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RESEARCH NOTE

A Short Note on Modeling Damage in Peridynamics

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Abstract We extend the peridynamic model to inherit irreversible damage. The governing equation is both nonlocal in time and in space and yields an abstract differential equation of Volterra type. We present conditions under which unique global solutions exist.

Keywords Nonlocal continuum mechanics · Nonlinear models · Peridynamics · Damage · Existence

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1 Modeling Damage in Peridynamics

According to Newton's law, the peridynamic equation of motion reads

$$\mathbf{y}_{tt}(\mathbf{x},t) = \int_{\Omega\cap B(\mathbf{x};\delta)} \mathbf{f}\big(\hat{\mathbf{x}}-\mathbf{x}, \mathbf{y}(\hat{\mathbf{x}},t)-\mathbf{y}(\mathbf{x},t)\big)\,\mathrm{d}\hat{\mathbf{x}} + \mathbf{b}(\mathbf{x},t), \quad (\mathbf{x},t)\in\Omega\times(0,T), \quad (1.1)$$

where $\mathbf{y} : \overline{\Omega}\times[0,T]\to\mathbb{R}^d$, $d\in\mathbb{N}$, denotes the deformation of an elastic body, which in the reference configuration takes the volume $\overline{\Omega}\subset\mathbb{R}^d$. The function $\mathbf{b} : \overline{\Omega}\times[0,T]\to\mathbb{R}^d$ is an external force density, and the integral term represents the stress. The integrand $\mathbf{f} : \overline{B}(\mathbf{0};\delta)\times\mathbb{R}^d\to\mathbb{R}^d$ is called pairwise force function and depends on the material under investigation. Here, $B(\mathbf{x};r)$ denotes the open ball with radius $r>0$ centered in $\mathbf{x}\in\mathbb{R}^d$, and the parameter $\delta>0$ is called peridynamic horizon. The pairwise force function describes the interaction of two neighbored particles $\hat{\mathbf{x}},\mathbf{x}\in\Omega$ for which the vector $\hat{\mathbf{x}}-\mathbf{x}$ is called *bond*. For more information on peridynamics, we refer to [1, 6] and the references cited therein.

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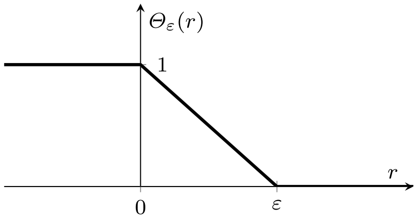
Fig. 1 The function $\Theta_\varepsilon : \mathbb{R} \to \mathbb{R}$ used for modeling irreversible damage.



In [6], damage is modeled via the bondstretch

$$s\big(\hat{\mathbf{x}} - \mathbf{x}, \mathbf{y}(\hat{\mathbf{x}}, t) - \mathbf{y}(\mathbf{x}, t)\big) = \frac{|\mathbf{y}(\hat{\mathbf{x}}, t) - \mathbf{y}(\mathbf{x}, t)| - |\hat{\mathbf{x}} - \mathbf{x}|}{|\hat{\mathbf{x}} - \mathbf{x}|}, \tag{1.2}$$

where $|\cdot|$ denotes the Euclidean norm. If the bondstretch exceeds a critical stretch $s_0 > 0$ for some time $t_0 \in [0, T]$ then the bond breaks and as a result there is no interaction (i.e., no force) between the particles $\hat{\mathbf{x}}, \mathbf{x} \in \Omega$ of the bond for any $t > t_0$. Lots of these broken bonds may lead to cracks which can be traced as spatial discontinuities in the deformation. In order to model bond breaking, in [6] it is suggested to multiply the pairwise force function in (1.1) by a factor, which is one if there holds $s(\hat{\mathbf{x}} - \mathbf{x}, \mathbf{y}(\hat{\mathbf{x}}, \tau) - \mathbf{y}(\mathbf{x}, \tau)) \le s_0$ for all $\tau \le t$ and zero if there was some time t_0 with $s(\hat{\mathbf{x}} - \mathbf{x}, \mathbf{y}(\hat{\mathbf{x}}, t_0) - \mathbf{y}(\mathbf{x}, t_0)) > s_0$. However, no explicit form of this factor is given. Observe that an incorporation of this multiplier in (1.1) changes the character of (1.1) completely since then, additionally to the nonlocality in space, there will also be a nonlocality in time. In [2], existence of local and global classical solutions to the initial-value problem for (1.1) is proven. In this note, we give extensions of some of the results of [2] inheriting the ability of bond breaking.

