsymbol{p}, \psi \rangle\,\mathrm{d}t + \mu_2 \int_0^T (\Delta \boldsymbol{p}, \Delta \psi)\,\mathrm{d}t + \gamma_2 \int_0^T (\nabla \boldsymbol{p}, \nabla \psi)\,\mathrm{d}t + \lambda_2 \int_0^T ((\boldsymbol{p} \cdot \nabla)\boldsymbol{p}, \psi)\,\mathrm{d}t & \\ + \alpha \int_0^T (|\boldsymbol{p}|^2 \boldsymbol{p}, \psi)\,\mathrm{d}t + \beta \int_0^T (\boldsymbol{p}, \psi)\,\mathrm{d}t + \int_0^T ((\boldsymbol{u} \cdot \nabla)\boldsymbol{p}, \psi)\,\mathrm{d}t & \\ + \frac{1}{2} \int_0^T ((\boldsymbol{p} \cdot \nabla)\psi - (\psi \cdot \nabla)\boldsymbol{p}, \boldsymbol{u})\,\mathrm{d}t = 0 & \end{aligned} \tag{4}$$

hold for all test functions $\varphi, \psi \in \mathscr{C}^\infty_{c,\sigma}(\Omega \times (0,T); \mathbb{R}^3)$ and the initial condition $\boldsymbol{p}(0) = \boldsymbol{p}_0$ is fulfilled in the weak sense, that is,

$$\boldsymbol{p}(t) \rightharpoonup \boldsymbol{p}_0 \ \text{ in } \ \boldsymbol{L}^2_\sigma(\Omega) \ \text{ as } \ t \to 0.$$

Remark 1. In order to justify that this weak formulation is well defined, we only consider the nonlinear terms. Since $\boldsymbol{p}$ is divergence-free, the identity

$$(\boldsymbol{p} \cdot \nabla)\boldsymbol{p} = \nabla \cdot (\boldsymbol{p} \otimes \boldsymbol{p}),$$

where $\otimes$ denotes the dyadic product, holds almost everywhere in $\Omega \times (0,T)$. Performing an integration-by-parts and using Hölder's inequality as well as the continuous embedding $\boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{W}^{1,6}(\Omega)$ then yields

$$|((\boldsymbol{p} \cdot \nabla)\boldsymbol{p}, \boldsymbol{w})| \le c \|\boldsymbol{p}\|^2_{\boldsymbol{L}^{\frac{12}{5}}} \|\boldsymbol{w}\|_{\boldsymbol{H}^2}$$

for any $\boldsymbol{w} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$. Therefore, we can estimate the norm of the functional $\boldsymbol{w} \mapsto ((\boldsymbol{p} \cdot \nabla)\boldsymbol{p}, \boldsymbol{w})$ by

$$\|(\boldsymbol{p} \cdot \nabla)\boldsymbol{p}\|_{(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma})^*} \le c \|\boldsymbol{p}\|^2_{\boldsymbol{L}^{\frac{12}{5}}}$$

almost everywhere in $(0, T)$. Hence,

$$\int_0^T \left|((\boldsymbol{p}\cdot\nabla)\boldsymbol{p}, A^{-1}\varphi)\right| \,\mathrm{d}t \le \int_0^T \|(\boldsymbol{p}\cdot\nabla)\boldsymbol{p}\|_{(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*}\|A^{-1}\varphi\|_{\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma}}\,\mathrm{d}t$$
$$\le c\|\boldsymbol{p}\|^2_{L^4\left(\boldsymbol{L}^{\frac{12}{5}}\right)}\|\varphi\|_{L^2(\boldsymbol{L}^2)},$$

where we also used the boundedness of $A^{-1} : \boldsymbol{L}^2_\sigma(\Omega) \to \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$ (see Temam[12, Section 2.5]). For the remaining nonlinear terms, applications of Hölder's inequality yield

$$\begin{aligned}
\int_0^T |((\boldsymbol{p}\cdot\nabla)\boldsymbol{p},\psi)|\,\mathrm{d}t &\le \|\boldsymbol{p}\|_{L^2(\boldsymbol{L}^\infty)}\|\nabla\boldsymbol{p}\|_{L^2(\boldsymbol{L}^6)}\|\psi\|_{L^\infty\left(\boldsymbol{L}^{\frac{6}{5}}\right)},\\
\int_0^T \left|(|\boldsymbol{p}|^2\boldsymbol{p},\psi)\right|\,\mathrm{d}t &\le \|\boldsymbol{p}\|^3_{L^4(\boldsymbol{L}^4)}\|\psi\|_{L^4(\boldsymbol{L}^4)},\\
\int_0^T |((\boldsymbol{u}\cdot\nabla)\boldsymbol{p},\psi)|\,\mathrm{d}t &\le \|\boldsymbol{u}\|_{L^2(\boldsymbol{L}^2)}\|\nabla\boldsymbol{p}\|_{L^2(\boldsymbol{L}^6)}\|\psi\|_{L^\infty(\boldsymbol{L}^3)},\\
\int_0^T |((\boldsymbol{p}\cdot\nabla)\psi-(\psi\cdot\nabla)\boldsymbol{p},\boldsymbol{u})|\,\mathrm{d}t &\le \|\boldsymbol{p}\|_{L^2(\boldsymbol{L}^\infty)}\|\nabla\psi\|_{L^\infty(\boldsymbol{L}^2)}\|\boldsymbol{u}\|_{L^2(\boldsymbol{L}^2)}\\
&\quad+\|\psi\|_{L^\infty(\boldsymbol{L}^3)}\|\nabla\boldsymbol{p}\|_{L^2(\boldsymbol{L}^6)}\|\boldsymbol{u}\|_{L^2(\boldsymbol{L}^2)}.
\end{aligned} \tag{5}$$

Together with the continuous embedding $\boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$, we see that this notion of a weak solution is well defined for all test functions $\psi \in L^4(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}) \cap L^\infty(\boldsymbol{H}^1_{0,\sigma})$ and $\varphi \in L^2(\boldsymbol{L}^2_\sigma)$.

We can prove the existence of such a solution via a Galerkin approximation.

Theorem 1. *Let $\boldsymbol{p}_0 \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$. Then there exists a weak solution to (1) in the sense of Definition 1.*

Proof. First, we choose a Galerkin basis $\{\boldsymbol{w}_m\}_{m\in\mathbb{N}} \subset \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$ consisting of eigenfunctions of the Stokes operator $A : \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega) \subset \boldsymbol{L}^2_\sigma(\Omega) \to \boldsymbol{L}^2_\sigma(\Omega)$. For any $n \in \mathbb{N}$, let W_n be the finite dimensional subspace of $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$ spanned by $\boldsymbol{w}_1, \dots, \boldsymbol{w}_n$ and $\Pi_n : \boldsymbol{L}^2_\sigma(\Omega) \to W_n$ the $\boldsymbol{L}^2(\Omega)$-orthogonal projection onto W_n. For the initial value $\boldsymbol{p}_{0,n} := \Pi_n\boldsymbol{p}_0$, we are then looking for a solution $(\boldsymbol{u}_n, \boldsymbol{p}_n) \in \mathscr{C}^1([0,T]; W_n)\times\mathscr{C}^1([0,T]; W_n)$ to the discretised problem

$$(\boldsymbol{u}_n,\boldsymbol{v}) - \mu_1(\Delta\boldsymbol{p}_n, \Delta A^{-1}\boldsymbol{v}) + \gamma_1(\Delta\boldsymbol{p}_n, A^{-1}\boldsymbol{v}) - \lambda_1(\boldsymbol{p}_n\cdot\nabla)\boldsymbol{p}_n, A^{-1}\boldsymbol{v}) = 0, \tag{6a}$$

$$\begin{aligned}
(\partial_t\boldsymbol{p}_n,\boldsymbol{w}) + \mu_2(\Delta\boldsymbol{p}_n,\Delta\boldsymbol{w}) + \gamma_2(\nabla\boldsymbol{p}_n,\nabla\boldsymbol{w}) + \lambda_2((\boldsymbol{p}_n\cdot\nabla)\boldsymbol{p}_n,\boldsymbol{w}) + \alpha(|\boldsymbol{p}_n|^2\boldsymbol{p}_n,\boldsymbol{w})&\\
+\beta(\boldsymbol{p}_n,\boldsymbol{w}) + ((\boldsymbol{u}_n\cdot\nabla)\boldsymbol{p}_n,\boldsymbol{w}) + \frac{1}{2}((\boldsymbol{p}_n\cdot\nabla)\boldsymbol{w} - (\boldsymbol{w}\cdot\nabla)\boldsymbol{p}_n,\boldsymbol{u}_n) &= 0,
\end{aligned} \tag{6b}$$

$$\boldsymbol{p}_n(0) = \boldsymbol{p}_{0,n}$$

in $(0, T)$ and for all $\boldsymbol{v}, \boldsymbol{w} \in W_n$. In order to show the existence of a solution to this system, we fix $n \in \mathbb{N}$ and represent each function $\boldsymbol{p}_n : [0, T] \to W_n$ via the corresponding basis of W_n, that is,

$$\boldsymbol{p}_n(t) = \sum_{i=1}^n P_{ni}(t)\boldsymbol{w}_i.$$

We make use of the inverse $J^{-1} : (\boldsymbol{L}^2_\sigma(\Omega))^* \to \boldsymbol{L}^2_\sigma(\Omega)$ of the Riesz isomorphism on the Hilbert space $\boldsymbol{L}^2_\sigma(\Omega)$ in order to define

$$\boldsymbol{u}_n = \Pi_n J^{-1}\boldsymbol{g}_{\boldsymbol{p}_n}, \tag{7}$$

where $\boldsymbol{g}_{\boldsymbol{p}_n} \in (\boldsymbol{L}^2_\sigma(\Omega))^*$ is given as

$$\langle \boldsymbol{g}_{\boldsymbol{p}_n}, \boldsymbol{v}\rangle = \mu_1(\Delta\boldsymbol{p}_n, \Delta A^{-1}\boldsymbol{v}) - \gamma_1(\Delta\boldsymbol{p}_n, A^{-1}\boldsymbol{v}) + \lambda_1((\boldsymbol{p}_n\cdot\nabla)\boldsymbol{p}_n, A^{-1}\boldsymbol{v})$$

for any $\boldsymbol{p}_n \in W_n$ and $\boldsymbol{v} \in \boldsymbol{L}^2_\sigma(\Omega)$. In order to transform the problem to an autonomous system of ordinary differential equations, we let $\boldsymbol{y} : [0, T] \to \mathbb{R}^n$ and $\boldsymbol{y}_0 \in \mathbb{R}^n$ be defined as

$$(\boldsymbol{y}(t))_i = P_{ni}(t), \ (\boldsymbol{y}_0)_i = (\boldsymbol{p}_{0,n}, \boldsymbol{w}_i)$$

for $i = 1, \dots, n$. Then the discretized problem (6) can equivalently be written as

$$\begin{aligned} \boldsymbol{y}' &= \boldsymbol{f}(\boldsymbol{y}) \quad \text{on } [0, T], \\ \boldsymbol{y}(0) &= \boldsymbol{y}_0, \end{aligned}$$

where the right-hand side $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is given as

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{y}) = &-\mu_2(\Delta \boldsymbol{p}_n, \Delta \boldsymbol{w}) - \gamma_2(\nabla \boldsymbol{p}_n, \nabla \boldsymbol{w}) - \lambda_2((\boldsymbol{p}_n \cdot \nabla)\boldsymbol{p}_n, \boldsymbol{w}) - \alpha(|\boldsymbol{p}_n|^2 \boldsymbol{p}_n, \boldsymbol{w}) \\ &- \beta(\boldsymbol{p}_n, \boldsymbol{w}) - ((\boldsymbol{u}_n \cdot \nabla)\boldsymbol{p}_n, \boldsymbol{w}) - \frac{1}{2}((\boldsymbol{p}_n \cdot \nabla)\boldsymbol{w} - (\boldsymbol{w} \cdot \nabla)\boldsymbol{p}_n, \boldsymbol{u}_n) \end{aligned}$$

for $i = 1, \dots, n$, as well as

$$\boldsymbol{p}_n = \sum_{i=1}^n \boldsymbol{y}_i \boldsymbol{w}_i,$$

and $\boldsymbol{u}_n$ as in (7). Note that the mass matrix in this case is just the identity since the eigenfunctions $\boldsymbol{w}_i$ are orthonormal with respect to the $\boldsymbol{L}^2$-inner product. Then $\boldsymbol{f}$ does not depend on t and is continuous with respect to $\boldsymbol{y}$. Therefore, the Peano theorem (see Hale[13, Chapter I, Theorem 5.1]) provides a maximally continued solution $(\boldsymbol{u}_n, \boldsymbol{p}_n) \in \mathscr{C}^1([0, T_n); W_n) \times \mathscr{C}^1([0, T_n); W_n)$ to (6), where $T_n \le T$ may depend on the dimension n of W_n. We now derive a priori estimates to show that there is no blow-up of these solution; hence, they exist globally in time.

