Error Analysis for the Second Order BDF Discretization of the Incompressible Navier-Stokes Problem

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Abstract An overview of some recent results for the temporal discretization of the incompressible Navier-Stokes problem by means of the two-step backward differentiation formula is given. The original nonlinear approximation as well as a variant based upon a linearization of the convective term are considered. After studying solvability and stability, convergence of a piecewise polynomial approximate solution towards a weak solution is shown. Furthermore, smoothing error estimates –under realistic assumptions on the problem's data– are presented for the velocity as well as the pressure.

Keywords Incompressible Navier-Stokes equation, time discretization, backward differentiation formula, stability, convergence, error estimate

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1. Introduction We consider the Navier-Stokes equations describing the non-stationary flow of an incompressible, homogeneous, viscous fluid with constant temperature,

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f\,, \quad \nabla\cdot u = 0 \quad \text{in } \Omega\times(0,T)\,,$$
$$u = 0 \quad \text{on } \partial\Omega\times(0,T)\,, \quad u(\cdot,0) = u_0 \quad \text{in } \Omega\,,$$

where $\Omega\subset\mathbb{R}^d$, $d=\dim\Omega\in\{2,3\}$, is a bounded domain with the locally Lipschitz continuous boundary $\partial\Omega$, $T>0$ is the time under consideration, $\nu = 1/\mathrm{Re} > 0$ denotes the inverse of the Reynolds number, $u=u(x,t)$ is the d-dimensional velocity vector with the prescribed initial velocity $u_0 = u_0(x)$, $p = p(x,t)$ is the pressure, and $f = f(x,t)$ is an outer force per unit mass.

We introduce the solenoidal function spaces

$$\mathcal{V} := \{v\in\mathcal{C}_0^\infty(\Omega)^d : \nabla\cdot v = 0\}\,,\ V := \mathrm{clo}_{\|\cdot\|_{1,2}}\,\mathcal{V} = \{v\in H_0^1(\Omega)^d : \nabla\cdot v = 0\}\,,$$
$$H := \mathrm{clo}_{\|\cdot\|_{0,2}}\,\mathcal{V} = \{v\in L^2(\Omega)^d : \nabla\cdot v = 0\,,\ \gamma_n v = 0\}\,,$$

where γ_n denotes the trace operator in normal direction. Furthermore, by L^p and $W^{m,p}$ ($W^{m,2}\equiv H^m$), we denote the usual Lebesgue and Sobolev spaces with the usual norms $\|\cdot\|_{0,p}$ and $\|\cdot\|_{m,p}$, respectively. With

$$((u,v)) := \sum_{i,j=1}^d \int_\Omega \frac{\partial u_i(x)}{\partial x_j}\frac{\partial v_i(x)}{\partial x_j}dx\,, \quad \|u\| := ((u,u))^{1/2}\,, \quad u,v\in V\,,$$

$$(u,v) := \sum_{i=1}^{d} \int_\Omega u_i(x) v_i(x) dx\,, \quad |u| := (u,u)^{1/2}\,, \quad u,v \in H\,,$$

the spaces V and H are Hilbert spaces. The space V is dense and continuously embedded in H. It holds the Poincaré-Friedrichs inequality

$$\exists \alpha > 0\, \forall v \in V: \quad |v| \le \alpha \,\|v\|\,.$$

Note that V, H, and the dual V^* form a Gelfand triple. The dual pairing between V and V^* is denoted by $\langle \cdot, \cdot \rangle$, the dual norm by $\|\cdot\|_*$.

We then consider the weak formulation of the Navier-Stokes problem:

Problem (P) *For given $u_0 \in H$ and $f \in L^2(0,T;V^*)$, find $u \in L^2(0,T;V)$ such that for all $v \in V$ and almost everywhere in $(0,T)$*

$$\frac{d}{dt}(u(t),v) + \nu\,(\!(u(t),v)\!) + b(u(t),u(t),v) = \langle f(t), v\rangle \tag{1}$$

holds with $u(0) = u_0$.

Here, $b(u,v,w) := ((u\cdot\nabla)v, w)$ describes the nonlinearity. By $L^p(0,T;X)$, we denote the usual space of Bochner integrable abstract functions $u: [0,T] \to X$, where X is some Banach space. The discrete counterparts are denoted by $l^p(0,T;X)$.