The multiplier described above can explicitly be formulated with

$$(H\mathbf{y})(\hat{\mathbf{x}}, \mathbf{x}, t) = \Theta\left(\int_0^t \max\big\{0, s\big(\hat{\mathbf{x}} - \mathbf{x}, \mathbf{y}(\hat{\mathbf{x}}, \tau) - \mathbf{y}(\mathbf{x}, \tau)\big) - s_0\big\}\,\mathrm{d}\tau\right), \tag{1.3}$$

where the function $\Theta : \mathbb{R} \to \mathbb{R}$ is given by $\Theta(r) = 1$ if $r \le 0$ and $\Theta(r) = 0$ otherwise. Due to the discontinuity of Θ, the mapping $\mathbf{y} \mapsto H\mathbf{y}$ cannot be continuous and in particular not Lipschitz continuous. The Lipschitz continuity property, however, will be needed to apply the techniques used in [2] in order to show existence and uniqueness of solutions. Therefore, we replace (1.3) by

$$(H_\varepsilon\mathbf{y})(\hat{\mathbf{x}}, \mathbf{x}, t) = \Theta_\varepsilon\left(\int_0^t \max\big\{0, s\big(\hat{\mathbf{x}} - \mathbf{x}, \mathbf{y}(\hat{\mathbf{x}}, \tau) - \mathbf{y}(\mathbf{x}, \tau)\big) - s_0\big\}\,\mathrm{d}\tau\right) \tag{1.4}$$

for fixed $\varepsilon > 0$. Here, $\Theta_\varepsilon : \mathbb{R} \to \mathbb{R}$ is given by (see also Fig. 1)

$$\Theta_\varepsilon(r) = \begin{cases} 1 & r \le 0, \\ 1 - \frac{r}{\varepsilon} & 0 < r < \varepsilon, \\ 0 & r \ge \varepsilon. \end{cases}$$

Observe that $H_\varepsilon\mathbf{y}$ is monotonically decreasing in time. Thus the damage is indeed irreversible.

The parameter $\varepsilon > 0$ in (1.4) does not necessarily need to be seen as a regularization parameter, which has to be sent to zero at the end. Instead, it can be seen as a material parameter. When the bondstretch exceeds the critical stretch s_0, the bond loses strength even though it does not break immediately. Hence, in order to break, there is either the possibility that the critical stretch is exceeded strongly, or it may break after exceeding the critical stretch softly over a certain period of time. In particular, this time period might be split into many short disjoint time intervals. Therefore, the effect of cycle fatigue is included into this notion of modeling damage. In that case, ε may also take large values.

In this paper, we prove global-in-time existence and uniqueness of solutions to the nonlinear peridynamic equation of motion for a class of pairwise force functions $\mathbf{f}$ multiplied by H_ε as given above. The assumptions on $\mathbf{f}$ we suppose are natural in the sense that we only require minimal measurability and (Lipschitz) continuity properties allowing us to interpret the underlying equation as an abstract ordinary differential equation of Volterra type. In a classical local theory, these assumptions correspond to the situation that the stress-strain relation is described by a (Lipschitz) continuous function. Our notion of solution may include spatially discontinuous deformations. Moreover, Theorem 2.2 can be interpreted as a regularity result: If there is no discontinuity in the initial deformation then, under suitable further assumptions, the deformation stays continuous.

The multiplication by H_ε allows us to take into account the time history in such a way that, although irreversible damage is incorporated, Lipschitz continuity is preserved. Here we model irreversible damage by comparing the bondstretch corresponding to the solution with a prescribed bondstretch. As already described above, the parameter ε can be seen as a regularization parameter as well as a parameter that describes a continuous transition from the undamaged to the damaged situation.

We would like to emphasize that nonlinear peridynamics in combination with fracture has been studied very recently in [5]. However, even though the effect of bonds losing strength is considered, no irreversible bond breaking is implemented in the formulation of [5].

2 Global Existence and Uniqueness

In what follows, we rely on the standard notation for function spaces. As usual, we identify the deformation $\mathbf{y} : \overline{\Omega} \times [0,T] \to \mathbb{R}^d$ with the abstract function $\mathbf{y} : [0,T] \to X$ by $[\mathbf{y}(t)](\mathbf{x}) = \mathbf{y}(\mathbf{x},t)$, where X is a suitable function space of mappings of $\overline{\Omega}$ into $\mathbb{R}^d$. Then, for initial data $\mathbf{y}_0, \mathbf{v}_0 \in X$, the peridynamic initial-value problem reads

$$\mathbf{y}''(t) = (K\mathbf{y})(t) + \mathbf{b}(t), \quad t \in (0,T), \qquad \mathbf{y}(0) = \mathbf{y}_0, \qquad \mathbf{y}'(0) = \mathbf{v}_0, \tag{2.1}$$

where $K : \mathcal{C}([0,T];X) \to \mathcal{C}([0,T];X)$ is given by

$$\big[(K\mathbf{y})(t)\big](\mathbf{x}) = \int_{\Omega \cap B(\mathbf{x};\delta)} \mathbf{f}\big(\hat{\mathbf{x}} - \mathbf{x}, \mathbf{y}(\hat{\mathbf{x}},t) - \mathbf{y}(\mathbf{x},t)\big)(H_\varepsilon \mathbf{y})(\hat{\mathbf{x}},\mathbf{x},t)\,\mathrm{d}\hat{\mathbf{x}}. \tag{2.2}$$

Note that K is a Volterra operator in the sense of [4, Definition 1.1 on p. 163], i.e., for $t \in [0,T]$, from $\mathbf{v}(\tau) = \mathbf{w}(\tau)$ for all $\tau \in [0,t]$ it follows $(K\mathbf{v})(t) = (K\mathbf{w})(t)$. For more information about Volterra operators in this generalized sense and corresponding existence theorems, we refer to [4].