To this end, we first test Equation (6b) with $\boldsymbol{p}_n$. Since $\boldsymbol{u}_n$ and $\boldsymbol{p}_n$ are both divergence-free, this results in

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{p}_n\|^2_{\boldsymbol{L}^2} + \mu_2\|\Delta \boldsymbol{p}_n\|^2_{\boldsymbol{L}^2} + \gamma_2\|\nabla \boldsymbol{p}_n\|^2_{\boldsymbol{L}^2} + \alpha\|\boldsymbol{p}_n\|^4_{\boldsymbol{L}^4} + \beta\|\boldsymbol{p}_n\|^2_{\boldsymbol{L}^2} = 0 \ \text{ in } (0, T).$$

Here, we also used the identity

$$(\partial_t \boldsymbol{p}_n, \boldsymbol{p}_n) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{p}_n\|^2_{\boldsymbol{L}^2} \ \text{ in } (0, T)$$

for continuously differentiable functions. If $\gamma_2, \beta \ge 0$, this already gives the desired a priori estimate for $\boldsymbol{p}$. We set

$$a^- := \begin{cases} 0, & \text{if } a \ge 0 \\ -a, & \text{if } a > 0. \end{cases}$$

for any $a \in \mathbb{R}$. Performing an integration-by-parts and using Cauchy–Schwarz's as well as Young's inequality then leads to

$$\gamma_2^-\|\nabla \boldsymbol{p}_n\|^2_{\boldsymbol{L}^2} \le \frac{(\gamma_2^-)^2}{2\mu_2}\|\boldsymbol{p}_n\|^2_{\boldsymbol{L}^2} + \frac{\mu_2}{2}\|\Delta \boldsymbol{p}_n\|^2_{\boldsymbol{L}^2},$$

and by another application of Young's inequality as well as the continuous embedding $\boldsymbol{L}^4(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$, it follows that

$$\left(\frac{(\gamma_2^-)^2}{2\mu_2} + \beta^-\right)\|\boldsymbol{p}_n\|^2_{\boldsymbol{L}^2} \le \frac{|\Omega|}{2\alpha}\left(\frac{(\gamma_2^-)^2}{2\mu_2} + \beta^-\right)^2 + \frac{\alpha}{2}\|\boldsymbol{p}_n\|^4_{\boldsymbol{L}^4}.$$

By absorbing the norms of $\boldsymbol{p}_n$ into the left-hand side, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{p}_n\|^2_{\boldsymbol{L}^2} + \mu_2\|\Delta \boldsymbol{p}_n\|^2_{\boldsymbol{L}^2} + \alpha\|\boldsymbol{p}_n\|^4_{\boldsymbol{L}^4} \le \frac{|\Omega|}{\alpha}\left(\frac{(\gamma_2^-)^2}{2\mu_2} + \beta^-\right)^2.$$

Integrating the above estimate leads to

$$\|\boldsymbol{p}_n(t)\|_{\boldsymbol{L}^2}^2 + \int_0^t \left(\mu_2 \|\Delta \boldsymbol{p}_n\|_{\boldsymbol{L}^2}^2 + \alpha \|\boldsymbol{p}_n\|_{\boldsymbol{L}^4}^4 \right) \mathrm{d}s \le \|\boldsymbol{p}_{n,0}\|_{\boldsymbol{L}^2}^2 + \frac{T|\Omega|}{\alpha} \left(\frac{(\gamma_2^-)^2}{2\mu_2} + \beta^- \right)^2 \tag{8}$$

for all $t \in [0, T_n)$. In order to derive an estimate for the velocity field, we now test (6a) with $\boldsymbol{u}_n$, which results in

$$\|\boldsymbol{u}_n\|_{\boldsymbol{L}^2(\Omega)}^2 = \mu_1(\Delta \boldsymbol{p}_n, \Delta A^{-1}\boldsymbol{u}_n) - \gamma_1(\Delta \boldsymbol{p}_n, A^{-1}\boldsymbol{u}_n) + \lambda_1((\boldsymbol{p}_n \cdot \nabla)\boldsymbol{p}_n, A^{-1}\boldsymbol{u}_n). \tag{9}$$

Just as in Remark 1, we can derive the estimate

$$\|(\boldsymbol{p}_n \cdot \nabla)\boldsymbol{p}_n\|_{(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma})^*} \le c\|\boldsymbol{p}_n\|_{\boldsymbol{L}^{\frac{12}{5}}}^2$$

and therefore, together with the embedding $\boldsymbol{L}^4(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{12}{5}}(\Omega)$,

$$((\boldsymbol{p}_n \cdot \nabla)\boldsymbol{p}_n, A^{-1}\boldsymbol{u}_n) \le c\|\boldsymbol{p}_n\|_{\boldsymbol{L}^4}^2 \|\boldsymbol{u}_n\|_{\boldsymbol{L}^2} \tag{10}$$

almost everywhere in $(0, T_n)$.

By the continuity of $A^{-1} : \boldsymbol{L}^2_\sigma(\Omega) \to \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$, it follows that

$$(\Delta \boldsymbol{p}_n, A^{-1}\boldsymbol{u}_n) \le c\|\boldsymbol{p}_n\|_{\boldsymbol{L}^2}\|\boldsymbol{u}_n\|_{\boldsymbol{L}^2} \le c\|\Delta \boldsymbol{p}_n\|_{\boldsymbol{L}^2}\|\boldsymbol{u}_n\|_{\boldsymbol{L}^2}, \tag{11}$$

and, similarly, we find

$$(\Delta \boldsymbol{p}_n, \Delta A^{-1}\boldsymbol{u}_n) \le c\|\Delta \boldsymbol{p}_n\|_{\boldsymbol{L}^2}\|\boldsymbol{u}_n\|_{\boldsymbol{L}^2}. \tag{12}$$

Applying Young's inequality in (10), (11), and (12) and absorbing the terms depending on $\boldsymbol{u}_n$ into the left-hand side of (9) yields

$$\|\boldsymbol{u}_n\|_{\boldsymbol{L}^2}^2 \le \varepsilon^2 c \left(\|\boldsymbol{p}_n\|_{\boldsymbol{L}^4}^4 + \|\Delta \boldsymbol{p}_n\|_{\boldsymbol{L}^2}^2 \right) \tag{13}$$

in $(0, T_n)$. Here, $\varepsilon > 0$ is given by (2). By integrating this inequality and using estimate (8) as well as the $\boldsymbol{L}^2$-stability of the projections Π_n, we arrive at

$$\|\boldsymbol{p}_n(t)\|_{\boldsymbol{L}^2}^2 + \int_0^t \left(\|\boldsymbol{u}_n(s)\|_{\boldsymbol{L}^2}^2 + \mu_2\|\Delta \boldsymbol{p}_n(s)\|_{\boldsymbol{L}^2}^2 + \alpha\|\boldsymbol{p}_n(s)\|_{\boldsymbol{L}^4}^4 \right) \mathrm{d}s \le c(\|\boldsymbol{p}_0\|_{\boldsymbol{L}^2}^2 + 1). \tag{14}$$

Since this estimate is uniform in n, the discrete solution $(\boldsymbol{u}_n, \boldsymbol{p}_n)$ actually exists on $[0, T]$ (see Hale[13, Chapter I, Theorem 5.2]).

We proceed with deriving an estimate for the time derivative of $\boldsymbol{p}_n$. By (6b) and Hölder's inequality, we find

$$\begin{aligned} |\langle \partial_t \boldsymbol{p}_n, \boldsymbol{v}\rangle| = |(\partial_t \boldsymbol{p}_n, \Pi_n \boldsymbol{v})| \le{}& \mu_2\|\Delta \boldsymbol{p}_n\|_{\boldsymbol{L}^2}\|\Delta \Pi_n \boldsymbol{v}\|_{\boldsymbol{L}^2} + |\gamma_2|\|\nabla \boldsymbol{p}_n\|_{\boldsymbol{L}^{\frac{6}{5}}}\|\nabla \Pi_n \boldsymbol{v}\|_{\boldsymbol{L}^6} \\ &+ \lambda_2\|\boldsymbol{p}_n\|_{\boldsymbol{L}^2}\|\nabla \boldsymbol{p}_n\|_{\boldsymbol{L}^2}\|\Pi_n \boldsymbol{v}\|_{\boldsymbol{L}^\infty} + \alpha\|\boldsymbol{p}_n\|_{\boldsymbol{L}^3}^3\|\Pi_n \boldsymbol{v}\|_{\boldsymbol{L}^\infty} \\ &+ |\beta|\|\boldsymbol{p}_n\|_{\boldsymbol{L}^1}\|\Pi_n \boldsymbol{v}\|_{\boldsymbol{L}^\infty} + \frac{3}{2}\|\boldsymbol{u}_n\|_{\boldsymbol{L}^2}\|\boldsymbol{p}_n\|_{\boldsymbol{L}^3}\|\nabla \Pi_n \boldsymbol{v}\|_{\boldsymbol{L}^6} \\ &+ \frac{1}{2}\|\Pi_n \boldsymbol{v}\|_{\boldsymbol{L}^\infty}\|\nabla \boldsymbol{p}_n\|_{\boldsymbol{L}^2}\|\boldsymbol{u}_n\|_{\boldsymbol{L}^2} \end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$. From the continuous embeddings $\boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ and the $\boldsymbol{H}^2$-stability of the orthogonal projections Π_n (cf. Málek et al.[14, Appendix, Theorem 4.11 and Lemma 4.26] together with Boyer and Fabrie[9, Proposition III.3.17]), it follows that

$$\|\partial_t \boldsymbol{p}_n\|_{(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma})^*} \le c\left(\|\Delta \boldsymbol{p}_n\|_{\boldsymbol{L}^2} + \|\boldsymbol{p}_n\|_{\boldsymbol{L}^2}\|\nabla \boldsymbol{p}_n\|_{\boldsymbol{L}^2} + \|\boldsymbol{p}_n\|_{\boldsymbol{L}^4}^3 + \|\boldsymbol{u}_n\|_{\boldsymbol{L}^2}\|\boldsymbol{p}_n\|_{\boldsymbol{L}^3} + \|\nabla \boldsymbol{p}_n\|_{\boldsymbol{L}^2}\|\boldsymbol{u}_n\|_{\boldsymbol{L}^2} \right).$$