For given $N \in \mathbb{N}$, let $\Delta t = T/N$, $t_n = n\Delta t$ $(n = 0,1,\ldots,N)$. We consider the time discretization of Problem (P) by means of the formally second order two-step backward differentiation formula:

Problem $(P_{\Delta t})$ *For given u^0, $u^1 \in H$ and $f \in L^2(0,T;V^*)$, find $u^n \in V$ $(n = 2, 3, \ldots, N)$ such that for all $v \in V$*

$$(D_2 u^n, v) + \nu(\!(u^n, v)\!) + b(u^n, u^n, v) = \langle R_2^n f, v\rangle\,. \tag{2}$$

By

$$D_2 u^n := \frac{1}{\Delta t}\left(\frac{3}{2}u^n - 2u^{n-1} + \frac{1}{2}u^{n-2}\right),$$

we denote the divided backward difference that satisfies $D_2 u(t_n) = u'(t_n) + \mathcal{O}((\Delta t)^2)$ for smooth u. We also introduce

$$R_2^n f := \frac{3}{2\Delta t}\int_{t_{n-1}}^{t_n} f(t)dt - \frac{1}{2\Delta t}\int_{t_{n-2}}^{t_{n-1}} f(t)dt\,.$$

Note that $R_2^n u' = D_2 u(t_n)$. Besides, we consider the following linearized variant of Problem $(P_{\Delta t})$:

Problem $(LP_{\Delta t})$ *For given* u^0, $u^1 \in V$ *and* $f \in L^2(0,T;V^*)$, *find* $u^n \in V$ *(*$n = 2, 3, \ldots, N$*) such that for all* $v \in V$

$$(D_2 u^n, v) + \nu((u^n, v)) + b(Eu^n, u^n, v) = \langle R_2^n f, v\rangle . \tag{3}$$

Here, $Eu^n := 2u^{n-1} - u^{n-2}$ is an extrapolation satisfying $Eu(t_n) = u(t_n) + \mathcal{O}((\Delta t)^2)$ for smooth u. The starting values can be obtained by $u^0 := u_0$ and computing u^1 from u^0 using the implicit Euler method.

Solvability and the velocity error $e^n := u(t_n) - u^n$ have been firstly studied in [2] for Problem $(LP_{\Delta t})$. However, the optimal second order estimate for the $l^\infty(0,T;H)$- and $l^2(0,T;V)$-norm of e^n given there relies upon higher regularity of the exact solution. As it was pointed out in [3] and [8], higher regularity is equivalent to compatibility conditions on the problem's data. In view of the divergence-free constraint, these conditions become global and, therefore, virtually uncheckable and hardly fulfillable. A more realistic estimate can be found in [5] where the sub-optimal order 1/4 in the $l^\infty(0,T;V)$-norm has been proven for the two-dimensional case with autonomous right hand side f.

Yet, the original Problem $(P_{\Delta t})$ has not been considered in the literature. For both the Problem $(P_{\Delta t})$ and $(LP_{\Delta t})$, error estimates under suitable assumptions on the data, convergence of an approximate solution towards a weak solution, and the quantification of appearing constants are the questions to be answered.

As higher regularity assumptions are improper but the Navier-Stokes operator possesses a smoothing property, we shall look for so-called smoothing or rough data error estimates. For the Navier-Stokes problem, such estimates are known from [4] for the Crank-Nicolson scheme, from [6] for the fractional-step θ-scheme, and from [7] for projection schemes.

For the exact solution to Problem (P), the following results are rather known from the literature (cf. [9], [4]):

Theorem 1 *Let* $\partial\Omega$ *be smooth,* $u_0 \in \mathcal{D}(A) := V \cap H^2(\Omega)^d$, *and*

$$f, tf', t^2 f'' \in L^2(0,T;V), \quad f', tf'' \in L^2(0,T;V^*).$$

Then there is (for $\dim\Omega = 3$ *only up to a time* $T^* \le T$*) a unique* u *with*

$$u \in \mathcal{C}([0,T];\mathcal{D}(A)),\ u' \in \mathcal{C}([0,T];H) \cap L^2(0,T;V),$$
$$\sqrt{t}\, u' \in \mathcal{C}((0,T];V) \cap L^\infty(0,T;V),$$
$$u'' \in L^2(0,T;V^*),\ tu'' \in L^2(0,T;V),$$
$$t(f'' - u''') \in L^2(0,T;V^*),\ t^{3/2}(f'' - u''') \in L^2(0,T;H)\ .$$

2. Solvability and stability The existence of a solution to $(P_{\Delta t})$ can be proved applying the main theorem on pseudomonotone operators by Brézis. For this, we observe that the nonlinearity is a strongly continuous operator from V into V^*. In the case of $(LP_{\Delta t})$, we may use the Lax-Milgram lemma. The stability results from energy type estimates. Note the identity

$$4\Delta t \sum_{j=2}^{n} (D_2 v^j, v^j) = |v^n|^2 + |E v^{n+1}|^2 + (\Delta t)^4 \sum_{j=1}^{n-1} |D^2 v^j|^2 - |v^1|^2 - |E v^2|^2 \,,$$

where $D^2 u^n := (u^{n+1} - 2u^n + u^{n-1})/(\Delta t)^2$ is the second divided difference.