As mentioned above, Lipschitz continuity of K is an essential ingredient of the existence proof. Indeed, H_ε is Lipschitz continuous, which is shown in the following lemma.

Lemma 2.1 *Let H_ε be given by (1.4) for $\varepsilon > 0$ with s given by (1.2). Then for $\mathbf{v} \in \mathcal{C}([0,T]; L^\infty(\Omega)^d)$ the mapping*

$$(\hat{\mathbf{x}}, \mathbf{x}) \mapsto (H_\varepsilon \mathbf{v})(\hat{\mathbf{x}}, \mathbf{x}, t) : \Omega \times \Omega \to \mathbb{R}$$

is measurable for every $t \in [0, T]$, *and the mapping*

$$t \mapsto (H_\varepsilon \mathbf{v})(\hat{\mathbf{x}}, \mathbf{x}, t) : [0, T] \to \mathbb{R}$$

is continuous for almost all $(\hat{\mathbf{x}}, \mathbf{x}) \in \Omega \times \Omega$. *Moreover, for almost all* $(\hat{\mathbf{x}}, \mathbf{x}) \in \Omega \times \Omega$ *and all* $t \in [0, T]$ *the mapping*

$$\mathbf{v} \mapsto (H_\varepsilon \mathbf{v})(\hat{\mathbf{x}}, \mathbf{x}, t) : \mathcal{C}\big([0, T]; L^\infty(\Omega)^d\big) \to \mathbb{R}$$

is Lipschitz continuous.

Proof The measurability is obvious. Concerning the continuity in time, for $\hat{t}, t \in [0, T]$, $\hat{\mathbf{x}}, \mathbf{x} \in \Omega$, $\hat{\mathbf{x}} \neq \mathbf{x}$, there holds

$$\begin{aligned}
&\big|(H_\varepsilon \mathbf{v})(\hat{\mathbf{x}}, \mathbf{x}, \hat{t}) - (H_\varepsilon \mathbf{v})(\hat{\mathbf{x}}, \mathbf{x}, t)\big| \\
&\leq \frac{1}{\varepsilon} \int_{\min\{\hat{t}, t\}}^{\max\{\hat{t}, t\}} \Big| s\big(\hat{\mathbf{x}} - \mathbf{x}, \mathbf{v}(\hat{\mathbf{x}}, \tau) - \mathbf{v}(\mathbf{x}, \tau)\big) - s_0 \Big| \, \mathrm{d}\tau \\
&\leq \frac{1}{\varepsilon} \int_{\min\{\hat{t}, t\}}^{\max\{\hat{t}, t\}} \left(\frac{|\mathbf{v}(\hat{\mathbf{x}}, \tau) - \mathbf{v}(\mathbf{x}, \tau)|}{|\hat{\mathbf{x}} - \mathbf{x}|} + 1 + s_0 \right) \mathrm{d}\tau,
\end{aligned}$$

which implies

$$\big|(H_\varepsilon \mathbf{v})(\hat{\mathbf{x}}, \mathbf{x}, \hat{t}) - (H_\varepsilon \mathbf{v})(\hat{\mathbf{x}}, \mathbf{x}, t)\big| \leq \frac{|\hat{t} - t|}{\varepsilon} \left(1 + s_0 + \frac{2\|\mathbf{v}\|_{\mathcal{C}([0,T]; L^\infty(\Omega)^d)}}{|\hat{\mathbf{x}} - \mathbf{x}|} \right). \tag{2.3}$$

In order to show the Lipschitz continuity asserted, we observe that for $a, b \in \mathbb{R}$ there holds $|\max\{0, a\} - \max\{0, b\}| \leq |a - b|$, and for $\boldsymbol{\xi}, \hat{\boldsymbol{\zeta}}, \boldsymbol{\zeta} \in \mathbb{R}^d$, $\boldsymbol{\xi} \neq \mathbf{0}$, there holds

$$\big| s(\boldsymbol{\xi}, \hat{\boldsymbol{\zeta}}) - s(\boldsymbol{\xi}, \boldsymbol{\zeta}) \big| \leq \frac{|\hat{\boldsymbol{\zeta}} - \boldsymbol{\zeta}|}{|\boldsymbol{\xi}|}.$$

Hence, for $\mathbf{v}, \mathbf{w} \in \mathcal{C}([0, T]; L^\infty(\Omega)^d)$, we find

$$\big|(H_\varepsilon \mathbf{v})(\hat{\mathbf{x}}, \mathbf{x}, t) - (H_\varepsilon \mathbf{w})(\hat{\mathbf{x}}, \mathbf{x}, t)\big| \leq \frac{2t}{\varepsilon |\hat{\mathbf{x}} - \mathbf{x}|} \|\mathbf{v} - \mathbf{w}\|_{\mathcal{C}([0,T]; L^\infty(\Omega)^d)}. \tag{2.4}$$

