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The peridynamic model in non-local elasticity theory

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The peridynamic model is described and an overview of some recent results concerning the analysis of the peridynamic equation of motion is given. Moreover, its numerical solution is discussed.

1 Peridynamic model

During the last few years, non-local theories in solid mechanics that account for effects of long-range interactions have become topical again. One of these theories is the so-called peridynamic modelling, introduced by Silling [3].

The governing equation of motion is the partial integro-differential equation

$$\rho(\boldsymbol{x})\partial_t^2\boldsymbol{u}(\boldsymbol{x},t) = \int_{\mathcal{V}} \boldsymbol{f}(\boldsymbol{x},\hat{\boldsymbol{x}},\boldsymbol{u}(\boldsymbol{x},t),\boldsymbol{u}(\hat{\boldsymbol{x}},t),t)\,\mathrm{d}\hat{\boldsymbol{x}} + \boldsymbol{b}(\boldsymbol{x},t)\,, \quad \boldsymbol{x}\in\mathcal{V}\,,\ t>0\,, \tag{1}$$

for the displacement field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t)$ of a body with the mass density ρ that occupies the reference volume $\mathcal{V} \subseteq \mathbb{R}^d$ with dimension $d \in \{1,2,3\}$, supplemented by initial conditions. Here, $\boldsymbol{f}$ is the pairwise force field that describes the interaction of material particles and $\boldsymbol{b}$ collects external forces. The interaction is often restricted to the peridynamic horizon $B_\delta(\boldsymbol{x}) := \{\hat{\boldsymbol{x}} \in \mathcal{V} : \|\hat{\boldsymbol{x}} - \boldsymbol{x}\| < \delta\}$ for prescribed $\delta > 0$, where $\|\cdot\|$ denotes the Euclidean norm, such that $\boldsymbol{f} = 0$ if $\hat{\boldsymbol{x}} \notin B_\delta(\boldsymbol{x})$. For the study of the dynamics of an infinite bar in [1, 6, 7, 8], however, no horizon is assumed and the interaction takes place between all material particles but decreases with increasing distance. Note that the peridynamic model implies a Poisson ratio $\nu = 1/4$.

An essential feature of the peridynamic theory is that the pairwise force function $\boldsymbol{f}$ is independent of any spatial derivative. It is, therefore, a promising approach for problems in which discontinuities emerge and has recently been successfully applied in numerical simulations of the fracture of a plate with notches, the undirected growth of cracks, the wrinkling and tearing of membranes, the deformation of composite materials etc. (see e.g. [4, 5]). As an example, Figure 1 shows a simulation of the evolution of a damage in a hard rubber plate, computed with the Emu code that has been developed by Silling (see also [4]).

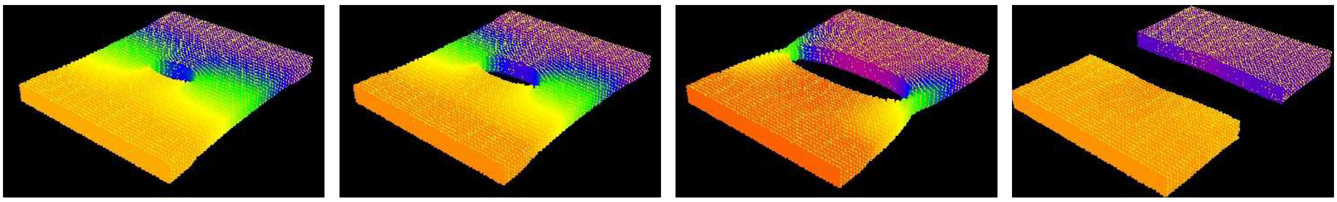

Fig. 1 Damaging of a hard rubber plate under tension

For modelling a so-called proportional microelastic material, a pairwise force function whose norm is proportional to the bond stretch has been suggested (see [3, 4, 5]):

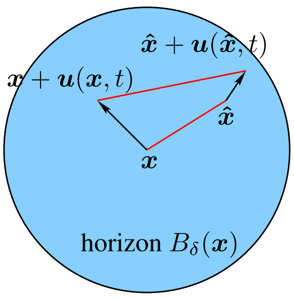

Fig. 2 Bond stretch model

$$\boldsymbol{f} = \begin{cases} c_{d,\delta}\, s\, \dfrac{\boldsymbol{e}}{\|\boldsymbol{e}\|} & \text{if } \hat{\boldsymbol{x}} \in B_\delta(\boldsymbol{x})\,, \\ 0 & \text{if } \hat{\boldsymbol{x}} \notin B_\delta(\boldsymbol{x})\,, \end{cases}$$

where $\boldsymbol{e} := [\hat{\boldsymbol{x}} + \boldsymbol{u}(\hat{\boldsymbol{x}},t)] - [\boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x},t)]$, $s := \dfrac{\|\boldsymbol{e}\| - \|\hat{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|}$. The constant $c_{d,\delta}$ can be determined by comparing the deformation energy in the peridynamic theory with that of the classical theory. So it holds e.g. $c_{3,\delta} = 12E/(\pi\delta^4)$ with Young's modulus E.