Now, using the embedding $\boldsymbol{H}_0^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we find

$$\|\boldsymbol{p}_n\|_{\boldsymbol{L}^3} \leq \|\boldsymbol{p}_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}} \|\boldsymbol{p}_n\|_{\boldsymbol{L}^6}^{\frac{1}{2}} \leq c\|\boldsymbol{p}_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}} \|\nabla \boldsymbol{p}_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}}.$$

Together with

$$\|\nabla \boldsymbol{p}_n\|_{\boldsymbol{L}^2}^2 = (\nabla \boldsymbol{p}_n, \nabla \boldsymbol{p}_n) = -(\boldsymbol{p}_n, \Delta \boldsymbol{p}_n) \leq \|\boldsymbol{p}_n\|_{\boldsymbol{L}^2} \|\Delta \boldsymbol{p}_n\|_{\boldsymbol{L}^2}$$

as well as Young's inequality, we arrive at

$$\|\partial_t \boldsymbol{p}_n\|_{(\boldsymbol{H}^2 \cap \boldsymbol{H}_{0,\sigma}^1)^*} \leq c \Bigg(\|\Delta \boldsymbol{p}_n\|_{\boldsymbol{L}^2} + \|\boldsymbol{p}_n\|_{\boldsymbol{L}^2} \|\nabla \boldsymbol{p}_n\|_{\boldsymbol{L}^2} + \|\boldsymbol{p}_n\|_{\boldsymbol{L}^4}^3$$
$$\|\boldsymbol{p}_n\|_{\boldsymbol{L}^2}^{\frac{3}{4}} \left(\|\boldsymbol{u}_n\|_{\boldsymbol{L}^2}^{\frac{5}{4}} + \|\Delta \boldsymbol{p}_n\|_{\boldsymbol{L}^2}^{\frac{5}{4}} \right) + \|\boldsymbol{p}_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}} \left(\|\boldsymbol{u}_n\|_{\boldsymbol{L}^2}^{\frac{3}{2}} + \|\Delta \boldsymbol{p}_n\|_{\boldsymbol{L}^2}^{\frac{3}{2}} \right) \Bigg).$$

Taking the estimate above to the power $\frac{4}{3}$, integrating and applying the a priori estimate (14) then yields

$$\int_0^t \|\partial_t \boldsymbol{p}_n\|_{(\boldsymbol{H}^2 \cap \boldsymbol{H}_{0,\sigma}^1)^*}^{\frac{4}{3}} \, \mathrm{d}s \leq c(\|\boldsymbol{p}_0\|_{\boldsymbol{L}^2}^2 + 1).$$

Together with (14), we can extract (not relabeled) subsequences $\{\boldsymbol{p}_n\}$ and $\{\boldsymbol{u}_n\}$ such that

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \quad \text{in } L^2(0,T;\boldsymbol{L}_\sigma^2), \tag{15a}$$

$$\boldsymbol{p}_n \overset{*}{\rightharpoonup} \boldsymbol{p} \quad \text{in } L^\infty(0,T;\boldsymbol{L}_\sigma^2), \tag{15b}$$

$$\boldsymbol{p}_n \rightharpoonup \boldsymbol{p} \quad \text{in } L^4(0,T;\boldsymbol{L}_\sigma^4), \tag{15c}$$

$$\boldsymbol{p}_n \rightharpoonup \boldsymbol{p} \quad \text{in } L^2(0,T;\boldsymbol{H}^2 \cap \boldsymbol{H}_{0,\sigma}^1), \tag{15d}$$

$$\partial_t \boldsymbol{p}_n \rightharpoonup \partial_t \boldsymbol{p} \quad \text{in } L^{\frac{4}{3}}(0,T;(\boldsymbol{H}^2 \cap \boldsymbol{H}_{0,\sigma}^1)^*), \tag{15e}$$

as $n \to \infty$. Let us show that $(\boldsymbol{u}, \boldsymbol{p})$ is a weak solution to (1) in the sense of Definition (1). Since $W_m \subset W_n$ for $m \leq n$, (6) also holds true for all $\phi(t)\boldsymbol{v}, \psi(t)\boldsymbol{w}$, where $t \in (0,T)$, $\phi, \psi \in \mathscr{C}_c^\infty(0,T)$ and $\boldsymbol{v}, \boldsymbol{w} \in W_m$. We fix m and let n go to ∞. In view of the compact embedding $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_{0,\sigma}^1(\Omega) \overset{c}{\hookrightarrow} \boldsymbol{H}_{0,\sigma}^1(\Omega)$, the Lions–Aubin lemma (see Lions[15, Théorème 1.5.2]) provides the strong convergence

$$\boldsymbol{p}_n \to \boldsymbol{p} \quad \text{in } L^2(0,T;\boldsymbol{H}_{0,\sigma}^1), \tag{16}$$

passing to a subsequence if necessary. This in turn allows us, after integrating, to pass to the limit also in the nonlinear terms of (6). Hence, we come up with

$$\int_0^T (\boldsymbol{u}, \phi\boldsymbol{v})\,\mathrm{d}t - \mu_1 \int_0^T (\Delta \boldsymbol{p}, \Delta A^{-1}(\phi\boldsymbol{v}))\,\mathrm{d}t + \gamma_1 \int_0^T (\Delta \boldsymbol{p}, A^{-1}(\phi\boldsymbol{v}))\,\mathrm{d}t$$
$$- \lambda_1 \int_0^T ((\boldsymbol{p} \cdot \nabla)\boldsymbol{p}, A^{-1}(\phi\boldsymbol{v}))\,\mathrm{d}t = 0$$

and

$$\int_0^T \langle \partial_t \boldsymbol{p}, \psi\boldsymbol{w} \rangle\,\mathrm{d}t + \mu_2 \int_0^T (\Delta \boldsymbol{p}, \Delta(\psi\boldsymbol{w}))\,\mathrm{d}t + \gamma_2 \int_0^T (\nabla \boldsymbol{p}, \nabla(\psi\boldsymbol{w}))\,\mathrm{d}t + \lambda_2 \int_0^T ((\boldsymbol{p} \cdot \nabla)\boldsymbol{p}, \psi\boldsymbol{w})\,\mathrm{d}t$$
$$+ \alpha \int_0^T (|\boldsymbol{p}|^2\boldsymbol{p}, \psi\boldsymbol{w})\,\mathrm{d}t + \beta \int_0^T (\boldsymbol{p}, \psi\boldsymbol{w})\,\mathrm{d}t + \int_0^T ((\boldsymbol{u} \cdot \nabla)\boldsymbol{p}, \psi\boldsymbol{w})\,\mathrm{d}t$$
$$+ \frac{1}{2} \int_0^T ((\boldsymbol{p} \cdot \nabla)(\psi\boldsymbol{w}) - ((\psi\boldsymbol{w}) \cdot \nabla)\boldsymbol{p}, \boldsymbol{u})\,\mathrm{d}t = 0$$

for all $\phi, \psi \in \mathscr{C}_c^\infty(0,T)$ and $\boldsymbol{v}, \boldsymbol{w} \in W_m$ with arbitrary m and thus for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$ since $\cup_{m\in\mathbb{N}} W_m$ is dense in $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$. It remains to show that $\boldsymbol{p}(0) = \boldsymbol{p}_0$. In order to show this, we observe that

$$\boldsymbol{p} \in W^{1,\frac{4}{3}}(0,T;(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma})^*) \hookrightarrow \mathscr{C}([0,T];(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma})^*) \hookrightarrow \mathscr{C}_w([0,T];(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma})^*),$$

making point evaluations of $\boldsymbol{p}$ in the dual pairing well defined. Since also $\boldsymbol{p} \in L^\infty(0,T;\boldsymbol{L}^2_\sigma)$, it follows from a well-known lemma (see Lions and Magenes[16, Chapter 3, Lemma 8.1]) that

$$\boldsymbol{p} \in \mathscr{C}_w([0,T];\boldsymbol{L}^2_\sigma)$$

and in particular

$$\boldsymbol{p}(t) \rightharpoonup \boldsymbol{p}(0) \ \text{ in } \ \boldsymbol{L}^2_\sigma(\Omega) \tag{17}$$

as $t \to 0$. Furthermore, for all $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$ and $\varphi(t) = (T-t)/T$, there holds

$$\langle \boldsymbol{p}_n(0) - \boldsymbol{p}(0), \boldsymbol{v} \rangle = -\int_0^T \left(\langle \boldsymbol{p}_n(t) - \boldsymbol{p}(t), \boldsymbol{v}\varphi'(t) \rangle + \langle \partial_t \boldsymbol{p}_n(t) - \partial_t \boldsymbol{p}(t), \boldsymbol{v}\varphi(t) \rangle \right) \mathrm{d}t \to 0$$

as $n \to \infty$, by the convergence in (15b) and (15e). This implies

$$\boldsymbol{p}_n(0) \to \boldsymbol{p}(0) \ \text{ in } \ (\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega))^*. \tag{18}$$

Since also

$$\boldsymbol{p}_n(0) = \Pi_n \boldsymbol{p}_0 \to \boldsymbol{p}_0 \in \boldsymbol{L}^2_\sigma(\Omega),$$

it follows that

$$\boldsymbol{p}(0) = \boldsymbol{p}_0 \ \text{ in } \ (\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega))^*. \tag{19}$$

By (17) and (19), it finally follows that

$$\boldsymbol{p}(t) \rightharpoonup \boldsymbol{p}_0 \ \text{ in } \ \boldsymbol{L}^2_\sigma(\Omega)$$

as $t \to 0$. □

From the above proof of existence, we can easily see that the following energy inequality as well as an estimate for $\boldsymbol{u}$ hold true.

Lemma 1 (Energy inequality). *Let $(\boldsymbol{u}, \boldsymbol{p})$ be the weak solution to (1) in the sense of Definition 1 that was constructed in the proof of Theorem (1). Then the estimates*

$$\frac{1}{2}\|\boldsymbol{p}(t)\|^2_{\boldsymbol{L}^2} + \int_0^t \left(\mu_2 \|\Delta \boldsymbol{p}\|^2_{\boldsymbol{L}^2} + \gamma_2 \|\nabla \boldsymbol{p}\|^2_{\boldsymbol{L}^2} + \alpha \|\boldsymbol{p}\|^4_{\boldsymbol{L}^4} + \beta \|\boldsymbol{p}\|^2_{\boldsymbol{L}^2} \right) \mathrm{d}s \le \frac{1}{2}\|\boldsymbol{p}_0\|^2_{\boldsymbol{L}^2}, \tag{20}$$

$$\|\boldsymbol{p}(t)\|^2_{\boldsymbol{L}^2} + \int_0^t \left(\mu_2 \|\Delta \boldsymbol{p}\|^2_{\boldsymbol{L}^2} + \alpha \|\boldsymbol{p}\|^4_{\boldsymbol{L}^4} \right) \mathrm{d}s \le \|\boldsymbol{p}_0\|^2_{\boldsymbol{L}^2} + \frac{T|\Omega|}{\alpha} \left(\frac{(\gamma_2^-)^2}{2\mu_2} + \beta^- \right)^2, \tag{21}$$

and

$$\int_0^t \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2} \mathrm{d}s \le \varepsilon^2 c(\|\boldsymbol{p}_0\|^2_{\boldsymbol{L}^2} + 1) \tag{22}$$

hold for almost all $t \in (0,T)$.