Note that the moduli of continuity strongly depend on ε, $\hat{\mathbf{x}}$ and $\mathbf{x}$. □

Theorem 2.1 *Let the pairwise force function* $f : B(\mathbf{0}; \delta) \times \mathbb{R}^d \to \mathbb{R}^d$ *be measurable in its first argument and continuous in its second. Suppose there exists* λ *such that the mapping* $\boldsymbol{\xi} \mapsto \lambda(\boldsymbol{\xi})/|\boldsymbol{\xi}|$ *is in* $L^1(B(\mathbf{0}; \delta))$ *and that there exists* $\ell \in L^1(B(\mathbf{0}; \delta))$ *such that for all* $\hat{\boldsymbol{\zeta}}, \boldsymbol{\zeta} \in \mathbb{R}^d$ *and almost all* $\boldsymbol{\xi} \in \mathbb{R}^d$, $|\boldsymbol{\xi}| < \delta$, *there holds*

$$\big|\mathbf{f}(\boldsymbol{\xi}, \hat{\boldsymbol{\zeta}}) - \mathbf{f}(\boldsymbol{\xi}, \boldsymbol{\zeta})\big| \leq \ell(\boldsymbol{\xi}) |\hat{\boldsymbol{\zeta}} - \boldsymbol{\zeta}|, \quad \big|\mathbf{f}(\boldsymbol{\xi}, \boldsymbol{\zeta})\big| \leq \lambda(\boldsymbol{\xi}). \tag{2.5}$$

Then, for $\mathbf{y}_0, \mathbf{v}_0 \in L^\infty(\Omega)^d$ *and* $\mathbf{b} \in L^1(0, T; L^\infty(\Omega)^d)$, *the initial-value problem* (2.1) *with* H_ε *given by* (1.4) *for* $\varepsilon > 0$ *and* s *given by* (1.2) *possesses a unique solution* $\mathbf{y} \in \mathcal{C}^1([0, T]; L^\infty(\Omega)^d)$ *with* $\mathbf{y}'' \in L^1(0, T; L^\infty(\Omega)^d)$. *If* $\mathbf{b} \in \mathcal{C}([0, T]; L^\infty(\Omega)^d)$ *then* $\mathbf{y} \in \mathcal{C}^2([0, T]; L^\infty(\Omega)^d)$.

Proof We show that under the above assumptions on the pairwise force function, and by Lemma 2.1, the Volterra operator $K : \mathcal{C}([0,T]; L^\infty(\Omega)^d) \to \mathcal{C}([0,T]; L^\infty(\Omega)^d)$ is well-defined and Lipschitz continuous.

On the one hand, we have for all $t \in [0,T]$ that $\|(K\mathbf{v})(t)\|_{L^\infty(\Omega)^d} \le \|\lambda\|_{L^1(B(\mathbf{0};\delta))}$. On the other hand, for $\hat{t}, t \in [0,T]$, there holds

$$\begin{aligned}
&\big|\big[(K\mathbf{v})(\hat{t})\big](\mathbf{x}) - \big[(K\mathbf{v})(t)\big](\mathbf{x})\big| \\
&\quad\le \int_{\Omega\cap B(\mathbf{x};\delta)} \big|\mathbf{f}\big(\hat{\mathbf{x}}-\mathbf{x}, \mathbf{v}(\hat{\mathbf{x}},\hat{t}) - \mathbf{v}(\mathbf{x},\hat{t})\big) - \mathbf{f}\big(\hat{\mathbf{x}}-\mathbf{x}, \mathbf{v}(\hat{\mathbf{x}},t) - \mathbf{v}(\mathbf{x},t)\big)\big| \,\mathrm{d}\hat{\mathbf{x}} \\
&\qquad + \int_{\Omega\cap B(\mathbf{x};\delta)} \lambda(\hat{\mathbf{x}}-\mathbf{x}) \big|(H_\varepsilon\mathbf{v})(\hat{\mathbf{x}},\mathbf{x},\hat{t}) - (H_\varepsilon\mathbf{v})(\hat{\mathbf{x}},\mathbf{x},t)\big| \,\mathrm{d}\hat{\mathbf{x}},
\end{aligned}$$

and therefore, with (2.3),

$$\begin{aligned}
&\big\|(K\mathbf{v})(\hat{t}) - (K\mathbf{v})(t)\big\|_{L^\infty(\Omega)^d} \\
&\quad\le 2\|\ell\|_{L^1(B(\mathbf{0};\delta))} \big\|\mathbf{v}(\hat{t}) - \mathbf{v}(t)\big\|_{L^\infty(\Omega)^d} \\
&\qquad + \frac{|\hat{t}-t|}{\varepsilon}\left((1+s_0)\|\lambda\|_{L^1(B(\mathbf{0};\delta))} + 2\|\mathbf{v}\|_{\mathcal{C}([0,T];L^\infty(\Omega)^d)} \int_{B(\mathbf{0};\delta)} \frac{\lambda(\boldsymbol{\xi})}{|\boldsymbol{\xi}|}\,\mathrm{d}\boldsymbol{\xi} \right).
\end{aligned}$$

Due to the integrability condition assumed on λ, we deduce that the Volterra operator $K : \mathcal{C}([0,T]; L^\infty(\Omega)^d) \to \mathcal{C}([0,T]; L^\infty(\Omega)^d)$ is well-defined and bounded.

