Supraconvergence of a Non-Uniform Discretisation for an Elliptic Third-Kind Boundary-Value Problem with Mixed Derivatives

Etienne Emmrich

Technische Universität Berlin, Institut für Mathematik,
Straße des 17. Juni 136, 10623 Berlin, Germany
emmrich@math.tu-berlin.de

Abstract. The third-kind boundary-value problem for an elliptic equation of second order with variable coefficients and mixed as well as first-order terms on a domain that is the union of rectangles is approximated by a linear finite element method with first-order accurate quadrature. The scheme is equivalent to a standard finite difference method.
Although the discretisation is in general only first-order consistent, supraconvergence, i.e. convergence of higher order, is shown to take place even on non-uniform grids. Local error estimates of optimal order $\min(s, 3/2)$ in the $H^1(\Omega)$-norm can be derived for $s \in [1, 2]$ if the exact solution is in the Sobolev-Slobodetskij space $H^{1+s}(\Omega)$. If no mixed derivatives occur then optimal order s can be achieved. The supraconvergence also implies the supercloseness of the gradient.

Key words: elliptic PDE, fully discrete FEM, FDM, non-uniform grid, supraconvergence, supercloseness of gradient

MSC (2000): 65N12; 65N15; 65N06; 65N30

1 Introduction

We consider the discretisation of the differential equation

$$Au := -(au_x)_x - (bu_x)_y - (bu_y)_x - (cu_y)_y + du_x + eu_y + fu = g \quad \text{in } \Omega \subset \mathbb{R}^2 \quad (1)$$

with variable coefficients subject to Robin boundary conditions

$$Bu := au_x\eta_x + bu_x\eta_y + bu_y\eta_x + cu_y\eta_y + \alpha u = \psi \quad \text{on } \Gamma := \partial\Omega\,, \quad (2)$$

where the domain Ω is the union of rectangles and (η_x, η_y) denotes the outer normal on Γ. The coefficients in A and B as well as ψ are assumed to be sufficiently smooth such that all are at least uniformly continuous and $g \in L^2(\Omega)$. Moreover, A is assumed to be uniformly elliptic and the corresponding homogeneous problem is supposed to be uniquely solvable. The corresponding bilinear form needs not to be strongly positive.

The discretisation is obtained from linear finite elements on a triangulation $\mathcal{T}_H$ of Ω, which relies upon a non-uniform rectangular grid $\overline{\Omega}_H$, in combination with an appropriate quadrature that is of first order.

Although the scheme is in general only first-order consistent, higher-order convergence can be proven: Let u_H denote the discrete solution on $\overline{\Omega}_H$, P_H the piecewise linear interpolation with respect to $\mathcal{T}_H$ (note that $P_H u_H$ is the finite element solution), R_H the pointwise restriction on $\overline{\Omega}_H$, and $\|\cdot\|_{1,2}$ the usual $H^1(\Omega)$-norm. For a quasi-uniform sequence of grids $\overline{\Omega}_H$ with a maximum mesh-size $H_{\max}$ tending to zero, we derive a local error estimate from which the global error estimate $\|P_H R_H u - P_H u_H\|_{1,2} = \mathcal{O}(H_{\max}^{\min(s,3/2)})$ for $s \in [1,2]$ if $u \in H^{1+s}(\Omega)$ follows. In the case of no mixed derivatives, optimal order $\mathcal{O}(H_{\max}^s)$ can be proven. For $s \in \{1,2\}$, the estimates even hold without the assumption of quasi-uniformity. As a consequence, a global estimate of order $\mathcal{O}(H_{\max}^{(s+1)/2})$ in the case of mixed derivatives and of order $\mathcal{O}(H_{\max}^s)$ if no mixed derivatives occur is obtained for $s \in [1,2]$ even for an arbitrary sequence of non-uniform grids. Note that these supraconvergence results also show the supercloseness property for the gradient in the context of finite elements. Convergence results of order $\mathcal{O}(H_{\max}^s)$ for $s \in (1/2, 1)$ can also be obtained but are not included here.

Super- and supraconvergence of finite element and finite difference solutions have been considered by many authors; we refer to [1, 2, 7–12] and the references cited therein. In particular, we continue the work presented in [4–6]. For the third-kind boundary-value problem (1), (2) on a domain Ω that is the union of rectangles but without mixed or first-order derivatives ($b = d = e = 0$), local error estimates showing supraconvergence of order s for $s \in (1/2, 2]$ if $u \in H^{1+s}(\Omega)$ are proved in [4]. The aim of this paper is to present a generalisation of the results of [4] that allows mixed and first-order derivatives in the differential operator A. An essential step in the analysis is the equivalence of the fully discrete finite element method with quadrature to a finite difference method.