Proof. As seen in the proof of Theorem 1, we obtain the energy inequality for the discrete solution $\boldsymbol{p}_n$ by testing the discretized equation (6b) with it. The estimate then also holds in the limit, as $n \to \infty$, by the sequential lower semicontinuity of the norm and employing the strong convergence (16) for those terms with possibly negative coefficients. In the same fashion as we derived the a priori estimate (8), we then deduce (21) from (20). Finally, the third estimate can be obtained by testing the very weak formulation of the Stokes equation (3) with the solution $\boldsymbol{u}$, applying the same steps as to reach (13) and using (21). □

4 | A RELATIVE ENERGY INEQUALITY

In the previous section, we have constructed a weak solution $(\boldsymbol{u}, \boldsymbol{p})$ to (1) for which the time derivative $\partial_t \boldsymbol{p}$ of the polar ordering field is only in $L^{\frac{4}{3}}(0, T; (\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma})^*)$. Similarly as in the case of the incompressible Navier–Stokes equations, we are therefore not allowed to test the weak formulation (4) with the solution $\boldsymbol{p} \in L^2(0, T; \boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}) \cap L^\infty(0, T; \boldsymbol{L}^2_\sigma) \cap L^4(0, T; \boldsymbol{L}^4_\sigma)$ itself, meaning that the obvious path for proving uniqueness of a solution is blocked. A way to at least partially compensate for the missing uniqueness is the concept of weak–strong uniqueness. A weak solution possesses the weak–strong uniqueness property if it coincides with the unique strong solution as long as it exists. The fact that Leray–Hopf solutions to the Navier–Stokes equations have this property is well-known (see, e.g., Leray,[17] Prodi,[18] and Serrin[19]). The goal of this section is now to prove the same for weak solutions to (1) that fulfill the energy inequality (20). The main argument here will be a suitable relative energy inequality.

We first define two functionals: the relative energy $\mathscr{E} : \boldsymbol{L}^2_\sigma(\Omega) \times \boldsymbol{L}^2_\sigma(\Omega) \to \mathbb{R}_{\geq 0}$ by

$$\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}] = \frac{1}{2}\|\boldsymbol{p} - \tilde{\boldsymbol{p}}\|^2_{\boldsymbol{L}^2}$$

and the relative dissipation $\mathscr{W} : \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega) \times \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega) \to \mathbb{R}_{\geq 0}$ by

$$\mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}] = \mu_2\|\Delta\boldsymbol{p} - \Delta\tilde{\boldsymbol{p}}\|^2_{\boldsymbol{L}^2} + \gamma_2^+\|\nabla\boldsymbol{p} - \nabla\tilde{\boldsymbol{p}}\|^2_{\boldsymbol{L}^2} + \alpha\|\boldsymbol{p} - \tilde{\boldsymbol{p}}\|^4_{\boldsymbol{L}^4} + \beta^+\|\boldsymbol{p} - \tilde{\boldsymbol{p}}\|^2_{\boldsymbol{L}^2},$$

where for any real number a we set $a^+ := a + a^-$. Both the relative energy and the relative dissipation are supposed to provide a way to measure the "distance" between two functions. Hence, we have to ensure their nonnegativity. Since we do not know the sign of β and γ_2, we therefore have to correct those possibly negative terms in the relative dissipation $\mathscr{W}$ by taking only their positive parts γ_2^+ and β^+, respectively. For $\boldsymbol{p}, \tilde{\boldsymbol{p}} \in L^\infty(\boldsymbol{L}^2_\sigma) \cap L^4(\boldsymbol{L}^4_\sigma) \cap \boldsymbol{L}^2(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma})$, setting

$$\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}](t) := \mathscr{E}[\boldsymbol{p}(t)|\tilde{\boldsymbol{p}}(t)] \text{ and } \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}](t) := \mathscr{W}[\boldsymbol{p}(t)|\tilde{\boldsymbol{p}}(t)], \qquad t \in [0, T],$$

we conclude that $\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}] \in L^\infty(0, T)$ and $\mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}] \in L^1(0, T)$. Furthermore, we introduce the mapping

$$\mathscr{K} : L^2(0, T; \boldsymbol{L}^\infty) \cap L^\infty\left(0, T; \boldsymbol{L}^{\frac{6}{5}}_\sigma\right) \cap L^2(0, T; \boldsymbol{W}^{1,\infty}) \cap L^4(0, T; \boldsymbol{L}^4_\sigma) \to L^1(0, T)$$

with

$$\mathscr{K}[\tilde{\boldsymbol{p}}](t) := \varepsilon c\left(1 + \|\nabla\tilde{\boldsymbol{p}}(t)\|^2_{\boldsymbol{L}^\infty} + \|\tilde{\boldsymbol{p}}\|^2_{L^\infty\left(\boldsymbol{L}^{\frac{6}{5}}\right)}\|\nabla\tilde{\boldsymbol{p}}(t)\|^2_{\boldsymbol{L}^\infty} + \|\tilde{\boldsymbol{p}}(t)\|^4_{\boldsymbol{L}^4}\right), \tag{23}$$

where $\varepsilon > 0$ is explained by (2) and $c > 0$ is a constant only depending on the coefficients $\alpha, \beta, \tilde{\gamma}_1, \gamma_2, \tilde{\lambda}_1, \lambda_2, \tilde{\mu}_1, \mu_2$ and the domain Ω.

Finally, we introduce an operator encoding the strong formulation of the equations. Let

$$\mathscr{R} : L^2(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}) \times (W^{1,2}(\boldsymbol{L}^2_\sigma) \cap L^6(\boldsymbol{L}^6_\sigma) \cap L^2(\boldsymbol{H}^4 \cap \boldsymbol{H}^1_{0,\sigma})) \to L^2(\boldsymbol{L}^2) \times L^2(\boldsymbol{L}^2)$$

be defined as

$$\mathscr{R}[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}] = \begin{pmatrix} \mathscr{R}_1[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}] \\ \mathscr{R}_2[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}] \end{pmatrix},$$

where

$$\mathscr{R}_1[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}] = -\Delta\tilde{\boldsymbol{u}} + \mu_1\Delta^2\tilde{\boldsymbol{p}} - \gamma_1\Delta\tilde{\boldsymbol{p}} + \lambda_1(\tilde{\boldsymbol{p}}\cdot\nabla)\tilde{\boldsymbol{p}}$$

and

$$\mathscr{R}_2[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}] = \partial_t\tilde{\boldsymbol{p}} + \mu_2\Delta^2\tilde{\boldsymbol{p}} + (\tilde{\boldsymbol{u}}\cdot\nabla)\tilde{\boldsymbol{p}} - \gamma_2\Delta\tilde{\boldsymbol{p}} + \lambda_2(\tilde{\boldsymbol{p}}\cdot\nabla)\tilde{\boldsymbol{p}} + \alpha|\tilde{\boldsymbol{p}}|^2\tilde{\boldsymbol{p}} + \beta\tilde{\boldsymbol{p}} - (\nabla\tilde{\boldsymbol{u}})_{\mathrm{skw}}\tilde{\boldsymbol{p}}.$$

Multiplying $\mathscr{R}_2[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}]$ by $\tilde{\boldsymbol{p}}$ and integrating in space and time yields

$$\begin{aligned}\frac{1}{2}\|\tilde{\boldsymbol{p}}(t)\|_{\boldsymbol{L}^2}^2 + \int_0^t \left(\mu_2\|\Delta\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2 + \gamma_2\ \|\nabla\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2 + \alpha\|\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^4}^4 + \beta\|\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s \\ = \frac{1}{2}\|\tilde{\boldsymbol{p}}(0)\|_{\boldsymbol{L}^2}^2 + \int_0^t (\mathscr{R}_2[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}],\tilde{\boldsymbol{p}})\,\mathrm{d}s.\end{aligned} \tag{24}$$

Note that if we choose $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{p}})$ as a strong solution to (1) then $\mathscr{R}[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}]$ vanishes and the identity (24) becomes the energy equality in (20). In this sense, we can think about (24) as a shifted energy equality.

We now can state the desired relative energy inequality in a way that allows us to compare a weak solution that fulfills the energy inequality (20) with an arbitrary pair of functions with additional regularity, which cannot be expected for weak solutions.

Theorem 2. *Let $(\boldsymbol{u},\boldsymbol{p})$ be a weak solution to (1) in the sense of Definition 1 that additionally fulfills the energy inequality (20). Then for any pair $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{p}})$ of test functions such that*

$$\tilde{\boldsymbol{u}} \in L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma}),\ \tilde{\boldsymbol{p}} \in W^{1,2}(\boldsymbol{L}^2_\sigma)\cap L^6(\boldsymbol{L}^6_\sigma)\cap L^2(\boldsymbol{H}^4\cap\boldsymbol{H}^1_{0,\sigma}),$$

the relative energy inequality

$$\begin{aligned}\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}](t) + \frac{1}{2}\int_0^t \mathscr{W}\,[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s \le \mathscr{E}[\boldsymbol{p}_0|\tilde{\boldsymbol{p}}(0)] + \int_0^t \mathscr{K}[\tilde{\boldsymbol{p}}]\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s \\ + \int_0^t \left(\mathscr{R}[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], \left(A^{-1}(\tilde{\boldsymbol{u}}-\boldsymbol{u})\tilde{\boldsymbol{p}}-\boldsymbol{p}\right)\right)\mathrm{d}s\end{aligned} \tag{25}$$

holds for almost all $t \in (0,T)$.

In order to prove the relative energy inequality above, we first need two lemmata: an integration-by-parts formula and a lower bound for the relative dissipation $\mathscr{W}$ in terms of the distance of $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$.

Lemma 2 (Integration-by-parts formula)**.** *For all*

$$\boldsymbol{p} \in L^\infty(\boldsymbol{L}^2)\cap L^4(\boldsymbol{L}^4)\cap L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})$$

such that $\partial_t\boldsymbol{p} \in L^{\frac{4}{3}}((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)$ and all

$$\tilde{\boldsymbol{p}} \in L^4(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})\cap W^{1,2}((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*),$$

there holds

$$\int_s^t (\langle\partial_t\boldsymbol{p},\tilde{\boldsymbol{p}}\rangle + \langle\boldsymbol{p},\partial_t\tilde{\boldsymbol{p}}\rangle)\,\mathrm{d}\tau = (\boldsymbol{p}(t),\tilde{\boldsymbol{p}}(t)) - (\boldsymbol{p}(s),\tilde{\boldsymbol{p}}(s)) \tag{26}$$

for all $s, t \in [0,T]$.