To show Lipschitz continuity, we observe for $\mathbf{v}, \mathbf{w} \in \mathcal{C}([0,T]; L^\infty(\Omega)^d)$ and $t \in [0,T]$, $\mathbf{x} \in \Omega$, that

$$\begin{aligned}
&\big|\big[(K\mathbf{v})(t)\big](\mathbf{x}) - \big[(K\mathbf{w})(t)\big](\mathbf{x})\big| \\
&\quad\le \int_{\Omega\cap B(\mathbf{x};\delta)} \big|\mathbf{f}\big(\hat{\mathbf{x}}-\mathbf{x}, \mathbf{v}(\hat{\mathbf{x}},t) - \mathbf{v}(\mathbf{x},t)\big) - \mathbf{f}\big(\hat{\mathbf{x}}-\mathbf{x}, \mathbf{w}(\hat{\mathbf{x}},t) - \mathbf{w}(\mathbf{x},t)\big)\big| \,\mathrm{d}\hat{\mathbf{x}} \\
&\qquad + \int_{\Omega\cap B(\mathbf{x};\delta)} \lambda(\hat{\mathbf{x}}-\mathbf{x}) \big|(H_\varepsilon\mathbf{v})(\hat{\mathbf{x}},\mathbf{x},t) - (H_\varepsilon\mathbf{w})(\hat{\mathbf{x}},\mathbf{x},t)\big| \,\mathrm{d}\hat{\mathbf{x}},
\end{aligned}$$

and therefore, by (2.4),

$$\begin{aligned}
\|K\mathbf{v} - K\mathbf{w}\|_{\mathcal{C}([0,T];L^\infty(\Omega)^d)} &\le 2\|\ell\|_{L^1(B(\mathbf{0};\delta))} \|\mathbf{v}-\mathbf{w}\|_{\mathcal{C}([0,T];L^\infty(\Omega)^d)} \\
&\quad + \frac{2T}{\varepsilon} \|\mathbf{v}-\mathbf{w}\|_{\mathcal{C}([0,T];L^\infty(\Omega)^d)} \int_{B(\mathbf{0};\delta)} \frac{\lambda(\boldsymbol{\xi})}{|\boldsymbol{\xi}|}\,\mathrm{d}\boldsymbol{\xi}.
\end{aligned}$$

Hence, there exists $L > 0$ such that

$$\|K\mathbf{v} - K\mathbf{w}\|_{\mathcal{C}([0,T];L^\infty(\Omega)^d)} \le L\|\mathbf{v}-\mathbf{w}\|_{\mathcal{C}([0,T];L^\infty(\Omega)^d)}.$$

Note that since $K : \mathcal{C}([0,T]; L^\infty(\Omega)^d) \to \mathcal{C}([0,T]; L^\infty(\Omega)^d)$ is a Volterra operator in the sense of [4], there holds in particular for all $t \in [0,T]$

$$\|K\mathbf{v} - K\mathbf{w}\|_{\mathcal{C}([0,t];L^\infty(\Omega)^d)} \le L\|\mathbf{v}-\mathbf{w}\|_{\mathcal{C}([0,t];L^\infty(\Omega)^d)}, \tag{2.6}$$

see [4, Lemma 1.4 on p. 163]. This allows us to apply Banach's fixed-point principle to $\mathbf{v} = S\mathbf{v}$, where the fixed-point mapping $S : \mathcal{C}([0,T]; L^\infty(\Omega)^d) \to \mathcal{C}([0,T]; L^\infty(\Omega)^d)$ is given by

$$(S\mathbf{v})(t) = \mathbf{y}_0 + t\mathbf{v}_0 + \int_0^t (t-s)\big((K\mathbf{v})(s) + \mathbf{b}(s)\big)\,\mathrm{d}s.$$

Indeed, S is a contraction on $\mathcal{C}([0,T];L^\infty(\Omega)^d)$ with respect to the equivalent norm

$$|||\mathbf{v}|||_{\mathcal{C}([0,T];L^\infty(\Omega)^d)} := \max_{0\le t\le T} e^{-2LTt}\big\|\mathbf{v}(t)\big\|_{L^\infty(\Omega)^d}$$

since by (2.6) there holds for $t \in [0,T]$

$$\begin{aligned}\big\|(S\mathbf{v})(t) - (S\mathbf{w})(t)\big\|_{L^\infty(\Omega)^d} &\le \int_0^t (t-s)\big\|(K\mathbf{v})(s) - (K\mathbf{w})(s)\big\|_{L^\infty(\Omega)^d}\,\mathrm{d}s \\ &\le LT\int_0^t \|\mathbf{v}-\mathbf{w}\|_{\mathcal{C}([0,s];L^\infty(\Omega)^d)}\,\mathrm{d}s \\ &\le LT\int_0^t \max_{0\le\tau\le s}\big\{\big\|\mathbf{v}(\tau)-\mathbf{w}(\tau)\big\|e^{-2LT\tau}\big\}e^{2LTs}\,\mathrm{d}s \\ &\le \frac{e^{2LTt}-1}{2}|||\mathbf{v}-\mathbf{w}|||_{\mathcal{C}([0,T];L^\infty(\Omega)^d)},\end{aligned}$$