2 Discretisation

The corresponding variational problem reads as

$$\text{find } u \in H^1(\Omega) \text{ such that } A(u,v) = (g,v) + (\psi, v)_\Gamma \text{ for all } v \in H^1(\Omega)$$

with the sesquilinear form

$$\begin{aligned} A(v,w) := {} & (av_x, w_x) + (bv_x, w_y) + (bv_y, w_x) + (cv_y, w_y) \\ & + (dv_x, w) + (ev_y, w) + (fv, w) + (\alpha v, w)_\Gamma , \quad v, w \in H^1(\Omega) . \end{aligned}$$

Here and in the following, we employ the usual notation for Lebesgue-, Sobolev-, Sobolev-Slobodetskij spaces and spaces of continuously differentiable functions. In particular, we denote by $(\cdot,\cdot)$ and $(\cdot,\cdot)_\Gamma$ the inner product on $L^2(\Omega)$ and $L^2(\Gamma)$, respectively, and by $\|\cdot\|_{r,p,D}$ the usual norm on $W^{r,p}(D)$ for a domain D (where we omit the subscript D if $D = \Omega$). As the boundary Γ is only

Lipschitz, the norm of Sobolev-Slobodetskij spaces on Γ shall be defined through summing up over disjoint straight boundary sections.

For the discretisation, let $\boldsymbol{h} = \{h_j\}_{j\in\mathbb{Z}}$ and $\boldsymbol{k} = \{k_\ell\}_{\ell\in\mathbb{Z}}$ be two sequences of positive real numbers and consider the two-dimensional grid $\mathbb{R}_H := \mathbb{R}^x_{\boldsymbol{h}} \times \mathbb{R}^y_{\boldsymbol{k}}$, where

$$\mathbb{R}^x_{\boldsymbol{h}} := \{x_j \in \mathbb{R} : x_{j+1} := x_j + h_j \, , \; j \in \mathbb{Z}\}$$

for $x_0 \in \mathbb{R}$ given; $\mathbb{R}^y_{\boldsymbol{k}}$ is defined analogously. Moreover, let $x_{j+1/2} := x_j + h_j/2 = x_{j+1} - h_j/2 := x_{(j+1)-1/2}$ with an analogous notation in the y-direction. We define

$$\Omega_H := \Omega \cap \mathbb{R}_H \, , \quad \Gamma_H := \Gamma \cap \mathbb{R}_H \, , \quad \overline{\Omega}_H := \overline{\Omega} \cap \mathbb{R}_H = \Omega_H \cup \Gamma_H$$

and consider a sequence $\{\overline{\Omega}_H\}_{H\in\Lambda}$ of grids with $H_{\max} := \max\{h_j, k_\ell : j, \ell \in \mathbb{Z}\}$ tending to zero. We say that $\{\overline{\Omega}_H\}_{H\in\Lambda}$ is quasi-uniform if all possible quotients of mesh sizes of $\overline{\Omega}_H$ are bounded independently of H. The vertices of $\overline{\Omega}$ are assumed to be in Γ_H. The triangulation $\mathcal{T}_H$ is supposed to be a set of open triangles in which the vertices are the grid points of $\overline{\Omega}_H$. Throughout this paper, we assume $H_{\max}$ being sufficiently small.

By W_H, we denote the space of grid functions on $\overline{\Omega}_H$. For convenience, we tacitly assume that a function $v_H \in W_H$ is extended on $\mathbb{R}_H$ by zero. We often write v_P instead of $v_H(P)$. For $P = (x_j, y_\ell) \in \overline{\Omega}_H$, let

$$\begin{aligned} \square_P &:= (x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Omega \, , \\ \Gamma_P &:= (x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Gamma \, . \end{aligned}$$

Then

$$(v_H, w_H)_H := \sum_{P\in\overline{\Omega}_H} |\square_P| v_P \overline{w}_P \quad \text{and} \quad (\phi_H, \chi_H)_{\Gamma,H} := \sum_{P\in\Gamma_H} |\Gamma_P| \phi_P \overline{\chi}_P$$

defines an inner product on W_H and on the space of grid functions on Γ_H, respectively.