Proof. First, we choose sequences $\{\boldsymbol{p}_n\}, \{\tilde{\boldsymbol{p}}_n\} \subset \mathscr{C}^1([0,T];\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})$ such that

$$\begin{aligned}\boldsymbol{p}_n \to \boldsymbol{p} \ \text{ in } \ L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})\cap W^{1,\frac{4}{3}}((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*), \\ \tilde{\boldsymbol{p}}_n \to \tilde{\boldsymbol{p}} \ \text{ in } \ L^4(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})\cap W^{1,2}((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*).\end{aligned}$$

Such sequences exist since $\mathscr{C}^1([0,T];\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})$ is dense in both $L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})\cap W^{1,\frac{4}{3}}((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)$ and $L^4(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})\cap W^{1,2}((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)$. By the higher time regularity of the approximating sequence, we find

$$-\int_0^T \varphi'(\boldsymbol{p}_n,\tilde{\boldsymbol{p}}_n)\,\mathrm{d}t=\int_0^T \varphi\left(\langle\partial_t\boldsymbol{p}_n,\tilde{\boldsymbol{p}}_n\rangle+(\boldsymbol{p}_n,\partial_t\tilde{\boldsymbol{p}}_n)\right)\,\mathrm{d}t \tag{27}$$

for all $\varphi\in\mathscr{C}_c^\infty(0,T)$. From the estimates

$$\begin{aligned}\int_0^T &|\langle\partial_t\boldsymbol{p},\tilde{\boldsymbol{p}}\rangle-\langle\partial_t\boldsymbol{p}_n,\tilde{\boldsymbol{p}}_n\rangle|\,\mathrm{d}t\\&\le\|\partial_t\boldsymbol{p}-\partial_t\boldsymbol{p}_n\|_{L^{\frac{4}{3}}((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)}\|\tilde{\boldsymbol{p}}\|_{L^4(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})}\\&\quad+\|\partial_t\boldsymbol{p}_n\|_{L^{\frac{4}{3}}((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)}\|\tilde{\boldsymbol{p}}-\tilde{\boldsymbol{p}}_n\|_{L^4(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})}\end{aligned}$$

and

$$\begin{aligned}\int_0^T &|(\boldsymbol{p},\partial_t\tilde{\boldsymbol{p}})-(\boldsymbol{p}_n,\partial_t\tilde{\boldsymbol{p}}_n)|\,\mathrm{d}t\\&\le\|\boldsymbol{p}\|_{L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})}\|\partial_t\tilde{\boldsymbol{p}}-\partial_t\tilde{\boldsymbol{p}}_n\|_{L^2((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)}\\&\quad+\|\boldsymbol{p}-\boldsymbol{p}_n\|_{L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})}\|\partial_t\tilde{\boldsymbol{p}}_n\|_{L^2((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)}\end{aligned}$$

as well as

$$\begin{aligned}\int_0^T &|(\boldsymbol{p},\tilde{\boldsymbol{p}})-(\boldsymbol{p}_n,\tilde{\boldsymbol{p}}_n)|\,\mathrm{d}t\\&\le\|\boldsymbol{p}\|_{L^2(\boldsymbol{L}^2)}\|\tilde{\boldsymbol{p}}-\tilde{\boldsymbol{p}}_n\|_{L^2(\boldsymbol{L}^2)}+\|\boldsymbol{p}-\boldsymbol{p}_n\|_{L^2(\boldsymbol{L}^2)}\|\tilde{\boldsymbol{p}}_n\|_{L^2(\boldsymbol{L}^2)},\end{aligned}$$

together with the embedding $L^4(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})\hookrightarrow L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})\hookrightarrow L^2(\boldsymbol{L}^2)$, we conclude that the equality (27) holds also in the limit as $n\to\infty$, that is,

$$-\int_0^T \varphi'(\boldsymbol{p},\tilde{\boldsymbol{p}})\,\mathrm{d}t=\int_0^T \varphi\left(\langle\partial_t\boldsymbol{p},\tilde{\boldsymbol{p}}\rangle+(\boldsymbol{p},\partial_t\tilde{\boldsymbol{p}})\right)\,\mathrm{d}t \tag{28}$$

for all $\phi\in\mathscr{C}_c^\infty(0,T)$. Note that $\|\partial_t\boldsymbol{p}_n\|_{L^{\frac{4}{3}}((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)}$, $\|\partial_t\tilde{\boldsymbol{p}}_n\|_{L^2((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)}$, and $\|\tilde{\boldsymbol{p}}_n\|_{L^2(\boldsymbol{L}^2)}$ are bounded uniformly in n. Using the same embedding as above, we find

$$\int_0^T |(\boldsymbol{p},\tilde{\boldsymbol{p}})|\,\mathrm{d}t\le c\|\boldsymbol{p}\|_{L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})}\|\tilde{\boldsymbol{p}}\|_{L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})}.$$

Since also

$$\begin{aligned}\int_0^T &|\langle\partial_t\boldsymbol{p},\tilde{\boldsymbol{p}}\rangle+(\boldsymbol{p},\partial_t\tilde{\boldsymbol{p}})|\,\mathrm{d}t\\&\le\|\partial_t\boldsymbol{p}\|_{L^{\frac{4}{3}}((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})*)}\|\tilde{\boldsymbol{p}}\|_{L^4(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})}+\|\boldsymbol{p}\|_{L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})}\|\partial_t\tilde{\boldsymbol{p}}\|_{L^2((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)},\end{aligned}$$

it follows from (28) that the mapping

$$t\mapsto(\boldsymbol{p}(t),\tilde{\boldsymbol{p}}(t)) \tag{29}$$

is in $W^{1,1}(0,T)$ since the weak derivative

$$t\mapsto\langle\partial_t\boldsymbol{p}(t),\tilde{\boldsymbol{p}}(t)\rangle+(\boldsymbol{p}(t),\partial_t\tilde{\boldsymbol{p}}(t))$$

is in $L^1(0,T)$. In particular, the function (29) is absolutely continuous on $[0,T]$ and we can apply the fundamental theorem of calculus to finally prove the statement. □

Lemma 3. *Let $(\boldsymbol{u},\boldsymbol{p})$ be a weak solution to (1) in the sense of Definition 1 and ε be chosen as in (2). Then there exists a constant $C>0$ such that for all*

$$\tilde{\boldsymbol{u}}\in L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma}),\ \ \tilde{\boldsymbol{p}}\in W^{1,2}(\boldsymbol{L}^2_\sigma)\cap L^6(\boldsymbol{L}^6_\sigma)\cap L^2(\boldsymbol{H}^4\cap\boldsymbol{H}^1_{0,\sigma}),$$

there holds

$$\frac{1}{2}\int_0^t \|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2}^2 \le \varepsilon C\left(1+\|\tilde{\boldsymbol{p}}\|^2_{L^\infty\left(\boldsymbol{L}^{\frac{6}{5}}\right)}\right)\int_0^t \mathscr{W}\,[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s + \int_0^t (\mathscr{R}_1[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], A^{-1}(\tilde{\boldsymbol{u}}-\boldsymbol{u}))\,\mathrm{d}s \tag{30}$$

for all $t \in [0, T]$.

Note that we could allow for a larger class of test functions $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{p}})$ in this lemma by shifting some of the derivatives contained in $\mathscr{R}_1[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}]$ onto $A^{-1}(\tilde{\boldsymbol{u}}-\boldsymbol{u})$ using an integration-by-parts formula. Since we only need to be able to apply estimate (30) to strong solutions, we restrict ourselves here to the regularity assumptions above.

Proof. Using Equation (3) as well as adding and subtracting $(\mathscr{R}_1[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], A^{-1}(\tilde{\boldsymbol{u}}-\boldsymbol{u}))$ yields

$$\begin{aligned}
&\int_0^t \|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2}^2\,\mathrm{d}s\\
&\quad= \mu_1\int_0^t (\Delta\tilde{\boldsymbol{p}}-\Delta\boldsymbol{p}, \Delta A^{-1}(\boldsymbol{u}-\tilde{\boldsymbol{u}}))\,\mathrm{d}s - \gamma_1\int_0^t (\Delta\tilde{\boldsymbol{p}}-\Delta\boldsymbol{p}, A^{-1}(\boldsymbol{u}-\tilde{\boldsymbol{u}}))\,\mathrm{d}s\\
&\qquad+ \lambda_1\int_0^t \big(((\tilde{\boldsymbol{p}}\cdot\nabla)\tilde{\boldsymbol{p}}, A^{-1}(\boldsymbol{u}-\tilde{\boldsymbol{u}})) - ((\boldsymbol{p}\cdot\nabla)\boldsymbol{p}, A^{-1}(\boldsymbol{u}-\tilde{\boldsymbol{u}}))\big)\,\mathrm{d}s\\
&\qquad+ \int_0^t (\mathscr{R}_1[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], A^{-1}(\tilde{\boldsymbol{u}}-\boldsymbol{u}))\,\mathrm{d}s\\
&\quad=: \mu_1 J_1 - \gamma_1 J_2 + \lambda_1 J_3 + \int_0^t (\mathscr{R}_1[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], A^{-1}(\tilde{\boldsymbol{u}}-\boldsymbol{u}))\,\mathrm{d}s.
\end{aligned}$$

Since $\boldsymbol{p}$ and $\tilde{\boldsymbol{p}}$ are divergence-free, we can rewrite J_1, perform an integration-by-parts, apply Hölder's inequality, and use the continuous embeddings $\boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ and $\boldsymbol{L}^4(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{12}{5}}(\Omega)$ as well as the continuity of $A^{-1} : \boldsymbol{L}^2_\sigma(\Omega) \to \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$ to obtain

$$\begin{aligned}
|J_3| &= \left|\int_0^t (\tilde{\boldsymbol{p}}\otimes(\tilde{\boldsymbol{p}}-\boldsymbol{p}) - (\tilde{\boldsymbol{p}}-\boldsymbol{p})\otimes(\tilde{\boldsymbol{p}}-\boldsymbol{p}) + (\tilde{\boldsymbol{p}}-\boldsymbol{p})\otimes\tilde{\boldsymbol{p}}, \nabla A^{-1}(\tilde{\boldsymbol{u}}-\boldsymbol{u}))\,\mathrm{d}s\right|\\
&\le c\int_0^t \|\tilde{\boldsymbol{p}}\otimes(\tilde{\boldsymbol{p}}-\boldsymbol{p}) - (\tilde{\boldsymbol{p}}-\boldsymbol{p})\otimes(\tilde{\boldsymbol{p}}-\boldsymbol{p}) + (\tilde{\boldsymbol{p}}-\boldsymbol{p})\otimes\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^{\frac{6}{5}}}\|\tilde{\boldsymbol{u}}-\boldsymbol{u}\|_{\boldsymbol{L}^2}\,\mathrm{d}s\\
&\le c\int_0^t \left(2\|\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^{\frac{6}{5}}}\|\tilde{\boldsymbol{p}}-\boldsymbol{p}\|_{\boldsymbol{L}^\infty} + \|\tilde{\boldsymbol{p}}-\boldsymbol{p}\|^2_{\boldsymbol{L}^{\frac{12}{5}}}\right)\|\tilde{\boldsymbol{u}}-\boldsymbol{u}\|_{\boldsymbol{L}^2}\,\mathrm{d}s\\
&\le c\int_0^t \left(1+\|\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^{\frac{6}{5}}}\right)\left(\|\Delta\tilde{\boldsymbol{p}}-\Delta\boldsymbol{p}\|_{\boldsymbol{L}^2} + \|\tilde{\boldsymbol{p}}-\boldsymbol{p}\|^2_{\boldsymbol{L}^4}\right)\|\tilde{\boldsymbol{u}}-\boldsymbol{u}\|_{\boldsymbol{L}^2}\,\mathrm{d}s.
\end{aligned}$$

In a similar way, we can estimate the term J_1 by

$$|J_1| \le c\int_0^t \|\Delta\tilde{\boldsymbol{p}}-\Delta\boldsymbol{p}\|_{\boldsymbol{L}^2}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2}\,\mathrm{d}s.$$

and the term J_2 by

$$\begin{aligned}
|J_2| &\le \int_0^t \|\Delta\tilde{\boldsymbol{p}}-\Delta\boldsymbol{p}\|_{\boldsymbol{L}^2}\|A^{-1}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\|_{\boldsymbol{L}^2}\,\mathrm{d}s\\
&\le c\int_0^t \|\Delta\tilde{\boldsymbol{p}}-\Delta\boldsymbol{p}\|_{\boldsymbol{L}^2}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2}\,\mathrm{d}s.
\end{aligned}$$

Using the above estimates in (30), applying Young's inequality and absorbing the terms depending on $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ into the left-hand side then proves the assertion. □

We are now able to prove the relative energy inequality.