Theorem 2 *There is at least one solution to Problem $(P_{\Delta t})$ and there exists a unique solution to Problem $(LP_{\Delta t})$. For both, the following stability estimates hold true ($n = 2, 3, \ldots, N$):*

$$|u^n|^2 + |E u^{n+1}|^2 + (\Delta t)^4 \sum_{j=1}^{n-1} |D^2 u^j|^2 + 2\nu \Delta t \sum_{j=2}^{n} \|u^j\|^2 \le C \,,$$

$$\Delta t \sum_{j=2}^{N} \|D_2 u^j\|_*^p \le C \,, \quad p = \begin{cases} 2 & \text{if } \dim \Omega = 2 \\ 4/3 & \text{if } \dim \Omega = 3 \end{cases} .$$

If, in addition, $u^0, u^1 \in V$ and $f \in L^1(0,T;H)$ then

$$(\Delta t)^q \sum_{j=2}^{n} |D_2 u^j|^2 \le C \,, \quad q = \begin{cases} 2 & \text{if } \dim \Omega = 2 \\ 9/4 & \text{if } \dim \Omega = 3 \end{cases} .$$

Moreover, if $\dim \Omega = 2$ and the data are sufficiently small then $q = 1$.

Here, C denotes a generic constant depending on ν, T, norms of u^0, u^1, and f, as well as on embedding constants.

The boundedness of $\Delta t \sum_j |D_2 u^j|^2$ in the two-dimensional case relies upon a solution of a nonlinear difference inequality with a quadratic term. However, in the three-dimensional case, we would have to consider a difference inequality with a cubic term (analogous to a differential inequality that appears in the proof of the local existence of a strong solution) which we cannot resolve.

We shall remark that a solution to Problem $(P_{\Delta t})$ is unique for small data. In the two-dimensional case, it is possible to show further stability results.

3. Convergence From the discrete values u^n $(n = 0, 1, \ldots, N)$, computed by solving $(P_{\Delta t})$ or $(LP_{\Delta t})$, we construct piecewise polynomial functions $U_{\Delta t}$, $V_{\Delta t}$, defined on $[0, T]$: For $t \in (t_{n-1}, t_n]$ $(n = 1, \ldots, N)$, let

$$U_{\Delta t}(t) = u^n\,, \quad V_{\Delta t}(t) = \begin{cases} \frac{1}{2}\left(u^n + Eu^{n+1}\right) + D_2 u^n\,(t - t_n) & \text{if } t > t_1 \\ \frac{1}{2}\left(u^1 + Eu^2\right) + \frac{u^1 - u^0}{\Delta t}\,(t - t_1) & \text{if } t \in [0, t_1] \end{cases}.$$

There are other possible constructions we will not consider here. The construction of $V_{\Delta t}$ reflects the choice of the method: The value u^1 is thought to be computed by the implicit Euler method. The slope of $V_{\Delta t}$ in $(t_{n-1}, t_n]$ is $D_2 u^n$ for $n = 2, 3, \ldots, N$.

Proposition 1 *Let $u^0 \in V$ be given, u^1 be computed by the implicit Euler method, and u^n $(n = 2, 3, \ldots, N)$ be the solution to $(P_{\Delta t})$ and $(LP_{\Delta t})$, respectively. Then for any sequence of step sizes $\{\Delta t\}$, there exist subsequences $\{U_{\Delta t'}\}$ and $\{V_{\Delta t'}\}$ that are weakly convergent in $L^2(0, T; V)$ and weakly-$*$ convergent in $L^\infty(0, T; H)$. Moreover, $\{V_{\Delta t'}\}$ is strongly convergent in $L^q(0, T; H)$, $q \in [2, \infty)$. If $\{\Delta t\}$ is a null sequence and $f \in L^1(0, T; H)$ then $\{U_{\Delta t'}\}$ is strongly convergent in $L^q(0, T; H)$, $q \in [2, \infty)$, with the same limit as $\{V_{\Delta t'}\}$.*

Proof The weak convergence in $L^2(0, T; V)$ and the weak-$*$ convergence in $L^\infty(0, T; H)$ follow from usual compactness arguments because of the boundedness of $\{U_{\Delta t}\}, \{V_{\Delta t}\}$, which is a direct consequence of Thm. 2.