Let $A_H := a_H + b_H + c_H + d_H + e_H + f_H + \alpha_H$ be the sesquilinear form defined by

$$\begin{aligned} a_H(v_H, w_H) &:= \sum_{\triangle\in\mathcal{T}_H} a_{\triangle,x} \int_\triangle (P_H v_H)_x (P_H \overline{w}_H)_x dV \, , \\ d_H(v_H, w_H) &:= \sum_{\triangle\in\mathcal{T}_H} (dP_H \overline{w}_H)_{\triangle,x} \int_\triangle (P_H v_H)_x dV \, , \\ f_H(v_H, w_H) &:= (R_H f v_H, w_H)_H \, , \quad \alpha_H(v_H, w_H) := (R_H \alpha v_H, w_H)_{\Gamma,H} \, , \end{aligned}$$

and with forms c_H and e_H defined analogously to a_H and d_H, respectively. Here, the subscript $\triangle, x$ denotes the value at the midpoint of the side of $\triangle \in \mathcal{T}_H$ parallel to the x-axis. The definition of the form b_H is based upon two special triangulations $\mathcal{T}^{(1)}_H$ and $\mathcal{T}^{(2)}_H$.

Let $\triangle_{j,\ell}^{(\perp)}$ denote an open triangle having an angle $\pi/2$ at $(x_j, y_\ell) \in \mathrm{IR}_H$ and two adjacent grid points as further vertices. We then define

$$\mathcal{T}_H^{(\nu)} := \left\{ \triangle_{j,\ell}^{(\perp)} \subset \Omega : (x_j, y_\ell) \in \mathrm{IR}_H \text{ with } j+\ell+\nu \text{ being odd} \right\}, \quad \nu = 1, 2,$$

and associate the piecewise linear interpolation $P_H^{(\nu)}$. Then

$$b_H(v_H, w_H) := \frac{1}{2} \left(b_H^{(1)}(v_H, w_H) + b_H^{(2)}(v_H, w_H) \right),$$

where for $\nu = 1, 2$

$$b_H^{(\nu)}(v_H, w_H) := \sum_{\triangle \in \mathcal{T}_H^{(\nu)}} \int_\triangle b_\triangle \left((P_H^{(\nu)} v_H)_x (P_H^{(\nu)} \overline{w}_H)_y + (P_H^{(\nu)} v_H)_y (P_H^{(\nu)} \overline{w}_H)_x \right) dV$$

with $b_\triangle$ being the value of b at the vertex of $\triangle \in \mathcal{T}_H^{(\nu)}$ that corresponds with the angle $\pi/2$.

The fully discrete Galerkin approximation now reads as

$$\begin{aligned} &\text{find } u_H \in W_H \text{ such that} \\ &A_H(u_H, v_H) = (g_H, v_H)_H + (\psi_H, v_H)_{\Gamma,H} \text{ for all } v_H \in W_H \end{aligned} \tag{3}$$

with right-hand side g_H and boundary value ψ_H given by

$$g_P := |\square_P|^{-1} \int_{\square_P} g dV \quad (P \in \overline{\Omega}_H), \quad \psi_P := \psi(P) \quad (P \in \Gamma_H). \tag{4}$$

It can be shown that (3) is equivalent to the finite difference approximation

$$\begin{aligned} A_H u_H := &-\delta_x^{(1/2)} \left(a \delta_x^{(1/2)} u_H \right) - \delta_y (b \delta_x u_H) - \delta_x (b \delta_y u_H) - \delta_y^{(1/2)} \left(c \delta_y^{(1/2)} u_H \right) \\ &+ R_H d\, \delta_x u_H + R_H e\, \delta_y u_H + R_H f\, u_H = g_H \quad \text{in } \Omega_H, \end{aligned}$$

supplemented by an appropriate approximation of the boundary condition (see also [4]). In particular, we have

$$A_H(v_H, w_H) = (A_H v_H, w_H)_H$$

for all $v_H, w_H \in W_H$ with $w_H = 0$ on Γ_H. Here, we use the divided differences

$$\delta_x^{(1/2)} v_{j,\ell} := \frac{v_{j+1/2,\ell} - v_{j-1/2,\ell}}{x_{j+1/2} - x_{j-1/2}}, \quad \delta_x^{(1/2)} v_{j+1/2,\ell} := \frac{v_{j+1,\ell} - v_{j,\ell}}{x_{j+1} - x_j},$$
$$\delta_x v_{j,\ell} := \frac{v_{j+1,\ell} - v_{j-1,\ell}}{x_{j+1} - x_{j-1}}$$

and corresponding differences in the y-direction.