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A linearisation of the proportional microelastic material above leads to

$$\boldsymbol{f} = \begin{cases} c_{d,\delta} \, \dfrac{(\hat{\boldsymbol{x}} - \boldsymbol{x}) \otimes (\hat{\boldsymbol{x}} - \boldsymbol{x})}{\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|^3} \, (\boldsymbol{u}(\hat{\boldsymbol{x}}, t) - \boldsymbol{u}(\boldsymbol{x}, t)) & \text{if } \hat{\boldsymbol{x}} \in B_\delta(\boldsymbol{x}) \, , \\ 0 & \text{if } \hat{\boldsymbol{x}} \notin B_\delta(\boldsymbol{x}) \, . \end{cases} \tag{2}$$

Damaging can be modelled relying upon the proportional microelastic material model by allowing bond breakage such that $\boldsymbol{f}$ vanishes if s is larger than a given bond stretch s_0.

2 Theoretical results

For suitable stiffness distributions C in the autonomous linear case $f(x, \hat{x}, u, \hat{u}, t) := C(x, \hat{x})(\hat{u} - u)$ with $\mathcal{V} = \mathbb{R}$, unique solvability in L^∞ as well as stability results have been proven in [1]. If the stiffness distribution is of convolution type, one may solve the initial-value problem for the peridynamic equation of motion exactly by means of Fourier transforms (see [1, 7]). It then can be shown that different stiffness distributions correlate with different nonlinear dispersion relations. In a forthcoming paper (see [2]), we prove well-posedness in L^p for the autonomous linear case $\boldsymbol{f}(\boldsymbol{x}, \hat{\boldsymbol{x}}, \boldsymbol{u}, \hat{\boldsymbol{u}}, t) := \boldsymbol{C}(\boldsymbol{x}, \hat{\boldsymbol{x}})(\hat{\boldsymbol{u}} - \boldsymbol{u})$ on a two- or three-dimensional bounded domain. This result can be applied to the linearisation (2) of the proportional microelastic material and provides its unique solvability in L^p with $p > 2$ in the two- and with $p > 3/2$ in the three-dimensional case. The solution is also shown to be stable against perturbations of the initial values, right-hand side, and stiffness tensor $\boldsymbol{C}$.

Regarding discontinuous solutions, we observe a major difference between the peridynamic equation of motion and e.g. the classical wave equation as jump discontinuities remain at their spatial position if $\boldsymbol{f}$ is sufficiently smooth.

Moreover, it can be shown that the weak formulation of (1) is –under suitable assumptions– the necessary condition for attaining the minimum of the corresponding Lagrangian. From the weak formulation, the conservation of the total energy follows.

The relation between the non-local peridynamic model and the classical local elasticity theory is a topic of ongoing research. By means of a Taylor expansion it can be seen that the peridynamic equation of motion linearised for small relative deformations reduces to the classical Navier-Lamé equation as the horizon radius δ tends to zero.

3 Numerical approximation

In [1, 2, 8], the authors suggested the quadrature formula method relying upon different composite quadratures. Besides, linear finite elements have been considered. For solving (1) in the unbounded one-dimensional case, for instance, the Gauß-Hermite quadrature can be combined with the composite trapezoidal rule.

In our numerical studies for a one-dimensional test problem (without peridynamic horizon) for which an exact solution is known, it turns out that the finite element method requires more computations (as a second integral in the weak formulation has to be approximated) but achieves higher accuracy than the composite midpoint rule which is not so costly. The same applies to the Gauß-Hermite quadrature that requires the accurate computation of the roots of the Hermite polynomials.

Two-dimensional simulations for (1) with $\boldsymbol{f}$ given by the proportional microelastic material model or its linearisation (2) have been carried out with the composite midpoint and a composite 4-point-Gauß rule on rectangles.

For the time discretisation in our simulations, we have employed a predefined time integrator based upon a variable step size/variable order multistep or a Runge-Kutta method.

The quadrature formula method has also been used in the simulations in [4, 5] relying upon the composite midpoint rule on cubes. It is described in detail in [4]. The time discretisation there is of leap-frog-type.

The afore-mentioned numerical schemes lead to a meshfree approximation that can efficiently be parallelised.

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