Proof of Theorem 2. We start by using the integration-by-parts formula from Lemma 2 to obtain

$$\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}](t) = \frac{1}{2}\|\boldsymbol{p}(t)\|_{\boldsymbol{L}^2}^2 - (\boldsymbol{p}_0, \tilde{\boldsymbol{p}}(0)) - \int_0^t (\langle \partial_t \boldsymbol{p}, \tilde{\boldsymbol{p}}\rangle + \langle \boldsymbol{p}, \partial_t \tilde{\boldsymbol{p}}\rangle)\,\mathrm{d}s + \frac{1}{2}\|\tilde{\boldsymbol{p}}(t)\|_{\boldsymbol{L}^2}^2 \tag{31}$$

for all $t \in [0, T]$. Note that by (18), the weak solution $\boldsymbol{p}$ takes the initial value in $(\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega))^*$. Since $\tilde{\boldsymbol{p}} \in L^4(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}) \cap L^\infty(\boldsymbol{H}^1)$ by Remark 1, it is an admissible test function in (4). Hence, by using (4) in (31) above as well as by adding and subtracting $(\mathscr{R}_2[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], \boldsymbol{p})$, we obtain

$$\begin{aligned}
\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}](t) =& \frac{1}{2}\|\boldsymbol{p}(t)\|_{\boldsymbol{L}^2}^2 - (\boldsymbol{p}_0, \tilde{\boldsymbol{p}}(0)) + \mu_2 \int_0^t (\Delta \boldsymbol{p}, \Delta \tilde{\boldsymbol{p}})\,\mathrm{d}s + \gamma_2 \int_0^t (\nabla \boldsymbol{p}, \nabla \tilde{\boldsymbol{p}})\,\mathrm{d}s \\
&+ \lambda_2 \int_0^t ((\boldsymbol{p}\cdot\nabla)\boldsymbol{p}, \tilde{\boldsymbol{p}})\,\mathrm{d}s + \alpha \int_0^t (|\boldsymbol{p}|^2\boldsymbol{p}, \tilde{\boldsymbol{p}})\,\mathrm{d}s + \beta \int_0^t (\boldsymbol{p}, \tilde{\boldsymbol{p}})\,\mathrm{d}s + \int_0^t ((\boldsymbol{u}\cdot\nabla)\boldsymbol{p}, \tilde{\boldsymbol{p}})\,\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t ((\boldsymbol{p}\cdot\nabla)\tilde{\boldsymbol{p}} - (\tilde{\boldsymbol{p}}\cdot\nabla)\boldsymbol{p}, \boldsymbol{u})\,\mathrm{d}s + \mu_2 \int_0^t (\Delta \tilde{\boldsymbol{p}}, \Delta \boldsymbol{p})\,\mathrm{d}s + \gamma_2 \int_0^t (\nabla \tilde{\boldsymbol{p}}, \nabla \boldsymbol{p})\,\mathrm{d}s \\
&+ \lambda_2 \int_0^t ((\tilde{\boldsymbol{p}}\cdot\nabla)\tilde{\boldsymbol{p}}, \boldsymbol{p})\,\mathrm{d}s + \alpha \int_0^t (|\tilde{\boldsymbol{p}}|^2\tilde{\boldsymbol{p}}, \boldsymbol{p})\,\mathrm{d}s + \beta \int_0^t (\tilde{\boldsymbol{p}}, \boldsymbol{p})\,\mathrm{d}s + \int_0^t ((\tilde{\boldsymbol{u}}\cdot\nabla)\tilde{\boldsymbol{p}}, \boldsymbol{p})\,\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t ((\tilde{\boldsymbol{p}}\cdot\nabla)\boldsymbol{p} - (\boldsymbol{p}\cdot\nabla)\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{u}})\,\mathrm{d}s - \int_0^t (\mathscr{R}_2[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], \boldsymbol{p})\,\mathrm{d}s + \frac{1}{2}\|\tilde{\boldsymbol{p}}(t)\|_{\boldsymbol{L}^2}^2.
\end{aligned} \tag{32}$$

We observe that we can write the relative dissipation $\mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]$ as

$$\begin{aligned}
\int_0^t \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s =& \int_0^t \left(\mu_2\|\Delta \boldsymbol{p}\|_{\boldsymbol{L}^2}^2 + \gamma_2\|\nabla \boldsymbol{p}\|_{\boldsymbol{L}^2}^2 + \alpha\|\boldsymbol{p}\|_{\boldsymbol{L}^4}^4 + \beta\|\boldsymbol{p}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s \\
&+ \int_0^t \left(\mu_2\|\Delta \tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2 + \gamma_2\|\nabla \tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2 + \alpha\|\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^4}^4 + \beta\|\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s \\
&- \alpha \int_0^t \left(4(|\boldsymbol{p}|^2\boldsymbol{p}, \tilde{\boldsymbol{p}}) - 6(|\boldsymbol{p}|^2, |\tilde{\boldsymbol{p}}|^2) + 4(\boldsymbol{p}, |\tilde{\boldsymbol{p}}|^2\tilde{\boldsymbol{p}})\right)\mathrm{d}s \\
&- 2\int_0^t (\mu_2(\Delta \boldsymbol{p}, \Delta \tilde{\boldsymbol{p}}) + \gamma_2(\nabla \boldsymbol{p}, \nabla \tilde{\boldsymbol{p}}) + \beta(\boldsymbol{p}, \tilde{\boldsymbol{p}}))\,\mathrm{d}s \\
&+ \int_0^t \left(\gamma_2^-\|\nabla \boldsymbol{p} - \nabla \tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2 + \beta^-\|\boldsymbol{p} - \tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s.
\end{aligned}$$

Employing the energy inequality (20) and the shifted energy equality (24), we thus come up with

$$\begin{aligned}
\int_0^t \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s \leq& \frac{1}{2}\left(\|\boldsymbol{p}_0\|_{\boldsymbol{L}^2}^2 - \|\boldsymbol{p}(t)\|_{\boldsymbol{L}^2}^2 + \|\tilde{\boldsymbol{p}}(0)\|_{\boldsymbol{L}^2}^2 - \|\tilde{\boldsymbol{p}}(t)\|_{\boldsymbol{L}^2}^2\right) + \int_0^t (\mathscr{R}_2[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], \tilde{\boldsymbol{p}})\,\mathrm{d}s \\
&- \alpha \int_0^t \left(4(|\boldsymbol{p}|^2\boldsymbol{p}, \tilde{\boldsymbol{p}}) - 6(|\boldsymbol{p}|^2, |\tilde{\boldsymbol{p}}|^2) + 4(\boldsymbol{p}, |\tilde{\boldsymbol{p}}|^2\tilde{\boldsymbol{p}})\right)\mathrm{d}s \\
&- 2\int_0^t (\beta(\boldsymbol{p}, \tilde{\boldsymbol{p}}) + \gamma_2(\nabla \boldsymbol{p}, \nabla \tilde{\boldsymbol{p}}) + \mu_2(\Delta \boldsymbol{p}, \Delta \tilde{\boldsymbol{p}}))\,\mathrm{d}s \\
&+ \int_0^t \left(\gamma_2^-\|\nabla \boldsymbol{p} - \nabla \tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2 + \beta^-\|\boldsymbol{p} - \tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s.
\end{aligned} \tag{33}$$

Here, we have employed the binomial formula

$$|\boldsymbol{a} - \boldsymbol{b}|^4 = |\boldsymbol{a}|^4 - 4|\boldsymbol{a}|^2\boldsymbol{a}\cdot\boldsymbol{b} + 6|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - 4\boldsymbol{a}\cdot\boldsymbol{b}|\boldsymbol{b}|^2 + |\boldsymbol{b}|^4$$

for vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. Together with the identity

$$\begin{aligned} & -4(|\boldsymbol{p}|^2\boldsymbol{p}, \tilde{\boldsymbol{p}}) + 6(|\boldsymbol{p}|^2, |\tilde{\boldsymbol{p}}|^2) - 4(\boldsymbol{p}, |\tilde{\boldsymbol{p}}|^2\tilde{\boldsymbol{p}}) \\ = & -(|\boldsymbol{p}|^2\boldsymbol{p}, \tilde{\boldsymbol{p}}) - (|\tilde{\boldsymbol{p}}|^2\tilde{\boldsymbol{p}}, \boldsymbol{p}) + 3((|\boldsymbol{p}|^2 - |\tilde{\boldsymbol{p}}|^2)(\tilde{\boldsymbol{p}} - \boldsymbol{p}), \tilde{\boldsymbol{p}}) + 3(|\tilde{\boldsymbol{p}}|^2(\tilde{\boldsymbol{p}} - \boldsymbol{p}), \tilde{\boldsymbol{p}} - \boldsymbol{p}), \end{aligned}$$

adding (32) and (33) results in

$$\begin{aligned} & \mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}](t) + \int_0^t \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s \\ \le & \mathscr{E}[\boldsymbol{p}_0|\tilde{\boldsymbol{p}}(0)] + \int_0^t (\mathscr{R}_2[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], \tilde{\boldsymbol{p}} - \boldsymbol{p})\,\mathrm{d}s + \int_0^t ((\boldsymbol{u}\cdot\nabla)\boldsymbol{p}, \tilde{\boldsymbol{p}}) + ((\tilde{\boldsymbol{u}}\cdot\nabla)\tilde{\boldsymbol{p}}, \boldsymbol{p})\,\mathrm{d}s \\ & + \frac{1}{2}\int_0^t (((\boldsymbol{p}\cdot\nabla)\tilde{\boldsymbol{p}} - (\tilde{\boldsymbol{p}}\cdot\nabla)\boldsymbol{p}, \boldsymbol{u}) + ((\tilde{\boldsymbol{p}}\cdot\nabla)\boldsymbol{p} - (\boldsymbol{p}\cdot\nabla)\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{u}}))\,\mathrm{d}s \\ & + \lambda_2 \int_0^t (((\boldsymbol{p}\cdot\nabla)\boldsymbol{p}, \tilde{\boldsymbol{p}}) + ((\tilde{\boldsymbol{p}}\cdot\nabla)\tilde{\boldsymbol{p}}, \boldsymbol{p}))\,\mathrm{d}s \\ & + \int_0^t \left(\gamma_2^- \|\nabla\boldsymbol{p} - \nabla\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2 + \beta^- \|\boldsymbol{p} - \tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s \\ & + 3\alpha \int_0^t \left(((|\boldsymbol{p}|^2 - |\tilde{\boldsymbol{p}}|^2)(\boldsymbol{p} - \tilde{\boldsymbol{p}}), \tilde{\boldsymbol{p}}) + (|\tilde{\boldsymbol{p}}|^2(\tilde{\boldsymbol{p}} - \boldsymbol{p}), \tilde{\boldsymbol{p}} - \boldsymbol{p})\right)\mathrm{d}s \\ =: & \mathscr{E}[\boldsymbol{p}_0|\tilde{\boldsymbol{p}}(0)] + \int_0^t (\mathscr{R}_2[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], \tilde{\boldsymbol{p}} - \boldsymbol{p})\,\mathrm{d}s + I_1 + \frac{1}{2}I_2 + \lambda_2 I_3 + I_4 + 3\alpha I_5. \end{aligned} \tag{34}$$

It remains to estimate the integral expressions $I_1, \ldots, I_5$ against the integral over terms of the relative energy $\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]$ and the relative dissipation $\mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]$.