The following stability result can be proven similarly as [5, Thm. 2].

Proposition 1. *For $\overline{\Omega}_H$ $(H \in \Lambda)$ and $v_H \in W_H$, the following estimate holds true:*

$$\|P_H v_H\|_{1,2} \leq C \sup_{0 \neq w_H \in W_H} \frac{|A_H(v_H, w_H)|}{\|P_H w_H\|_{1,2}} .$$

By C, we denote a generic constant that is independent of significant quantities such as the grid size. Proposition 1 implies the unique solvability of the discrete problem.

3 Supraconvergence of the Discretisation

For $P = (x_j, y_\ell) \in \Gamma_H^y$, the set of grid points lying on Γ^y which is the part of Γ that is parallel to the y-axis, we define $\Gamma_P := \{x_j\} \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Gamma$. Moreover, we use the convention that, depending on the location of the domain Ω, η_x and η_y always take the value $+1$ or -1 on the sections of the boundary Γ even at the vertices of the domain.

The main result reads as

Theorem 1. *Let $u \in H^2(\Omega)$, $a, b, c, d, e, f \in W^{1,\infty}(\Omega)$, $\alpha \in W^{1,\infty}(\Gamma)$, $\psi \in H^1(\Gamma)$ if $s = 1$ and let $u \in H^{1+s}(\Omega)$, $a, b, c, d, e, f \in W^{2,2/(2-s)}(\Omega)$, $\alpha \in W^{2,1/(2-s)}(\Gamma)$, $\psi \in H^s(\Gamma)$ if $s \in (1, 2]$. Moreover, assume that $\{\overline{\Omega}_H\}_{H \in \Lambda}$ is quasi-uniform if $s \notin \{1, 2\}$. The discretisation error then satisfies the estimate*

$$\begin{aligned}\|P_H R_H u - P_H u_H\|_{1,2} \leq C\Bigg(&\sum_{P \in \overline{\Omega}_H} (\text{diam } \square_P)^{2s} \|u\|^2_{1+s,2,\square_P} \\ &+ \sum_{P \in \Gamma_H} \left(|\Gamma_P|^{2\min(s,3/2)} \|u\|^2_{1/2+s,2,\Gamma_P} + |\Gamma_P|^{2s} \|\psi\|^2_{s,2,\Gamma_P}\right)\Bigg)^{1/2} \\ &\leq C H_{\max}^{\min(s,3/2)} \left(\|u\|_{1+s,2} + \|u\|_{1/2+s,2,\Gamma} + \|\psi\|_{s,2,\Gamma}\right) \qquad (5)\end{aligned}$$

in the case of mixed derivatives. If $b \equiv 0$ then

$$\begin{aligned}\|P_H R_H u - P_H u_H\|_{1,2} \leq C\Bigg(&\sum_{P \in \overline{\Omega}_H} (\text{diam } \square_P)^{2s} \|u\|^2_{1+s,2,\square_P} \\ &+ \sum_{P \in \Gamma_H} |\Gamma_P|^{2s} \left(\|u\|^2_{s,2,\Gamma_P} + \|\psi\|^2_{s,2,\Gamma_P}\right)\Bigg)^{1/2} \\ &\leq C H_{\max}^s \left(\|u\|_{1+s,2} + \|u\|_{s,2,\Gamma} + \|\psi\|_{s,2,\Gamma}\right) . \qquad (6)\end{aligned}$$

Sketch of Proof In what follows, we sketch the main steps in the proof of Theorem 1 focussing only on the main part of the differential operator as the lower-order terms can be dealt with somewhat simpler.