and hence $|||S\mathbf{v} - S\mathbf{w}|||_{\mathcal{C}([0,T];L^\infty(\Omega)^d)} \le \frac{1}{2}|||\mathbf{v}-\mathbf{w}|||_{\mathcal{C}([0,T];L^\infty(\Omega)^d)}$. The fixed-point problem is equivalent to (2.1). Thus existence and uniqueness of the solution follows. The regularity of $\mathbf{y}$ is immediate. □

Of course, the Lipschitz and growth condition (2.5) on the pairwise force function are quite restrictive, although they allow for a weak singularity in $\boldsymbol{\xi}$ if $d \ge 2$.

The choice of the function space $X = L^\infty(\Omega)^d$ provides the opportunity for the deformation to experience spatial discontinuities. Therefore, the globally existing classical solution may suffer from a strong type of damage that results in cracks. However, if the initial values are taken as elements of the function space $X = \mathcal{C}(\overline{\Omega})^d$, then, even though bonds are allowed to break irreversibly, there exists a global classical solution taking values in the space of continuous functions, i.e., no cracks will occur. In order to show this, we need the following estimate for the mapping $\mathbf{x} \mapsto (H_\varepsilon\mathbf{v})(\hat{\mathbf{x}},\mathbf{x},t)$ if $\mathbf{v} \in \mathcal{C}([0,T];\mathcal{C}(\overline{\Omega})^d)$.

Lemma 2.2 *Let H_ε be given by (1.4) for $\varepsilon > 0$ with s given by (1.2) and let $\mathbf{v} \in \mathcal{C}([0,T];\mathcal{C}(\overline{\Omega})^d)$. Then for $\mathbf{x},\hat{\mathbf{x}} \in \overline{\Omega}$, $t \in [0,T]$, there holds for all $\mathbf{h} \in \mathbb{R}^d$ with $\mathbf{x}+\mathbf{h} \in \overline{\Omega}$*

$$\begin{aligned}&\big|(H_\varepsilon\mathbf{v})(\hat{\mathbf{x}},\mathbf{x}+\mathbf{h},t) - (H_\varepsilon\mathbf{v})(\hat{\mathbf{x}},\mathbf{x},t)\big| \\ &\quad\le \frac{2t}{\varepsilon}\frac{|\mathbf{h}|}{|\hat{\mathbf{x}}-\mathbf{x}-\mathbf{h}||\hat{\mathbf{x}}-\mathbf{x}|}\|\mathbf{v}\|_{\mathcal{C}([0,T];\mathcal{C}(\overline{\Omega})^d)} \\ &\qquad + \frac{1}{\varepsilon}\frac{1}{|\hat{\mathbf{x}}-\mathbf{x}|}\int_0^t \big|\mathbf{v}(\mathbf{x}+\mathbf{h},\tau) - \mathbf{v}(\mathbf{x},\tau)\big|\,\mathrm{d}\tau.\end{aligned}$$

Proof We find

$$\begin{aligned}&\big|(H_\varepsilon\mathbf{v})(\hat{\mathbf{x}},\mathbf{x}+\mathbf{h},t) - (H_\varepsilon\mathbf{v})(\hat{\mathbf{x}},\mathbf{x},t)\big| \\ &\quad\le \frac{1}{\varepsilon}\int_0^t \big|s\big(\hat{\mathbf{x}}-\mathbf{x}-\mathbf{h},\mathbf{v}(\hat{\mathbf{x}},\tau)-\mathbf{v}(\mathbf{x}+\mathbf{h},\tau)\big) - s\big(\hat{\mathbf{x}}-\mathbf{x},\mathbf{v}(\hat{\mathbf{x}},\tau)-\mathbf{v}(\mathbf{x},\tau)\big)\big|\,\mathrm{d}\tau.\end{aligned}$$

Moreover, since for $\hat{\boldsymbol{\xi}}, \boldsymbol{\xi}, \hat{\boldsymbol{\zeta}}, \boldsymbol{\zeta} \in \mathbb{R}^d$, $\hat{\boldsymbol{\xi}}, \boldsymbol{\xi} \neq \mathbf{0}$, there holds

$$\left|s(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\zeta}}) - s(\boldsymbol{\xi}, \boldsymbol{\zeta})\right| \leq |\boldsymbol{\zeta}| \frac{|\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}|}{|\hat{\boldsymbol{\xi}}||\boldsymbol{\xi}|} + \frac{|\hat{\boldsymbol{\zeta}} - \boldsymbol{\zeta}|}{|\hat{\boldsymbol{\xi}}|},$$