Since $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ are divergence-free, we observe that $(((\boldsymbol{u} - \tilde{\boldsymbol{u}})\cdot\nabla)\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{p}}) = 0$ and we obtain

$$\begin{aligned} |I_1| & = \left|\int_0^t (((\boldsymbol{u}\cdot\nabla)\boldsymbol{p}, \tilde{\boldsymbol{p}}) + ((\tilde{\boldsymbol{u}}\cdot\nabla)\tilde{\boldsymbol{p}}, \boldsymbol{p}) + (((\boldsymbol{u} - \tilde{\boldsymbol{u}})\cdot\nabla)\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{p}}))\,\mathrm{d}s\right| \\ & = \left|\int_0^t (((\boldsymbol{u} - \tilde{\boldsymbol{u}})\cdot\nabla)\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{p}} - \boldsymbol{p})\,\mathrm{d}s\right|. \end{aligned}$$

Applying Hölder and Young's inequalities as well as Lemma 3 then yields

$$\begin{aligned} |I_1| \le & \frac{\delta}{2\varepsilon C\left(1 + \|\tilde{\boldsymbol{p}}\|^2_{L^\infty\left(\boldsymbol{L}^{\frac{6}{5}}\right)}\right)} \int_0^t \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2}^2\,\mathrm{d}s \\ & + \int_0^t c_\delta \varepsilon C\left(1 + \|\tilde{\boldsymbol{p}}\|^2_{L^\infty\left(\boldsymbol{L}^{\frac{6}{5}}\right)}\right) \|\nabla\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^\infty}^2 \|\boldsymbol{p} - \tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2\,\mathrm{d}s \\ \le & \delta \int_0^t \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s + \int_0^t \mathscr{K}[\tilde{\boldsymbol{p}}]\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s + \frac{1}{2}\int_0^t (\mathscr{R}_1[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], A^{-1}(\tilde{\boldsymbol{u}} - \boldsymbol{u}))\,\mathrm{d}s. \end{aligned} \tag{35}$$

Here, $0 < \delta \le 1$ is fixed in such a way that the factor in front of the terms on the right-hand side of (25) that depend on $\mathscr{W}$ will add up to 1/2 so that they can be absorbed into the left-hand side. Hence, the constant c_δ that stems from Young's inequality can be incorporated into the definition of $\mathscr{K}$.

An integration-by-parts yields

$$\|\nabla\boldsymbol{p} - \nabla\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2} \le \|\boldsymbol{p} - \tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^{\frac{1}{2}} \|\Delta\boldsymbol{p} - \Delta\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^{\frac{1}{2}},$$

and hence, by Hölder's inequality, we get

$$\begin{aligned}\frac{1}{2}|I_2| &= \frac{1}{2}\left|\int_0^t \left((((\boldsymbol{p}-\tilde{\boldsymbol{p}})\cdot\nabla)\tilde{\boldsymbol{p}}, \boldsymbol{u}-\tilde{\boldsymbol{u}}) + ((\tilde{\boldsymbol{p}}\cdot\nabla)(\tilde{\boldsymbol{p}}-\boldsymbol{p}), \boldsymbol{u}-\tilde{\boldsymbol{u}})\right) \mathrm{d}s\right| \\ &\leq \frac{1}{2}\int_0^t \|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}\|\nabla\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^\infty}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2}\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \|\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^\infty}\|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^{\frac{1}{2}}\|\Delta\boldsymbol{p}-\Delta\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^{\frac{1}{2}}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2}\,\mathrm{d}s.\end{aligned}$$

As in (35), we can then use Young's inequality and Lemma 3 to find

$$\frac{1}{2}|I_2| \leq \delta\int_0^t \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s + \int_0^t \mathscr{K}[\tilde{\boldsymbol{p}}]\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s + \frac{1}{2}\int_0^t (\mathscr{R}_1[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], A^{-1}(\tilde{\boldsymbol{u}}-\boldsymbol{u}))\,\mathrm{d}s. \tag{36}$$

Since also $\boldsymbol{p}-\tilde{\boldsymbol{p}}$ is divergence-free, we find, similarly as for I_1, that

$$\begin{aligned}\lambda_2|I_3| &= \lambda_2\left|\int_0^t (((\boldsymbol{p}-\tilde{\boldsymbol{p}})\cdot\nabla)\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{p}}-\boldsymbol{p})\,\mathrm{d}s\right| \\ &\leq \lambda_2\int_0^t \|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}\|\nabla\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^\infty}\|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}\,\mathrm{d}s \\ &\leq \int_0^t \mathscr{K}[\tilde{\boldsymbol{p}}]\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s + \delta\int_0^t \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s.\end{aligned} \tag{37}$$

Performing an integration-by-parts and using Hölder's inequality as well as Young's inequality give

$$\begin{aligned}|I_4| &\leq \int_0^t \left(\gamma_2^-\|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}\|\Delta\boldsymbol{p}-\Delta\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2} + \beta^-\|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s \\ &\leq \int_0^t c_\delta\|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2\,\mathrm{d}s + \delta\int_0^t \mu_2\|\Delta\boldsymbol{p}-\Delta\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2\,\mathrm{d}s \\ &\leq \int_0^t \mathscr{K}[\tilde{\boldsymbol{p}}]\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s + \delta\int_0^t \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s.\end{aligned} \tag{38}$$

The estimate

$$\begin{aligned}|\boldsymbol{p}|^2 - |\tilde{\boldsymbol{p}}|^2 &= 2(|\boldsymbol{p}| - |\tilde{\boldsymbol{p}}|)|\tilde{\boldsymbol{p}}| + (|\tilde{\boldsymbol{p}}| - |\boldsymbol{p}|)^2 \\ &\leq 2|\boldsymbol{p}-\tilde{\boldsymbol{p}}||\tilde{\boldsymbol{p}}| + |\tilde{\boldsymbol{p}}-\boldsymbol{p}|^2\end{aligned}$$

together with Hölder's inequality then gives

$$3\alpha|I_5| \leq 3\alpha\int_0^t \left(3\|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^2\|\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^\infty}^2 + \|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^4}^3\|\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^4}\right)\mathrm{d}s.$$

By a well-known interpolation inequality for $\boldsymbol{L}^p(\Omega)$ spaces and the Sobolev embedding $\boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$, we can further estimate

$$\|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^4}^3\|\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^4} \leq c\|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^4}^2\|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^{\frac{1}{2}}\|\Delta\boldsymbol{p}-\Delta\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^2}^{\frac{1}{2}}\|\tilde{\boldsymbol{p}}\|_{\boldsymbol{L}^4}.$$

With Young's inequality, we see that

$$3\alpha|I_5| \leq \int_0^t \mathscr{K}[\tilde{\boldsymbol{p}}]\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s + \delta\int_0^t \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s. \tag{39}$$

Applying (35)–(39) to (34) with an appropriate choice of δ finally yields

$$\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}](t) + \int_0^t \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s \le \mathscr{E}[\boldsymbol{p}_0|\tilde{\boldsymbol{p}}(0)] + \frac{1}{2}\int_0^t \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s + \int_0^t \mathscr{K}[\tilde{\boldsymbol{p}}]\mathscr{E}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}s + \int_0^t \left(\mathscr{R}[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}], \begin{pmatrix} A^{-1}(\tilde{\boldsymbol{u}} - \boldsymbol{u}) \\ \tilde{\boldsymbol{p}} - \boldsymbol{p} \end{pmatrix}\right) \mathrm{d}s,$$

which is the assertion. □

With the relative energy inequality available, an immediate consequence is then the weak–strong uniqueness of the weak solutions constructed in Section 3, which by Lemma 1 fulfill the necessary energy inequality. First, we make our understanding of strong solutions precise.

Definition 2 (Strong solution)**.** Let $\tilde{\boldsymbol{p}}_0 \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$. A pair $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{p}}) \in L^2(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}) \times W^{1,2}(\boldsymbol{H}^4 \cap \boldsymbol{H}^1_{0,\sigma})$ is called a strong solution to (1) if Equations 3 and (4) hold and $\tilde{\boldsymbol{p}}(0) = \tilde{\boldsymbol{p}}_0$ almost everywhere in Ω.

It is then clear that strong solutions lie in the class of possible test functions for the relative energy inequality from which the weak–strong uniqueness follows.

Corollary 1 (Weak–strong uniqueness)**.** *Let $(\boldsymbol{u}, \boldsymbol{p})$ be a weak solution to (1) in the sense of Definition 1 that additionally fulfills the energy inequality (20) and let $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{p}})$ be a strong solution to (1) in the sense of Definition 2, starting from the same initial datum $\tilde{\boldsymbol{p}}_0 \in \boldsymbol{H}^4(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$. Then $(\boldsymbol{u}, \boldsymbol{p})$ is unique and coincides with $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{p}})$.*

Proof. Since for any strong solution $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{p}})$ the expression $\mathscr{R}[\tilde{\boldsymbol{u}}|\tilde{\boldsymbol{p}}]$ vanishes, dropping the (positive) term $\frac{1}{2}\int_0^t \mathscr{W}[\boldsymbol{p}|\tilde{\boldsymbol{p}}]\,\mathrm{d}t$ on the left-hand side of (25), and an application of Gronwall's lemma yields the result. □

5 | RELATION TO A PHENOMENOLOGICAL MODEL

The numerical experiments carried out in Reinken et al[5] suggest that, for a very small parameter ε, the dynamics of the active fluid is dominated by the movement of the microswimmers, and the influence of the velocity field of the suspension fluid can be neglected. Hence, the behavior of the fluid as a whole is described by only the vector field $\boldsymbol{p}$. We see this reflected in the governing equations via the following formal calculations. Setting $\varepsilon = 0$ and thus $\mu_1 = \gamma_1 = \lambda_1 = 0$ in (1a) leaves us with the decoupled system

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla \pi_1 &= 0, \\ \partial_t \boldsymbol{p} + \mu_2 \Delta^2 \boldsymbol{p} - \gamma_2 \Delta \boldsymbol{p} + \lambda_2 (\boldsymbol{p} \cdot \nabla)\boldsymbol{p} + \alpha |\boldsymbol{p}|^2 \boldsymbol{p} + \beta \boldsymbol{p} & \\ + (\boldsymbol{u} \cdot \nabla)\boldsymbol{p} + \kappa (\nabla \boldsymbol{u})_{\mathrm{sym}} \boldsymbol{p} - (\nabla \boldsymbol{u})_{\mathrm{skw}} \boldsymbol{p} + \nabla \pi_2 &= 0, \\ \nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{p} &= 0, \end{aligned}$$

together with (1b). Inserting the trivial solution $\boldsymbol{u} = 0$ and π_1 constant reduces this to

$$\partial_t \boldsymbol{p} + \mu_2 \Delta^2 \boldsymbol{p} - \gamma_2 \Delta \boldsymbol{p} + \lambda_2 (\boldsymbol{p} \cdot \nabla)\boldsymbol{p} + \alpha |\boldsymbol{p}|^2 \boldsymbol{p} + \beta \boldsymbol{p} + \nabla \pi_2 = 0 \tag{40}$$

with (1b). The equations above coincide with the ones in a phenomenological model that was proposed in Wensink et al.[4] The existence of strong solutions to a generalization of these equations on the whole space was shown in Zanger et al[20] and can be adapted to the case of a bounded domain. The goal of this section is to make this relation rigorous by showing that weak solutions to (1) constructed in Section 3 converge to strong solutions to (40) with (1b) as $\varepsilon \to 0$ by employing the relative energy inequality (25) from Theorem 2.