Due to the boundedness of $\{V_{\Delta t}\}$ in $L^2(0, T; V)$ and of the derivatives $\{V'_{\Delta t}\}$ in $L^{4/3}(0, T; V^*)$, the strong convergence in $L^2(0, T; H)$ follows from a theorem by Lions and Aubin. Hence, the convergence is strong in $L^q(0, T; H)$ for any $q \in [2, \infty)$. Finally, we observe that

$$\|U_{\Delta t} - V_{\Delta t}\|^2_{L^2(0,T;H)} \le$$
$$\frac{\Delta t}{12}|u^1 - u^0|^2 + \frac{(\Delta t)^3}{6}\sum_{j=2}^{N}|D_2 u^j|^2 + \frac{(\Delta t)^5}{8}\sum_{j=1}^{N-1}|D^2 u^j|^2\,,$$

which shows, in view of Thm. 2, the strong convergence of $\{U_{\Delta t}\}$ in $L^2(0, T; H)$ if $\{V_{\Delta t}\}$ converges. Note that $|u^1 - u^0|$ is bounded (independently on Δt) if u^1 is computed by the implicit Euler method. #

Theorem 3 *Let $u^0 = u_0$. The common limit of the subsequences $\{U_{\Delta t'}\}$ and $\{V_{\Delta t'}\}$ from Prop. 1 is a weak solution to (P). If (P) possesses a unique solution then the whole sequences $\{U_{\Delta t}\}, \{V_{\Delta t}\}$ converge to it.*

The proof of the first part of the assertion follows from rewriting the scheme $(P_{\Delta t})$ and $(LP_{\Delta t})$, respectively, in terms of $U_{\Delta t}$, $V_{\Delta t}$, and studying the limit of the appearing terms. The proof of the second part then is clear.

4. Error estimates We commence with the error equation

$$\begin{aligned}(D_2 e^n, v) + \nu\,((e^n, v)) + b(u(t_n), e^n, v) + b(e^n, u(t_n), v) \\ -b(e^n, e^n, v) = \langle \rho^n, v\rangle \end{aligned} \tag{4}$$

corresponding to $(P_{\Delta t})$, where $\rho^n = D_2 u(t_n) - u'(t_n) + f(t_n) - R_2^n f$ is the consistency error of the linear Stokes problem. With standard arguments, it follows (with the notation $\tilde{a}^n := t_n a^n$)

Proposition 2 *Let $t(f'' - u''') \in L^2(0,T;V^*)$. Then there is a constant c (independent on the data) such that for $n = 2, 3, \ldots, N$*

$$\Delta t \sum_{j=2}^{n} \|t_j^q \rho^j\|_*^2 \le c\,(\Delta t)^{2(1+q)}\, \|t(f'' - u''')\|^2_{L^2(0,t_n;V^*)}, \quad q \in \{0,1\}.$$

We are now in the position to prove first order error estimates under suitable assumptions on the problem's data.

Theorem 4 *Let $u \in \mathcal{C}([0,T];\mathcal{D}(A))$, $t(f'' - u''') \in L^2(0,T;V^*)$, and let Δt or the data be sufficiently small such that*

$$a := 1 - c\nu^{-1/3}\Delta t \max_t \|u(t)\|_{2,2}^{4/3} > 0\,, \tag{5}$$

where c depends on embedding constants, only. Then the error e^n ($n = 2, 3, \ldots, N$) in the solution of $(P_{\Delta t})$ can be bounded by

$$\begin{aligned}|e^n|^2 + |Ee^{n+1}|^2 + (\Delta t)^4 \sum_{j=1}^{n-1} |D^2 e^j|^2 + 2\nu\Delta t \sum_{j=2}^{n} \|e^j\|^2 \\ \le a^{1-n}\, \left(|e^1|^2 + |Ee^2|^2 + \nu^{-1}(\Delta t)^2\, \|t(f'' - u''')\|^2_{L^2(0,t_n;V^*)}\right).\end{aligned}$$

Smoothing estimates for the time-weighted error $\tilde{e}^n$ can be derived after multiplying (4) by t_n and testing with $\tilde{e}^n$. However, this leads to the additional term $\Delta t \sum_j \|Ee^j\|_*^2$ in the error bound. In order to find estimates for this term, we consider an auxiliary problem that can be interpreted as the discrete dual to a linearization of Problem $(P_{\Delta t})$. Here, A denotes the Stokes operator.