the claim follows. □

Theorem 2.2 *Let the pairwise force function* $\mathbf{f} : B(\mathbf{0}; \delta) \times \mathbb{R}^d \to \mathbb{R}^d$ *be as in Theorem 2.1 with* H_ε *given by* (1.4) *for* $\varepsilon > 0$ *and* s *given by* (1.2). *Additionally, we assume that* $\boldsymbol{\xi} \mapsto \lambda(\boldsymbol{\xi})/(|\boldsymbol{\xi}||\boldsymbol{\xi} + \mathbf{h}|)$ *is in* $L^1(B(\mathbf{0}; \delta))$ *for all* $\mathbf{h} \in \mathbb{R}^d$ *with* $|\mathbf{h}| \leq h_0$, *for a fixed* $h_0 > 0$. *Then, for each* $\mathbf{y}_0, \mathbf{v}_0 \in \mathcal{C}(\overline{\Omega})^d$ *and* $\mathbf{b} \in L^1(0, T; \mathcal{C}(\overline{\Omega})^d)$, *the initial-value problem* (2.1) *possesses a unique solution* $\mathbf{y} \in \mathcal{C}^1([0, T]; \mathcal{C}(\overline{\Omega})^d)$ *with* $\mathbf{y}'' \in L^1(0, T; \mathcal{C}(\overline{\Omega})^d)$. *If* $\mathbf{b} \in \mathcal{C}([0, T]; \mathcal{C}(\overline{\Omega})^d)$ *then* $\mathbf{y} \in \mathcal{C}^2([0, T]; \mathcal{C}(\overline{\Omega})^d)$.

Proof Due to the bound $\|K\mathbf{v}\|_{\mathcal{C}([0,T];\mathcal{C}(\overline{\Omega})^d)} \leq \|\lambda\|_{L^1(B(\mathbf{0};\delta))}$, it remains to show continuity of $\mathbf{x} \mapsto [(K\mathbf{v})(t)](\mathbf{x})$ to conclude that the operator is well-defined. In order to shorten notation, we denote $\Omega \cap B(\mathbf{x}; \delta)$ by $\mathcal{H}(\mathbf{x})$. For $\mathbf{x}, \mathbf{x} + \mathbf{h} \in \overline{\Omega}$, we find

$$\left|\left[(K\mathbf{v})(t)\right](\mathbf{x} + \mathbf{h}) - \left[(K\mathbf{v})(t)\right](\mathbf{x})\right| \leq \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4$$

with

$$\begin{aligned}
\mathcal{I}_1 &= \int_{\mathcal{H}(\mathbf{x}+\mathbf{h}) \cap \mathcal{H}(\mathbf{x})} \Big|\mathbf{f}\big(\hat{\mathbf{x}} - \mathbf{x} - \mathbf{h}, \mathbf{v}(\hat{\mathbf{x}}, t) - \mathbf{v}(\mathbf{x} + \mathbf{h}, t)\big) \\
&\quad - \mathbf{f}\big(\hat{\mathbf{x}} - \mathbf{x}, \mathbf{v}(\hat{\mathbf{x}}, t) - \mathbf{v}(\mathbf{x}, t)\big)\Big| \, \mathrm{d}\hat{\mathbf{x}}, \\
\mathcal{I}_2 &= \int_{\mathcal{H}(\mathbf{x}+\mathbf{h}) \cap \mathcal{H}(\mathbf{x})} \lambda(\hat{\mathbf{x}} - \mathbf{x}) \big|(H_\varepsilon \mathbf{v})(\hat{\mathbf{x}}, \mathbf{x} + \mathbf{h}, t) - (H_\varepsilon \mathbf{v})(\hat{\mathbf{x}}, \mathbf{x}, t)\big| \, \mathrm{d}\hat{\mathbf{x}}, \\
\mathcal{I}_3 &= \int_{\mathcal{H}(\mathbf{x}+\mathbf{h}) \setminus \mathcal{H}(\mathbf{x})} \lambda(\hat{\mathbf{x}} - \mathbf{x} - \mathbf{h}) \, \mathrm{d}\hat{\mathbf{x}}, \\
\mathcal{I}_4 &= \int_{\mathcal{H}(\mathbf{x}) \setminus \mathcal{H}(\mathbf{x}+\mathbf{h})} \lambda(\hat{\mathbf{x}} - \mathbf{x}) \, \mathrm{d}\hat{\mathbf{x}}.
\end{aligned}$$