For the discretisation error, we find from Proposition 1

$$\|P_H R_H u - P_H u_H\|_{1,2} \leq C \sup_{0 \neq w_H \in W_H} \frac{|\tau_H(w_H)|}{\|P_H w_H\|_{1,2}}$$

with the truncation error

$$\tau_H(w_H) := A_H(R_H u, w_H) - A_H(u_H, w_H)$$
$$= \tau_H^{(a)}(w_H) + \tau_H^{(b)}(w_H) + \tau_H^{(c)}(w_H) + \tau_H^{(d)}(w_H) + \tau_H^{(e)}(w_H) + \tau_H^{(f)}(w_H)\,,$$

where

$$\tau_H^{(a)}(w_H) := a_H(R_H u, w_H) + \sum_{P\in\overline{\Omega}_H} \int_{\Box_P} (au_x)_x dV\,\overline{w}_P - \sum_{P\in\Gamma_H} |\Gamma_P|(au_x\eta_x)(P)\,\overline{w}_P$$

and the other parts of τ_H corresponding to $b_H, \dots, f_H$ are defined analogously.

We commence with the case $s = 1$. The estimate of $\tau_H^{(a)}(w_H)$ relies upon the decomposition

$$\tau_H^{(a)}(w_H) \;=\; \tau_{H,1}^{(a)}(w_H) + \tau_{H,2}^{(a)}(w_H) \text{ with}$$
$$\tau_{H,2}^{(a)}(w_H) := \sum_{P\in\Gamma_H} \left(\int_{\Gamma_P} au_x\eta_x d\sigma - |\Gamma_P|(au_x\eta_x)(P) \right) \overline{w}_P\,.$$

Analogous decompositions are at hand for $\tau_H^{(b)}(w_H)$ and $\tau_H^{(c)}(w_H)$. The terms $\tau_{H,1}^{(a)}(w_H)$ and $\tau_{H,1}^{(c)}(w_H)$ satisfy the estimate desired as is shown in [4, Sect. 4]. Also $\tau_{H,1}^{(b)}(w_H)$, the truncation error related to mixed derivatives, can be estimated as desired as is shown in the following.

The proof starts with an integration and summation by parts, rewriting $\tau_{H,1}^{(b)}(w_H)$ in terms of contributions on rectangles $\Box_{j+1/2,\ell+1/2} := (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1}) \cap \Omega$, and an application of the Bramble-Hilbert lemma.

We find (see also [4, Lemmata 4.7, 4.8])

$$|\tau_{H,2}^{(a)}(w_H) + \tau_{H,2}^{(b)}(w_H) + \tau_{H,2}^{(c)}(w_H)|$$
$$= \left| \sum_{P\in\Gamma_H} \left(\int_{\Gamma_P} (\psi - \alpha u) d\sigma - |\Gamma_P|(\psi - \alpha u)(P) \right) \overline{w}_P \right|$$
$$\le C \Bigg(\sum_{P\in\Gamma_H} |\Gamma_P|^2 \big(\|u\|_{1,2,\Gamma_P}^2 + \|\psi\|_{1,2,\Gamma_P}^2 \big) \Bigg)^{1/2} \|P_H w_H\|_{1,2}\,,$$

which finally proves the result for $s = 1$. Note that $\|u\|_{1,2,\Gamma_P} \le \|u\|_{3/2,2,\Gamma_P}$.

We come to the case $s \in (1, 2]$. For $P \in \Gamma_{1/2}^y$, the set of points $(x_j, y_{\ell+1/2})$ lying on parts of Γ parallel to the y-axis, let $P^- := (x_j, y_\ell)$, $P^+ := (x_j, y_{\ell+1})$, and

$$\Gamma_P := \{x_j\} \times (y_\ell, y_{\ell+1})\,,\;\; \Gamma_P^- := \{x_j\} \times (y_\ell, y_{\ell+1/2})\,,\;\; \Gamma_P^+ := \{x_j\} \times (y_{\ell+1/2}, y_{\ell+1})\,.$$

The estimates are based upon the decomposition

$$\tau_H^{(a)}(w_H) \;=\; \tau_{H,3}^{(a)}(w_H) + \tau_{H,4}^{(a)}(w_H) \text{ with}$$

$$\tau_{H,4}^{(a)}(w_H) := \tau_{H,2}^{(a)}(w_H) - \frac{1}{2}\sum_{P\in\Gamma_{1/2}^y} |\Gamma_P| \bigg(\int_{\Gamma_P^+} a u_x \eta_x d\sigma - \frac{|\Gamma_P|}{2}(a u_x \eta_x)(P^+)$$
$$- \int_{\Gamma_P^-} a u_x \eta_x d\sigma + \frac{|\Gamma_P|}{2}(a u_x \eta_x)(P^-)\bigg) \delta_y^{(1/2)} \overline{w}_P$$

and analogous decompositions for $\tau_H^{(b)}(w_H)$ and $\tau_H^{(c)}(w_H)$.