Problem $(P^*_{\Delta t})$ *For given* $\phi^{n+1} = \phi^n = 0$ *and* $g^j := A^{-1}e^j \in V$*, find* $\phi^j \in V$ $(j = n-1, \ldots, 0)$ *such that for all* $w \in V$

$$(w, D_2^*\phi^j) + \nu\left((w, \phi^j)\right) + b(u(t_j), w, \phi^j) + b(w, u(t_j), \phi^j) = (w, g^j)\,,$$

where $D_2^*\phi^j := \left(3w^j - 4w^{j+1} + w^{j+2}\right)/(2\Delta t)$.

The most difficult part in proving higher order smoothing error estimates consists in deriving optimal stability estimates for Problem $(P^*_{\Delta t})$ in higher norms. We shall omit this here and refer to [1]. After all, we can show

Theorem 5 *Let* $u \in \mathcal{C}([0,T];\mathcal{D}(A))$*,* $u' \in L^2(0,T;V)$*,* $u'', t(f''-u''') \in L^2(0,T;V^*)$*, and* $\sqrt{t}\,u' \in \mathcal{C}((0,T];V) \cap L^\infty(0,T;V)$*. Let, furthermore,* Δt *or the data be sufficiently small such that (5) and*

$$1 - c\nu^{-1}\left(\Delta t \max_t \|u(t)\|\,\|u(t)\|_{2,2} + \sqrt{\Delta t}\max_t \|\sqrt{t}u'(t)\|\right) > 0$$

are fulfilled and assume that $|A^{-s}e^{0,1}| = \mathcal{O}((\Delta t)^{1+s})$ $(s \in \{0, 1/2, 1\})$*. For the time-weighted error* $\tilde{e}^n$ $(n = 2, 3, \ldots, N)$ *to* $(P_{\Delta t})$*, the estimate*

$$|\tilde{e}^n|^2 + |E\tilde{e}^{n+1}|^2 + (\Delta t)^4 \sum_{j=1}^{n-1} |D^2\tilde{e}^j|^2 + 2\nu\Delta t \sum_{j=2}^{n} \|\tilde{e}^j\|^2 \le C\,(\Delta t)^4$$

then holds true, where C *depends, in a highly nonlinear way, on the data of the problem.*

Taking into account the error equation corresponding to $(LP_{\Delta t})$,

$$\begin{aligned}(D_2e^n, v) + \nu\left((e^n, v)\right) + b(Eu(t_n), e^n, v) + b(Ee^n, u(t_n), v) - b(Ee^n, e^n, v)\\ = \langle \rho^n, v\rangle - (\Delta t)^2 b(D^2u(t_{n-1}), u(t_n), v)\,,\end{aligned} \tag{6}$$

we may prove $e^n = \mathcal{O}(\Delta t)$, $\tilde{e}^n = \mathcal{O}((\Delta t)^{3/2})$ for Problem $(LP_{\Delta t})$. It is worth to mention that the first order result does not require any restriction on Δt. However, we are not able to prove an optimal second order estimate for $\tilde{e}^n$ because of the appearance of additional terms in (6).

Finally, we come to the error in the pressure. We assume that the approximation p^n of $p(t_n)$ $(n = 2, 3, \ldots, N)$ is determined by

$$(p^n, \nabla\cdot v) = (D_2u^n, v) + \nu((u^n, v)) + b(u^n, u^n, v) - \langle f^n, v\rangle \tag{7}$$

for all $v \in H_0^1(\Omega)^d \setminus V$. A problem in deriving estimates for the error $\pi^n := p(t_n) - p^n$ is the strict inclusion $V \subset H_0^1(\Omega)^d$ that leads to $H^{-1}(\Omega)^d \subset V^*$ and $\|f\|_* \le \|f\|_{-1,2}$. Nevertheless, because of the LBB condition, we find

Theorem 6 *Let* $t(f'' - u''') \in L^2(0,T;H^{-1}(\Omega)^d)$ *or let* $t^{3/2}(f'' - u''') \in L^2(0,T;H)$. *If* $\{u^n\}$ *is computed by* $(P_{\Delta t})$ *or* $(LP_{\Delta t})$*, and* $\{p^n\}$ *by (7) then*

$$\|\tilde{\pi}^n\|_{L^2(\Omega)/\mathbb{R}} \leq C\,\Delta t \quad or \quad \|\tilde{\pi}^n\|_{L^2(\Omega)/\mathbb{R}} \leq C\,(\Delta t)^{1/2}\,.$$

Note that $t(f'' - u''') \in L^2(0,T;H^{-1}(\Omega)^d)$ does not follow from Thm. 1. In opposite to [4] (Crank-Nicolson scheme), we do not need a higher time weight since we use the sub-optimal estimate $|t_n \rho^n| \leq c\,\Delta t$ rather than $|t_n^{3/2} \rho^n| \leq c\,(\Delta t)^{3/2}$.

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