For the first integral $\mathcal{I}_1$, there holds

$$\begin{aligned}
\mathcal{I}_1 &\leq \int_{\mathcal{H}(\mathbf{x}+\mathbf{h}) \cap \mathcal{H}(\mathbf{x})} \Big|\mathbf{f}\big(\hat{\mathbf{x}} - \mathbf{x} - \mathbf{h}, \mathbf{v}(\hat{\mathbf{x}}, t) - \mathbf{v}(\mathbf{x} + \mathbf{h}, t)\big) \\
&\quad - \mathbf{f}\big(\hat{\mathbf{x}} - \mathbf{x} - \mathbf{h}, \mathbf{v}(\hat{\mathbf{x}}, t) - \mathbf{v}(\mathbf{x}, t)\big)\Big| \, \mathrm{d}\hat{\mathbf{x}} \\
&\quad + \int_{\mathcal{H}(\mathbf{x}+\mathbf{h}) \cap \mathcal{H}(\mathbf{x})} \Big|\mathbf{f}\big(\hat{\mathbf{x}} - \mathbf{x} - \mathbf{h}, \mathbf{v}(\hat{\mathbf{x}}, t) - \mathbf{v}(\mathbf{x}, t)\big) \\
&\quad - \mathbf{f}\big(\hat{\mathbf{x}} - \mathbf{x}, \mathbf{v}(\hat{\mathbf{x}}, t) - \mathbf{v}(\mathbf{x}, t)\big)\Big| \, \mathrm{d}\hat{\mathbf{x}} \\
&= \mathcal{I}_{1,1} + \mathcal{I}_{1,2}.
\end{aligned}$$

With the transformation $\boldsymbol{\xi} = \hat{\mathbf{x}} - \mathbf{x} - \mathbf{h}$, we find

$$\mathcal{I}_{1,1} = \int_{\mathcal{H}(\mathbf{0})\cap\mathcal{H}(-\mathbf{h})} \big|\mathbf{f}\big(\boldsymbol{\xi}, \mathbf{v}(\boldsymbol{\xi}+\mathbf{x}+\mathbf{h}, t) - \mathbf{v}(\mathbf{x}+\mathbf{h}, t)\big) \\ - \mathbf{f}\big(\boldsymbol{\xi}, \mathbf{v}(\boldsymbol{\xi}+\mathbf{x}+\mathbf{h}, t) - \mathbf{v}(\mathbf{x}, t)\big)\big| \,\mathrm{d}\boldsymbol{\xi}.$$

Here, the integrand converges pointwise almost everywhere due to the continuity of the pairwise force function in its second argument and since $\mathbf{v}$ is continuous. Moreover, the pairwise force function is bounded by λ in view of (2.5). Therefore, by Lebesgue's theorem on dominated convergence, it follows $\mathcal{I}_{1,1} \to 0$ for $\mathbf{h} \to \mathbf{0}$. For $\mathcal{I}_{1,2}$, we observe convergence to zero since, in particular, the pairwise force function is continuous with respect to the L^1-mean. Regarding $\mathcal{I}_2$, with Lemma 2.2 there holds

$$\mathcal{I}_2 \leq \frac{2t}{\varepsilon}|\mathbf{h}|\|\mathbf{v}\|_{\mathcal{C}([0,T];\mathcal{C}(\overline{\Omega})^d)} \int_{\mathcal{H}(\mathbf{x}+\mathbf{h})\cap\mathcal{H}(\mathbf{x})} \frac{\lambda(\hat{\mathbf{x}}-\mathbf{x})}{|\hat{\mathbf{x}}-\mathbf{x}-\mathbf{h}||\hat{\mathbf{x}}-\mathbf{x}|}\,\mathrm{d}\hat{\mathbf{x}} \\ + \frac{1}{\varepsilon}\int_{\mathcal{H}(\mathbf{x}+\mathbf{h})\cap\mathcal{H}(\mathbf{x})} \frac{\lambda(\hat{\mathbf{x}}-\mathbf{x})}{|\hat{\mathbf{x}}-\mathbf{x}|}\,\mathrm{d}\hat{\mathbf{x}} \int_0^t \big|\mathbf{v}(\mathbf{x}+\mathbf{h},\tau) - \mathbf{v}(\mathbf{x},\tau)\big|\,\mathrm{d}\tau,$$

which yields continuity. Note that the integrals on the right-hand side exist due to our assumptions on λ. Finally, for the remaining integrals $\mathcal{I}_3$ and $\mathcal{I}_4$, we find

$$\mathcal{I}_3 = \int_{\mathcal{H}(\mathbf{0})\setminus\mathcal{H}(-\mathbf{h})} \lambda(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \to 0 \quad \text{for } \mathbf{h} \to \mathbf{0}$$

as well as

$$\mathcal{I}_4 = \int_{\mathcal{H}(\mathbf{0})\setminus\mathcal{H}(\mathbf{h})} \lambda(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \to 0 \quad \text{for } \mathbf{h} \to \mathbf{0}.$$

Therefore, $K : \mathcal{C}([0,T];\mathcal{C}(\overline{\Omega})^d) \to \mathcal{C}([0,T];\mathcal{C}(\overline{\Omega})^d)$ is well-defined. The Lipschitz continuity of K follows as before. Hence, with the same fixed-point operator as used in the proof of Theorem 2.1 but as a mapping of $\mathcal{C}([0,T];\mathcal{C}(\overline{\Omega})^d)$ into itself, the claim follows with Banach's fixed-point principle. The regularity of $\mathbf{y}$ follows immediately. □

Observe that no continuity with respect to the first argument of the pairwise force function was assumed. Hence, in that sense, this result generalizes the results in [2].

Note that incorporating irreversible bond breaking into the setting of weak solutions in peridynamics (see [3]) is a question of ongoing research.

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