Theorem 3. *Let $(\boldsymbol{p}_{0,\varepsilon})_{\varepsilon>0} \subset \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$ be a family of initial values, and for $\varepsilon > 0$, let $(\boldsymbol{u}_\varepsilon, \boldsymbol{p}_\varepsilon)$ be a weak solution to (1), in the sense of Definition 1, starting from the initial datum $\boldsymbol{p}_{0,\varepsilon} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$ that fulfills the energy inequality (20). Furthermore, let $\tilde{\boldsymbol{p}} \in W^{1,2}(\boldsymbol{H}^4 \cap \boldsymbol{H}^1_{0,\sigma})$ be a strong solution to (40) together with (1b) starting from the initial datum*

$\tilde{\boldsymbol{p}}_0 \in \boldsymbol{H}^4(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega)$. If $\boldsymbol{p}_{0,\varepsilon} \to \tilde{\boldsymbol{p}}_0$ in $\boldsymbol{L}^2_\sigma(\Omega)$, then for the corresponding family of weak solutions $(\boldsymbol{u}_\varepsilon, \boldsymbol{p}_\varepsilon)_{\varepsilon>0}$, there holds

$$(\boldsymbol{u}_\varepsilon, \boldsymbol{p}_\varepsilon) \to (0, \tilde{\boldsymbol{p}}) \quad in \quad L^2(\boldsymbol{L}^2_\sigma) \times L^\infty(\boldsymbol{L}^2) \cap L^4(\boldsymbol{L}^4) \cap L^2(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma})$$

as $\varepsilon \to 0$.

Proof. First, we denote by $\mathcal{K}_\varepsilon[\tilde{\boldsymbol{p}}] = \mathcal{K}[\tilde{\boldsymbol{p}}]$ the expression defined in (23), depending on ε via $(\boldsymbol{u}_\varepsilon, \boldsymbol{p}_\varepsilon)$. The pair $(0, \tilde{\boldsymbol{p}})$ then is admissible to test with in the relative energy inequality (25). An application of Gronwall's lemma yields

$$\begin{aligned} &\mathcal{E}[\boldsymbol{p}_\varepsilon|\tilde{\boldsymbol{p}}](t) + \frac{1}{2}\int_0^t \mathcal{W}[\boldsymbol{p}_\varepsilon|\tilde{\boldsymbol{p}}] \exp\left(\int_s^t \mathcal{K}_\varepsilon[\tilde{\boldsymbol{p}}]\mathrm{d}\tau\right)\mathrm{d}s \\ \leq &\mathcal{E}[\boldsymbol{p}_{0,\varepsilon}|\tilde{\boldsymbol{p}}_0] \exp\left(\int_0^t \mathcal{K}_\varepsilon[\tilde{\boldsymbol{p}}]\mathrm{d}s\right) + \int_0^t \left(\mathcal{R}[0|\tilde{\boldsymbol{p}}], \begin{pmatrix} A^{-1}(-\boldsymbol{u}_\varepsilon) \\ \tilde{\boldsymbol{p}} - \boldsymbol{p}_\varepsilon \end{pmatrix}\right) \exp\left(\int_s^t \mathcal{K}_\varepsilon[\tilde{\boldsymbol{p}}]\mathrm{d}\tau\right)\mathrm{d}s \end{aligned} \tag{41}$$

for almost all $t \in (0, T)$. Since $\mathcal{K}_\varepsilon[\tilde{\boldsymbol{p}}] \geq 0$, it suffices to show that the right-hand side of this inequality vanishes to obtain the desired convergences. First, we note that $\int_s^t \mathcal{K}_\varepsilon[\tilde{\boldsymbol{p}}]\mathrm{d}\tau$ is uniformly bounded in ε (in fact vanishes as $\varepsilon \to 0$) since ε only appears as the leading constant in $\mathcal{K}_\varepsilon[\tilde{\boldsymbol{p}}]$. Hence, with the convergence $\boldsymbol{p}_{0,\varepsilon} \to \tilde{\boldsymbol{p}}_0$ in $\boldsymbol{L}^2(\Omega)$, the first term on the right-hand side of (41) vanishes as $\varepsilon \to 0$. Furthermore, since $\tilde{\boldsymbol{p}}$ is a strong solution to (40) and therefore $\mathcal{R}_2[0|\tilde{\boldsymbol{p}}]$ vanishes, it only remains to show that

$$\int_0^t (\mathcal{R}_1[0|\tilde{\boldsymbol{p}}], A^{-1}\boldsymbol{u}_\varepsilon)\mathrm{d}s = \varepsilon \int_0^t (\tilde{\mu}_2\Delta^2\tilde{\boldsymbol{p}} - \tilde{\gamma}_1\Delta\tilde{\boldsymbol{p}} + \tilde{\lambda}_1(\tilde{\boldsymbol{p}}\cdot\nabla)\tilde{\boldsymbol{p}}, A^{-1}\boldsymbol{u}_\varepsilon)\mathrm{d}s$$

tends to zero as $\varepsilon \to 0$. Using the continuity of $A^{-1} : \boldsymbol{L}^2_\sigma(\Omega) \to \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{0,\sigma}(\Omega) \hookrightarrow \boldsymbol{L}^2_\sigma(\Omega)$ and estimate (22), we find

$$\|A^{-1}\boldsymbol{u}_\varepsilon\|_{L^2(\boldsymbol{L}^2)} \leq \varepsilon c(\|\boldsymbol{p}_{0,\varepsilon}\|_{\boldsymbol{L}^2} + 1).$$

Together with the Cauchy–Schwarz inequality and (2), we finally obtain

$$\begin{aligned} \int_0^t (\mathcal{R}_1[0|\tilde{\boldsymbol{p}}], A^{-1}\boldsymbol{u}_\varepsilon)\mathrm{d}s \leq &\varepsilon\|\tilde{\mu}_2\Delta^2\tilde{\boldsymbol{p}} - \tilde{\gamma}_1\Delta\tilde{\boldsymbol{p}} + \tilde{\lambda}_1(\tilde{\boldsymbol{p}}\cdot\nabla)\tilde{\boldsymbol{p}}\|_{L^2(\boldsymbol{L}^2)}\|A^{-1}\boldsymbol{u}_\varepsilon\|_{L^2(\boldsymbol{L}^2)} \\ \leq &\varepsilon^2 c\|\tilde{\mu}_2\Delta^2\tilde{\boldsymbol{p}} - \tilde{\gamma}_1\Delta\tilde{\boldsymbol{p}} + \tilde{\lambda}_1(\tilde{\boldsymbol{p}}\cdot\nabla)\tilde{\boldsymbol{p}}\|_{L^2(\boldsymbol{L}^2)}(\|\boldsymbol{p}_{0,\varepsilon}\|_{\boldsymbol{L}^2} + 1), \end{aligned}$$

which in turn yields the statement. □

ACKNOWLEDGEMENT

This work has been supported by Deutsche Forschungsgemeinschaft through Collaborative Research Center 910 "Control of self-organizing nonlinear systems: Theoretical methods and concepts of application."

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

ORCID

Etienne Emmrich https://orcid.org/0000-0001-9869-0334

REFERENCES

1. Vicsek T, Czirók A, Ben-Jacob E, Cohen I, Shochet O. Novel type of phase transition in a system of self-driven particles. *Phys Rev Lett*. 1995;75(6):1226-1229.
2. Toner J, Tu Y. Flocks, herds, and schools. A Quantitative Theory of Flocking. *Phys Rev E*. 1998;58:4828-4858.
3. Thampi SP, Yeomans JM. Active turbulence in active nematics. *Eur Phys J Spec Top*. 2016;225(4):651-662.

4. Wensink HH, Dunkel J, Heidenreich S, et al. Meso-scale turbulence in living fluids. *Proc Natl Acad Sci*. 2012;109(36):14308-14313.
5. Reinken H, Klapp SHL, Bär M, Heidenreich S. Derivation of a hydrodynamic theory for mesoscale dynamics in microswimmer suspensions. *Phys Rev E*. 2018;97(2):22613-22631.
6. Feireisl E. Relative entropies in thermodynamics of complete fluid systems. *Discrete Contin Dyn Syst*. 2012;32(9):3059-3080.
7. Fischer J. A posteriori modeling error estimates for the assumption of perfect incompressibility in the Navier–Stokes equation. *SIAM J Numer Anal*. 2015;53(5):2178-2205.
8. Lasarzik R. Dissipative solution to the Ericksen–Leslie system equipped with the Oseen–Frank energy. *Zeitschrift für angewandte Mathematik und Physik*. 2019;70(1):8.
9. Boyer F, Fabrie P. *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*. New York: Springer; 2013.
10. Dunkel J, Heidenreich S, Drescher K, Wensink HH, Bär M, Goldstein RE. Fluid dynamics of bacterial turbulence. *Phys Rev Lett*. 2013;110(22):228102-228107.
11. Temam R. *Navier–Stokes Equations. Theory and Numerical Analysis*. North-Holland, Amsterdam-New York; 1979.
12. Temam R. *Navier-Stokes Equations and Nonlinear Functional Analysis, Vol. 66 of CBMS-NSF Regional Conference Series in Applied Mathematics*. 2nd ed. Philadelphia: SIAM; 1995.
13. Hale RK. *Ordinary Differential Equations*. New York: Wiley-Interscience; 1960.
14. Málek J, Nečas J, Rokyta M, Růžička M. *Weak and Measure-Valued Solutions to Evolutionary PDEs*. London: Chapman & Hall; 1996.
15. Lions JL. *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires. Etudes Mathématiques*, Gauthier–Villars, 1969.
16. Lions J-L, Magenes E. *Non-Homogeneous Boundary Value Problems and Applications*. Vol. I. New York-Heidelberg: Springer; 1972.
17. Leray J. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math*. 1934;63(0):193-248.
18. Prodi G. Un teorema di unicità per le equazioni di Navier–Stokes. *Annali di Matematica Pura ed Applicata*. 1959;48(1):173-182.
19. Serrin J. The initial value problem for the Navier–Stokes equations. *Nonlin Probl*. 1963;21:69-98.
20. Zanger F, Löwen H, Saal J. Analysis of a living fluid continuum model. In: Maekawa Y, Jimbo S, eds. *Mathematics for Nonlinear Phenomena. Analysis and Computation*. Berlin: Springer; 2015:285-303.

How to cite this article: Emmrich E, Geuter L. Analysis of a model for the dynamics of microswimmer suspensions. *Math Meth Appl Sci*. 2021;1-18. https://doi.org/10.1002/mma.7673