The terms $\tau_{H,3}^{(a)}(w_H)$ and $\tau_{H,3}^{(c)}(w_H)$ satisfy local estimates of order s (see [4, Sect. 5]) and it remains to consider $\tau_{H,3}^{(b)}(w_H)$. With integration and summation by parts, $\tau_{H,3}^{(b)}(w_H)$ can again be rewritten in terms of contributions on rectangles $\square_{j+1/2,\ell+1/2}$. What follows, is a rather intrigued decomposition and multiple application of the (generalised) Bramble-Hilbert (see [3]) and bilinear lemma.

After all, the boundary contribution

$$- \sum_{P\in\Gamma_{1/2}^y} \frac{|\Gamma_P|^2}{4} \eta_x(P) \Big(b(P^+)\big(u_y(P^+) - u_y(P)\big) + b(P^-)\big(u_y(P) - u_y(P^-)\big)\Big) \times$$
$$\times \delta_y^{(1/2)} \overline{w}_P \tag{7}$$

(plus an analogous contribution from parts of Γ parallel to the x-axis) has to be estimated. Unfortunately, with the aid of the generalised Bramble-Hilbert lemma, we can only derive an estimate of maximum order 3/2 in terms of $|u|_{1/2+s,2,\Gamma}$. Here, we also employ the continuous embedding $H^1(\Omega) \hookrightarrow H^{1/2}(\Gamma)$ and an identification of $|P_H w_H|_{1/2,2,\Gamma}$ in terms of differences of values of w at the boundary Γ.

Some straightforward calculations finally show that

$$\tau_{H,4}^{(a)}(w_H) + \tau_{H,4}^{(b)}(w_H) + \tau_{H,4}^{(c)}(w_H) =$$
$$\sum_{P\in\Gamma_{1/2}^x} \bigg(\int_{\Gamma_P} (\psi - \alpha u) d\sigma - \frac{|\Gamma_P|}{2}\big((\psi - \alpha u)(P^+) + (\psi - \alpha u)(P^-)\big)\bigg) \frac{\overline{w}_{P^+} + \overline{w}_{P^-}}{2} +$$
$$\sum_{P\in\Gamma_{1/2}^y} \bigg(\int_{\Gamma_P} (\psi - \alpha u) d\sigma - \frac{|\Gamma_P|}{2}\big((\psi - \alpha u)(P^+) + (\psi - \alpha u)(P^-)\big)\bigg) \frac{\overline{w}_{P^+} + \overline{w}_{P^-}}{2} .$$

We are thus left with trapezoidal rules that lead to the estimate of order s (see also [4, Lemma 5.8] and note that $\|u\|_{s,2,\Gamma_P} \le C\|u\|_{1/2+s,2,\Gamma_P}$). Here, it is important to have the assumption $\psi \in H^s(\Gamma)$. □

By interpolation using the estimates (5), (6) of Theorem 1 for $s = 1$ and $s = 2$, which is then valid without the assumption of quasi-uniformity, the following corollary is derived.

Corollary 1. *Let $s \in [1, 2]$ and assume that $u \in H^{1+s}(\Omega)$, $a, b, c, d, e, f \in W^{2,\infty}(\Omega)$, $\alpha \in W^{2,\infty}(\Gamma)$, and $\psi \in H^s(\Gamma)$. The discretisation error then satisfies the estimate*

$$\|P_H R_H u - P_H u_H\|_{1,2} \le C H_{\max}^{(1+s)/2} \big(\|u\|_{1+s,2} + \|u\|_{1/2+s,2,\Gamma} + \|\psi\|_{s,2,\Gamma}\big)$$

in the case of mixed derivatives. If $b \equiv 0$ *then*

$$\|P_H R_H u - P_H u_H\|_{1,2} \le C H_{\max}^s \big(\|u\|_{1+s,2} + \|u\|_{s,2,\Gamma} + \|\psi\|_{s,2,\Gamma}\big)\,.$$

As is shown in [4, Remark 5.3, 5.4], the averaged restriction of the right-hand side g in (4) can be replaced by the pointwise restriction on $\overline{\Omega}_H$ if $g \in H^s(\Omega)$ for $s \in (1,2]$ retaining the order of convergence. If, however, g is not smooth enough then the pointwise restriction may destroy the higher order.

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